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FRAMED (2m+3)-DIMENSIONAL ALMOST KENMOTSU MANIFOLDS ENDOWED WITH A VERTICAL ξ-PARALLEL CONNECTION

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In the last decades contact, almost contact, paracontact and almost cosymplectic manifolds carrying $r\,(r > 1)$ Reeb vector fields ξ_r have been studied by a certain number of authors, as for instance M. Kobayashi [K], A. Bucki [B], S. Tachibana and W.N. Yu [TY], R. Rosca [R3], I. Mihai and R. Rosca [MR], F. Etayo and R. Rosca [ER], I. Mihai, R. Rosca and L. Verstraelen [MRV] and some others.

In the present paper, we consider a $(2m+3)$-dimensional framed manifold $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ carrying 3 Reeb vector fields ξ_r $(r \in \{2m+1, 2m+2, 2m+3\})$ satisfying the f-Kenmotsu condition, i.e.

$$\nabla \xi_r = f_r\, dp - \eta^r \otimes \xi\,, \quad f_r \in C^\infty M\,, \tag{0.1}$$

(dp: soldering form of M, $\xi = \sum f_r \xi_r$: principal vector field of M, $\eta^r = \xi_r^\flat$: Reeb covector), and, in addition, that the vertical connection forms θ_s^r define a *ξ-parallel connection* [R1], [MRV], i.e.

$$\theta_r^s = < \xi, \xi_s \wedge \xi_r > \tag{0.2}$$

($\wedge$: wedge product of vector fields).

It is shown that in this case, the structure 2-form Ω of M defines a locally cosymplectic 3-structure on M having $\eta = \xi^\flat$ as covector of Lee, that is:

$$d\Omega = 2\eta \wedge \Omega\,. \tag{0.3}$$

In Section 2, it is proved that any such manifold may be viewed as the local Riemannian product

$$M = M^\top \times M^\perp\,,$$

such that:

i) $M^\perp$ is a 3-dimensional submanifold of M, tangent to $\{\xi_r\}$ and which is a space form of curvature c;

ii) $M^\top$ is a $2m$-dimensional submanifold of M tangent to $\{\xi_r\}^\perp$ and having $-\xi$ as mean curvature vector field. In addition, the immersion $x : M^\top \to M$ is *pseudo-umbilical* and its associated *Gauss map* is conformal.

Moreover, the structure scalars f_r are *eigenfunctions* of the Laplacian Δ and the *spectrum* of Δ on $\wedge^1 M$ is $(2m+3)(c - \|\xi\|^2)$, and, in addition, f_r define an *isoparametric* system [W].

Next, it is proved that the manifold under consideration is a *framed CR-manifold* [Be] and some infinitesimal transformations defined by ξ are also discussed.

Finally, one studies some properties of horizontal skew-symmetric Killing vector fields V. Among others, it is shown that, as in the case of standard f-Kenmotsu manifolds, the vector fields $V, \phi V$ and ξ define a commutative triple, that is

$$[V,\xi] = 0\,, \quad [\phi V,\xi] = 0\,, \quad [V,\phi V] = 0$$

1. PRELIMINARIES

Let (M,g) be a Riemannian C^∞-manifold and ∇ the covariant differential operator defined by the metric tensor g. Let ΓTM be the set of sections of the tangent bundle TM and $\flat : TM \to T^*M$, $Z \mapsto Z^\flat$, the *musical isomorphism* [P] defined by g and $\sharp : T^*M \to TM$ its inverse.

Following a standard notation (see also [P]), we set $A^q(M,TM) = Hom\,(\wedge^q TM, TM)$ and notice that the elements of $A^q(M,TM)$ are vector valued q-forms ($q \leq \dim M$). Denote by $d^\nabla : A^q(M,TM) \to A^{q+1}(M,TM)$ the *exterior covariant derivative* operator with respect to ∇ (it should be noticed that generally $d^{\nabla^2} = d^\nabla \circ d^\nabla \neq 0$, unlike $d^2 = d \circ d = 0$). The identity tensor field dp of type (1.1) is a vector-valued 1-form and following [Di] is called the *soldering form* of M.

We will also recall the following definition (see [GL]). The operator $d^\omega = d + e(\omega)$, acting on $\wedge M$, is called the *cohomology operator*, i.e.

$$d^\omega u = du + \omega \wedge u\,, \tag{1.1}$$

for any $u \in \wedge M$. One has $d^\omega \circ d^\omega = 0$, and if $d^\omega u = 0$, u is said to be d^ω – *closed*. If ω is exact, then u is said to be d^ω – *exact*.

