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LOCALLY CONFORMAL KAEHLER STRUCTURES ON TANGENT MANIFOLD OF A SPACE FORM

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Dedicated to Professor Radu Rosca on his 90th birthday anniversary

Abstract. *A set of locally conformal Kaehler structures on tangent manifold TM of a space form M is pointed out. This is found in a study of a type of Sasaki metric whose second term is a special deformation of the first one. Introducing an adequate almost complex structure we find at first a large class of locally conformal almost Kaehler structures on TM for M a (pseudo)- Riemannian manifold. When M is a space form, a subset of it is made of locally conformal Kaehler structures. One of them was found by R. Miron in [3].*

1. INTRODUCTION

Let (M,g) be a (pseudo)- Riemannian manifold and ∇ its Levi-Civita connection. In a local chart $(U,(x^i))$ we set $g_{ij} = g(\partial_i, \partial_j)$, where $\partial_i : \frac{\partial}{\partial x^i}$ and we denote by $\gamma^i_{jk}(x)$ the Christoffel symbols giving ∇. Let $(x^i, y^i) \equiv (x,y)$ be the local coordinates on the manifold TM projected on M by τ. The indices $i, j, k...$ will run from 1 to $n = dimM$.

The functions $N^i_j(x,y) := \gamma^i_{jk}(x)y^k$ are the local coefficients of a nonlinear connection, that is the local vector fields $\delta_i = \partial_i - N^k_i(x,y)\,\dot\partial_k$, where $\dot\partial_k: \frac{\partial}{\partial y^k}$ span a distribution on TM called horizontal which is supplementary to the vertical distribution $u \to V_uTM = \ker \tau_{*,u}, u \in TM$. Let us denote by $u \to H_uTM$ the horizontal distribution and let $(\delta_i, \dot\partial_i)$ be the basis adapted to the decomposition $T_uTM = H_uTM \oplus V_uTM, u \in TM$. The basis dual of it is $(dx^i, \delta y^i)$ with $\delta y^i = dy^i + N^i_k(x,y)dx^k$

The Sasaki metric on TM is as follows

$$G_S = g_{ij}(x)dx^i \otimes dx^j + g_{ij}(x)\delta y^i \otimes \delta y^j. \tag{1.1}$$

If in the second term of G_S one replaces $g_{ij}(x)$ with the components $h_{ij}(x,y)$ of a generalized Lagrange metric (see Ch.X in [4]) one gets a type of Sasaki metric

$$G(x,y) = g_{ij}(x)dx^i \otimes dx^j + h_{ij}(x,y)\delta y^i \otimes \delta y^j \tag{1.2}$$

In particular, $h_{ij}(x,y)$ could be a deformation of $g_{ij}(x)$, a case studied by the present author and H. Shimada in [1].

In this paper we are concerning with the metrical structure (1.2) in the case when $h_{ij}(x,y)$ is the following special deformation of $g_{ij}(x)$

(1.3) $$h_{ij}(x,y) = a(L^2)g_{ij}(x) + b(L^2)y_i y_j,$$
where $L^2 = g_{ij}(x)y^i y^j, y_i = g_{ij}(x)y^j$ and $a, b : \mathrm{Im}(L^2) \subseteq \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $a > 0, b \geq 0$.

For $b = 0$ and $a = \frac{c^2}{L^2}$ for any constant c, the metrical structure (1.2), (1.3) was studied by R. Miron in [3] as an homogeneous lift of $g_{ij}(x)$ to TM.

In the following Section we introduce an almost complex structure which paired with G given by (1.2), (1.3) provides a large set of almost Hermitian structures on TM. Then, in Section 3 we show that all these structures are locally conformal almost Kaehler structures. Finally, we find in Section 4 that, when (M, g) is of constant curvature, a part of them are locally conformal Kaehler structures.

2. SOME ALMOST HERMITIAN STRUCTURES ON TM

Let F_S be the almost complex structure on TM given in the adapted basis $(\delta_i, \dot{\partial}_i)$ by

(2.1) $$F_S(\delta_i) = -\dot{\partial}_i, F_S(\dot{\partial}_i) = \delta_i.$$

It is well known that the pair (G_S, F_S) is an almost Kaehler structure on TM, that is $G_S(F_S X, F_S Y) = G_S(X, Y)$ and the 2-form
$\Omega(X, Y) = G_S(F_S(X), Y)$ is closed, $X, Y \in \chi(M)$.

The pair (G, F_S) with G given by (1.2), (1.3) is no longer an almost Hermitian structure. We look for a new almost complex structure which paired with G to provide an almost Hermitian structure. We modify F_S to a linear map F given in the basis $(\delta_i, \dot{\partial}_i)$ as follows

(2.2) $$F(\delta_i) = (\alpha\delta_i^k + \beta y_i y^k)\dot{\partial}_k, F(\dot{\partial}_j) = (\gamma\delta_j^h + \delta y_j y^h)\delta_h,$$
where $\alpha, \beta, \gamma, \delta$ are functions on TM to be determined. The condition $F^2 = -I$ (identity) leads to

(2.3) $$\alpha\gamma = -1, \alpha\delta + \beta\gamma + \beta\delta L^2 = 0.$$

Then the condition $G(F(X), F(Y)) = G(X, Y)$ gives

(2.4) $$a\alpha^2 = 1, \gamma^2 = a, 2\gamma\delta + \delta^2 L^2 = b, (2\alpha\beta + \beta^2 L^2)(a + bL^2) + b\alpha^2 = 0$$

The solution of the system of equations (2.3), (2.4) is

(2.5) $$\alpha = -\frac{1}{\sqrt{a}}, \beta = \frac{\sqrt{a} + \sqrt{a + bL^2}}{L^2\sqrt{a(a + bL^2)}}, \gamma = \sqrt{a}, \delta = -\frac{\sqrt{a} + \sqrt{a + bL^2}}{L^2}.$$

We notice that for $b = 0$, besides the solution provided by (2.5), that is

(2.6) $$\alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = \frac{2}{L^2\sqrt{a}}, \delta = -\frac{2\sqrt{a}}{L^2},$$

there exists also the solution

(2.7) $$\alpha = -\frac{1}{\sqrt{a}}, \gamma = \sqrt{a}, \beta = 0, \delta = 0.$$

Let us make the substitution $a \longrightarrow \frac{a^2}{L^2}, b \longrightarrow \frac{b^2 - a^2}{L^4}$.

Then (2.5) and (2.6) are unified to

$$\alpha = -\frac{L}{a}, \beta = \frac{a+b}{abL}, \gamma = \frac{a}{L}, \delta = -\frac{a+b}{L^3}, b \geq a > 0 \quad (2.8)$$

and (2.7) modifies to

$$\alpha = -\frac{L}{a}, , \gamma = \frac{a}{L}, \beta = \delta = 0. \quad (2.9)$$

The metric G takes the form

$$(2.10)\ G_{a,b}(x,y) = g_{ij}(x)dx^i \otimes dx^j + (\frac{a^2}{L^2}g_{ij}(x) + \frac{b^2 - a^2}{L^4}y_i y_j)\delta y^i \otimes \delta y^j, b \geq a > 0.$$

Let $F_{a,b}$ be the almost complex structures given by (2.2), (2.8) and F_a those given by (2.2), (2.9). Then the pairs $(G_{a,b}, F_{a,b})$ and $(G_{a,a}, F_a)$ are almost Hermitian structures on TM.

For $a^2 = \frac{L^2}{1+L^2}, b = L^2$, the metric $G_{a,b}(x,y)$ is the Cheeger- Gromoll metric ,[5],[6]

$$G_{CG}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{1+L^2}(g_{ij}(x) + y_i y_j)\delta y^i \otimes \delta y^j. \quad (2.11)$$

If $a^2 = \varphi' L^2, b^2 = L^2(\varphi' + 2\varphi'' L^2)$ for $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\varphi'(t) \neq 0, t \in \mathrm{Im}(L^2)$, one obtains the Antonelli - Hrimiuc metrical structure ,[2]

$$G_{AH}(x,y) = g_{ij}(x)dx^i \otimes dx^j + (\varphi' g_{ij}(x) + 2\varphi'' y_i y_j)\delta y^i \otimes \delta y^j. \quad (2.12)$$

3. LOCALLY CONFORMAL ALMOST KAEHLER STRUCTURES ON TM

Let $\Omega(X,Y) = G_{a,b}(F_{a,b}X, Y), X, Y \in \chi(TM)$ be the 2-form associated to the almost Hermitian structure $(G_{a,b}, F_{a,b})$.