Let Ω be a closed or an exterior recurrent 2-form. Then one defines the (1.1)-operator $L : u \mapsto u \wedge \Omega$, $u \in \wedge^1 M$ (note that one has $dLu = Ldu + u \wedge Lu$) and one sets [We]

$$u_q = L^q u = u \wedge \Omega^q\,. \tag{1.2}$$

Proposition 1.1. Let $u \in \wedge^1 M$ be a Pfaff form on a manifold M. Then in order that u be of *class* $2s$ [G] on M it is necessary and sufficient to have

$$(du)^{s+1} = 0\,, \quad u \wedge (du)^s = 0\,. \tag{1.3}$$

Let $\mathcal{O} = \text{vect}\,\{e_A |\, A = 1, ..., n\}$ be an adapted local field of orthonormal frames over M and let $\mathcal{O}^* = \text{covect}\,\{\omega^A\}$ be the associated coframe. With respect to $\mathcal{O}$ and $\mathcal{O}^*$, the soldering form dp and E. Cartan's structure equations can be written in indexless manner as:

$$dp = \omega^A \otimes e_A \in A^1(M,TM)\,, \tag{1.4}$$

$$\nabla e = \theta \otimes e\,, \tag{1.5}$$

$$d\omega = -\theta \wedge \omega\,, \tag{1.6}$$

$$d\theta = -\theta \wedge \theta + \Theta\,. \tag{1.7}$$

If $\mathcal{A}$ and ψ are a vector-valued 1-form and a vector-valued p-form respectively, then the covariant derivative of ψ with respect to $\mathcal{A}$ is expressed by

$$d_{\mathcal{A}}\psi = d\psi - c_p[\mathcal{A}, \psi] , \tag{1.8}$$

where c_p is a nonzero constant. Then Yang-Mills equations are

$$F = d_{\mathcal{A}}\mathcal{A} , \quad d_{\mathcal{A}} * F = 0 , \tag{1.9}$$

where $*$ means the Hodge operator and F is called the *field strengths* (see also [Br]).

2. VERTICAL ξ-PARALLEL CONNECTION

Let $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ be a $(2m+3)$-dimensional Riemannian framed manifold [YK] with soldering form dp and structure tensors $(\phi, \Omega, \xi_r, \eta^r)$. These tensors satisfy, as is known (see also [R3]):

$$\begin{cases} \phi^3 + \phi = 0 \ , \ \phi\xi_r = 0 \ , \ \eta^r \circ \phi = 0 \ , \\ \phi^2 = -Id + \Sigma_r \eta^r \otimes \xi_r \ , \\ g(Z, Z') = g(\phi Z, \phi Z') + \Sigma_r \eta^r(Z)\eta^r(Z') \ , \\ \Omega(Z, Z') = g(\phi Z, Z') \ , \ \Omega^m \wedge \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3} \neq 0 \ , \end{cases} \tag{2.1}$$

($r \in \{2m+1, 2m+2, 2m+3\}$), where ϕ, Ω, ξ_r and η^r represent the (1.1)-tensor field, the structure 2-form, the Reeb vector fields and the Reeb covector fields, respectively.

By reference to Section 1, we also write $\eta^r = \xi_r^\flat$ and $\xi_r = (\eta^r)^\flat$.

With respect to the cobasis $\mathcal{O}^* = \text{covect}\{\omega^a, \eta^r\}$ of the vector basis $\mathcal{O} = \text{vect}\{e_a, \xi_r ; a \in \{1, ..., 2m\}\}$, Ω and dp are expressed by

(2.2) $dp = \omega^a \otimes e_a + \eta^r \otimes \xi_r$.(2.3) $\qquad \Omega = \sum_{a=1}^{m} \omega^a \wedge \omega^{a^*} , \quad a^* = a + m ,$

Further, by E. Cartan's structure equations (1.5) and by (2.1), the *horizontal* connection forms θ_b^a [R3] satisfy the Kähler conditions:

$$\theta_b^a = \theta_{b^*}^{a^*} , \quad \theta_b^{a^*} = \theta_a^{b^*} . \tag{2.4}$$

One may decompose the tangent space T_pM at $p \in M$ as

$$T_pM = D_p^\top \oplus D_p^\perp , \tag{2.5}$$

where $D_p^\perp = \{\xi_r\}$ defines the *vertical* distribution spanned by ξ_r and its orthogonal complement $D_p^\top = \{e_a\}$ the *horizontal* distribution (or the almost contact distribution).