Theorem 3.1 *The almost Hermitian structures $(G_{a,b}, F_{a,b})$ are locally conformal almost Kaehlerian structures, that is*

$$d\Omega = \Omega \wedge \theta, \theta = \frac{2a'L + b}{aL}dL. \quad (3.1)$$

Proof. We shall check (3.1) on the basis (δ_i, ∂_i) . If we rewrite (2.2) in the form

$$F(\delta_i) = A_i^k \partial_k, F(\partial_i) = B_i^h \delta_h. \quad (3.2)$$

we easily get

$$\Omega(\delta_i, \delta_j) = 0, \Omega(\delta_i, \partial_j) = A_i^k h_{kj}, \Omega(\partial_j, \delta_i) = B_j^k g_{ki}, \Omega(\partial_i, \partial_j) = 0. \quad (3.3)$$

with $A_i^k h_{kj} + B_j^k g_{ki} = 0$.

Thus Ω is completely determined by

$$\Omega_{ij} := B_j^k g_{ki} = \gamma g_{ij} + \delta y_i y_j; \Omega = \Omega_{ij}\delta y^i \wedge dx^j. \quad (3.4)$$

Next we have the following essential components of $d\Omega$.

$$d\Omega(\delta_i,\delta_j,\dot\partial_k) = \delta_j\Omega_{ik} - \gamma^s_{ki}\Omega_{sj} - \delta_i\Omega_{jk} - \gamma^s_{kj}\Omega_{si},$$

$$d\Omega(\delta_i,\dot\partial_j,\dot\partial_k) = \dot\partial_j\,\Omega_{ik} - \dot\partial_k\,\Omega_{ij}.$$

Now we consider the Berwald connection $(N^i_j = \gamma^i_{kj}(x)y^k, \gamma^i_{kj}(x), 0)$ on TM (see Ch.8 in [4]) and denote by $|k$ its h- covariant derivative. Thus because of $\Omega_{jk|i} = \delta_i\Omega_{jk} - \gamma^s_{ji}\Omega_{sk} - \gamma^s_{ki}\Omega_{js}$, we have $d\Omega(\delta_i,\delta_j,\dot\partial_k) = \Omega_{ki|j} - \Omega_{kj|i}$.

The following formulae are verified by a direct calculation.

$$g_{ij|k} = 0, y^j_{|k} = 0, y_{i|k} = 0, \delta_k L^2 = 0, \delta_k\psi(L^2) = 0, \tag{3.5}$$

$$\dot\partial_k\, y_i = g_{ik}, \dot\partial_k\, L^2 = 2y_k, \dot\partial_k\,\psi(L^2) = 2y_k\psi'(L^2),$$

for any $\psi : \mathrm{Im}(L^2) \subseteq R_+ \longrightarrow R_+$.

Using (3.5) it immediately results $\Omega_{kj|i} = 0$ and so $d\Omega(\delta_i,\delta_j,\dot\partial_k) = 0$. Consequently, $d\Omega$ is completely determined by $d\Omega(\delta_i,\dot\partial_j,\dot\partial_k) = (\dot\partial_j\ \gamma)g_{ik} - (\dot\partial_k\ \gamma)g_{jk} + (\dot\partial_j\delta)y_k y_i - (\dot\partial_k\ \delta)y_j y_i + \delta(g_{ij}y_k - g_{ik}y_j)$

Inserting here $\dot\partial_j\ \gamma, \dot\partial_j\delta$ with γ,δ from (2.8) one arrives to

$$d\Omega(\delta_i,\dot\partial_j,\dot\partial_k) = (2\gamma' - \delta)(g_{ik}y_j - g_{ij}y_k) = \frac{2a'L^2 + b}{L^3}(g_{ik}y_j - g_{ij}y_k). \tag{3.6}$$

Let be $\theta_0 = dL^2 = 2y_i\delta y^i$. Thus $\theta_0(\delta_i) = 0$ and $\theta_0\ (\dot\partial_j) = 2y_j$. Evaluating $\Omega\wedge\theta_0$ on the basis $(\delta_i,\dot\partial_i)$ one finds the essential component

$$\Omega\wedge\theta_0(\delta_i,\dot\partial_j,\dot\partial_k) = 2(\Omega_{ik}y_j - \Omega_{ij}y_k) = \frac{2a}{L}(g_{ik}y_j - g_{ij}y_k). \tag{3.7}$$

Comparing (3.6) with (3.7) one obtains $d\Omega = \frac{2a'L^2+b}{2aL^2}\Omega\wedge\theta_0$ which is just (3.1). ■

Obviously θ is globally defined. Moreover, θ is closed. This fact can be directly verified using (3.5) or by differentiating (3.1).