In consequence of (2.5), any vector field Z may be split as $Z = Z^\top + Z^\perp$ (i.e. $Z^\top$ and $Z^\perp$ are the horizontal and vertical components of Z respectively), and similarly one may write

$$dp = dp^\top + dp^\perp . \tag{2.6}$$

In these conditions, if $f_r \in C^\infty M$ are nonvanishing scalars, we agree to call the vector field

$$\xi = \Sigma f_r \xi_r \tag{2.7}$$

the *principal* structure vector field of M, and we assume in a first step that the vertical connection forms θ^r_s define a ξ-*parallel* connection [R1], [MRV], i.e.

$$\theta^s_r = < \xi, \xi_s \wedge \xi_r > . \tag{2.8}$$

By (2.7), it follows that

$$\theta^s_r = f_r \eta^s - f_s \eta^r . \tag{2.9}$$

Next, in a second step, we assume as in [ER], that the transversal connection forms θ^a_r are expressed by

$$\theta^a_r = f_r \omega^a .$$

Then, making use of the structure equations (1.5), one finds by (2.9)

$$\nabla \xi_r = f_r dp - \eta^r \otimes \xi . \tag{2.10}$$

We notice that if $\xi_r = \xi$ and $f_r = 1$, the above equation is the same as in the case of Kenmotsu manifolds.

Therefore, a framed manifold M such that (2.8) and (2.10) hold good will be defined to be endowed with an almost K-structure, equipped with a vertical ξ-*parallel* connection.

One derives by (2.10)

$$[\xi_r, \xi_s] = f_s \xi_r - f_r \xi_s = (\theta^r_s)^\sharp \tag{2.11}$$

and one checks the Jacobi identity

$$\sum [\xi_r, [\xi_s, \xi_t]] = 0$$

((r, s, t): cyclic permutation of $2m+1, 2m+2, 2m+3$).

Further, setting $\eta = f_r \eta^r = \xi^\flat$ and making use of (2.4) and (2.10), one obtains by exterior differentiation of (2.2)

$$d\Omega = 2\eta \wedge \Omega . \tag{2.12}$$

This shows that Ω defines a locally cosymplectic 3-structure on M, having η as covector of Lee.

It should be noticed that this property agrees with the Kenmotsu character of the manifold under discussion.

Similarly, making use of the structure equations (1.6), we find by (2.10)

$$d\eta^r = \eta \wedge \eta^r . \tag{2.13}$$

Since, by the above, one has $\eta^r \wedge (d\eta^r) = 0$, it follows (see also [G]) that the Reeb covectors η^r are of *class* 2.

We notice that the fact that this class is not maximal is in accordance with the almost contact structure of M.

By (2.10) one derives $d\eta = 0$. Thus, in terms of d^ω-cohomology, (2.13) may be written as $d^{-\eta}\eta^r = 0$, i.e. the Pfaffians η^r are all $d^{-\eta}$-closed.

From (2.13) it follows that

$$df_r = \lambda \eta^r \Rightarrow d\lambda + \lambda\eta = 0 , \quad \lambda \in C^\infty M . \tag{2.14}$$

Then, setting $2f = ||\xi||^2$, we quickly derives $f + \lambda = c =$ constant.

Since, following a well-known definition, one has

$$\operatorname{div}\xi_r = \sum_a g(\nabla_{e_a}\xi_r, e_a) + g(\nabla_{\xi_r}\xi_r, \xi_r) + g(\nabla_{\xi_s}\xi_r, \xi_s) + g(\nabla_{\xi_t}\xi_r, \xi_t)\,,$$

one finds

$$\operatorname{div}\xi_r = (2m+2)f_r\,, \tag{2.15}$$

which shows that up to $2m+2$ the structure functions f_r represent the divergences of the Reeb vector fields ξ_r (the indices correspond).