Looking at (3.6) we notice that contracting $g_{ik}y_j - g_{ij}y_k = 0$ with g^{ik} one gets $(n-1)y_j = 0$ which is a contradiction. Thus we have

Theorem 3.2 *The almost Hermitian structures $(G_{a,b}, F_{a,b})$ are almost Kaehler structures if and only if*

$$2a'L^2 + b = 0, \tag{3.8}$$

holds good.

We put $t = L^2$ and think (3.8) as a first order differential equation : $2ta'(t) + b(t) = 0$. Its general solutions is $a(t) = c - \frac{1}{2}\int\frac{b(t)}{t}dt$ for a constant c. Thus for various functions b we find a set of pairs (a, b) for which (3.8) holds. Choosing among these pairs those which verify $b \geq a > 0$ we find a set of almost Kaehler structures on TM. For instance, if we take $b(t) = 2t$ it results $a(t) = c - t$ and $b \geq a > 0$ holds if $\frac{c}{3} \leq L^2(x,y) < c$, for $c > 0$. When $a = b$, the equation (3.8) has the general solution $a(t) = \frac{c}{\sqrt{t}}$. It follows

Corollary 3.1 *The almost Hermitian structures $(G_{a,a}, F_{a,a})$ are almost Kaehler structures if and only if $a(L^2) = \frac{c}{\sqrt{L^2}}, c > 0$.*

The almost Hermitian structures $(G_{a,a}, F_a)$ have to be separately considered. Repeating for them the proof of Theorem 3.1 one obtains

Theorem 3.3 *The almost Hermitian structures $(G_{a,a}, F_a)$ are locally conformal almost Kaehler structures, that is*

$$d\Omega = \Omega \wedge \theta, \theta = \frac{2a'L - a}{aL} dL.$$

The following corresponds to Theorem 3.2

Theorem 3.4 *The almost Hermitian structures $(G_{a,a}, F_a)$ are almost Kaehler structures if and only if $a = c\sqrt{L^2}, c > 0$.*

Proof. The almost Kaehler condition is now $2a'L^2 - a = 0$. Integrating the equation $2a't - a = 0$ one gets $a = c\sqrt{t}$. ■

Remark 3.1 *For $a = c\sqrt{L^2}, c > 0, G_{a,a}$ is very close to G_S which is obtained for $c = 1$.*

4. LOCALLY CONFORMAL KAEHLER STRUCTURES ON TM

In order to find conditions that $(G_{a,b}, F_{a,b})$ be a locally conformal Kaehler structure we have to put zero for the Nijenhuis tensor field of $F := F_{a,b}$,

$$N_F = [FX, F] - F[FX, Y] - F[X, FY] - [X, Y], X, Y \in \chi(TM). \tag{4.1}$$

As the evaluation of N_F on the basis $(\delta_i, \dot{\partial}_i)$ is in general very complicated we confine ourselves to the structures $(G_{a,a}, F_a)$. In this case, the conditions

$$N_F(\delta_i, \delta_j) = 0, N_F(\delta_i, \dot{\partial}_j) = 0, N_F(\dot{\partial}_j, \dot{\partial}_k) = 0, \tag{4.2}$$

are equivalent with six equations. Three of them are identities because of $\delta_i a = \delta_i \gamma = 0$ and the other three are each one equivalent with

$$R^k_{ij} = \frac{2a'L^2 - a}{a^3}(y_j \delta^k_i - y_i \delta^k_j), \tag{4.3}$$

where $R^k_{ij} = R^k_{sij}(x)y^s$ and R^k_{sij} is the curvature tensor of ∇.

By a contraction with g_{rk} the Eq. (4.3) reduces to

$$R_{srij}(x)y^s = \frac{2a'L^2 - a}{a^3}(g_{js}g_{ri} - g_{is}g_{rj})y^s. \tag{4.4}$$

The Eq. (4.4) remember us the condition that (M, g) is of constant curvature (space form). It suggests us to look for functions a such that $\frac{2a'L^2 - a}{a^3} = k$, where k is a constant. For $t = L^2$, solving the Bernoulli equation $a' = \frac{1}{2t}a + \frac{k}{2t}a^3$ one gets $a(L^2) = \sqrt{\frac{L^2}{c - kL^2}}$ for $c - kL^2 > 0$, where c is a constant of integration. For these functions a, the Eq. (4.4) becomes

$$R_{srij}(x)y^s = -k(g_{js}g_{ri} - g_{is}g_{rj})y^s, \tag{4.5}$$

which says that (M, g) is of constant curvature $-k$. Thus we have proved

Theorem 4.1 *If the (pseudo)- Riemannian manifold (M,g) is of constant curvature $k \in R$, for $a(L^2) = \sqrt{\frac{L^2}{c+kL^2}}$ with c a constant such that $c+kL^2 > 0$, the structures $(G_{a,a}, F_a)$ are locally conformal Kaehler structures on TM.*

The explicit form of these structures is as follows.

$$G_{a,a}(x,y) = g_{ij}(x)dx^i \otimes dx^j + \frac{1}{c+kL^2}(g_{ij}(x))\delta y^i \otimes \delta y^j. \tag{4.6}$$

$$F_a(\delta_i) = -\sqrt{c+kL^2}\,\dot{\partial}_i, F_a(\dot{\partial}_i) = \frac{1}{\sqrt{c+kL^2}}\delta_i, \tag{4.7}$$

The 1-form θ is

$$\theta = \frac{kL}{c+kL^2}dL. \tag{4.8}$$

Corollary 4.1 *For $a(L^2) = c_0\sqrt{L^2}$, with c_0 a strict positive constant, the pairs $(G_{a,a}, F_a)$ are Kaehler structures on TM if and only if (M,g) is flat.*

Proof. If (M,g) is flat, by the Theorem 4.1 for $a(L^2) = c_0\sqrt{L^2}, c_0 = \frac{1}{\sqrt{c}}$, the pair $(G_{a,a}, F_a)$ is a locally conformal Kaehler structure and by the Theorem 3.4 this is almost Kaehler. Thus $(G_{a,a}, F_a)$ is a Kaehler structure on TM. Conversely, if the pair $(G_{a,a}, F_a)$ with $a(L^2) = c_0\sqrt{L^2}$ is a Kaehler structure, the Eq. (4.3) gives $R^k_{ij} = 0$, equivalently $R_{srij}(x) = 0$, that is (M,g) is flat. ■

Looking at (4.6) and (4.7) we see that the structures $(G_{a,a}, F_a)$ from Corollary 4.1 are very close to (G_S, F_S) which is obtained for $c = 1$. Thus the Corollary 4.1 covers a well-known result: (G_S, F_S) is a Kaehlerian structure if and only if (M,g) is flat.

Finally, we notice that for $c = 0$ and $k \longrightarrow \frac{1}{k^2}$ in (4.6) and (4.7) one obtains the locally conformal Kaehler structure found by R.Miron in [3].

REFERENCES

[1] Anastasiei M., Shimada H., *Deformations of Finsler metrics*. To appear.

[2] Antonelli P.L., Hrimiuc D., *A new class of Spray-generating Lagrangians*. In the volume *Lagrange and Finsler geometry. Application to Physics and Biology*. Eds. P.L. Antonelli and R. Miron, Kluwer Academic Publishers, TFTPH 76, 1996, 81-92.

[3] Miron R., *The homogeneous lift of a Riemannian metric*. To appear.

[4] Miron R., Anastasiei M., *The geometry of Lagrange spaces: Theory and Applications*. Kluwer Academic Publishers. FTPH 59, 1994.

[5] Musso E., Tricerri F., *Riemannian metrics on tangent bundle*. Ann.Mat.Pura Appl.(4), 150, 1988, 1-20

[6] Sekizawa M., *Curvatures of tangent bundle with a Cheeger-Gromoll metric*. Tokyo J. Math.Vol. 14, (2), 1991, 407-417.