Now, denoting by $\nabla\tau = \operatorname{grad}\tau$, $\tau \in C^\infty M$, one gets by (2.14)

$$\nabla f_r = \lambda\xi_r \Rightarrow \|\nabla f_r\|^2 = \lambda^2 \tag{2.16}$$

and by the well-known formula $\Delta\tau = -\operatorname{div}\nabla\tau$, one finds

$$\Delta f_r = (2m+3)\lambda f_r\,. \tag{2.17}$$

This shows that f_r are *eigenfunctions* of Δ. Hence, $(2m+3)\lambda$ is the *spectrum* of Δ on $\wedge^1 M$ and $\{f_r\}$ form the corresponding *eigenspace*.

Also, one finds $g(\nabla f_r, \nabla f_s) = 0$, $r \neq s$, and

$$[\nabla f_r, \nabla f_s] = 2\lambda(f_s\nabla f_r - f_r\nabla f_s)\,.$$

Therefore, by reference to [W], the structure scalars f_r define an *isoparametric* system on M.

By the structure equations (1.5), (1.6) and by (2.7) and (2.10), one also derives

$$\begin{cases} \nabla e_a = \theta_a^b \otimes e_b - \omega^a \otimes \xi\,, \\ \\ d\omega^a = \omega^b \wedge \theta_b^a - \omega^a \wedge \eta\,. \end{cases} \tag{2.18}$$

Then, since $dp^\top = \omega^a \otimes e_a$, $dp^\perp = \eta^r \otimes \xi_r$, operating by d^∇ on the principal components $dp^\top$ and $dp^\perp$ of dp, one obtains using (2.18)

$$d^\nabla(dp^\top) = \eta \wedge dp^\top\,, \quad d^\nabla(dp^\perp) = -\eta \wedge dp^\top\,.$$

Making use of the Yang-Mills equation (1.9) (see also [Br]), it is worth to point-out the following fact:

Identifying $\mathcal{A}$ with $dp^\perp$ and making use of (2.11) and (2.13), then one derives

$$d_{\mathcal{A}}\mathcal{A} = (1 - c_p)\eta \wedge dp^\perp\,.$$

On the other hand, by (2.1) and the first equation (2.18), one obtains the following structure equation

$$(\nabla\phi)Z = -\eta(Z)\phi dp^\top - (\phi Z)^\flat \otimes \xi\,, \tag{2.19}$$

which is similar to that of standard Kenmotsu manifolds (see also [MMR]).

Next, set

$$\varphi^\top = \omega^1 \wedge \ldots \wedge \omega^{2m}\,,$$
$$\varphi^\perp = \eta^{2m+1} \wedge \eta^{2m+2} \wedge \eta^{2m+3}\,,$$

for the simple forms which correspond to the distributions $D^\top$ and $D^\perp$ respectively. Then, with the help of the second equation (2.18) and of (2.13), one finds

$$d\varphi^\top = 2m\eta \wedge \varphi^\top\,,$$

$$d\varphi^{\perp} = 0\,.$$

This shows that $\varphi^{\top}$ is a *conformal integral invariant* of $dp^{\top}$ and $\varphi^{\perp}$ is an *integral invariant* of $D^{\perp}$.

Hence, the following fact emerges: any manifold $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ may be viewed as the local Riemannian product

$$M = M^{\top} \times M^{\perp}\,,$$

where $M^{\top}$ is a $2m$-dimensional submanifold tangent to $\{\xi_r\}^{\perp}$ and $M^{\perp}$ is a 3-dimensional submanifold tangent to $\{\xi_r\}$.

We will now focus on the vertical and the transversal curvatures of M, i.e. on Θ^r_s and Θ^a_r respectively. Making use of the structure equations (1.7) and (2.9), one finds

$$\Theta^r_s = 2c\eta^r \wedge \eta^s\,.$$

Hence, following a known definition (see also [KN]), the above equation reveals that $M^{\perp}$ is a space form of curvature c.

Similarly, one has

$$\Theta^a_r = (c + f)\eta^r \wedge \omega^a\,,$$

which proves that the curvature forms Θ^a_r are monomial. By reference to [BCGG], it follows that the *retracting space* $C(I)$ (I ideal, $I \subset \wedge V^*$) associated with Θ^a_r is of dimension 1 and has as Grassmannian coordinate vectors the horizontal covectors ω^a. One defines the following 3 vector-valued 2-forms

$$\Pi_r = \sum_a \Theta^a_r \otimes e_a = (c + f)\eta^r \wedge dp^{\perp}\,.$$

We agree to denominate Π_r the associated vector-valued 2-forms with the transversal curvature forms.

On the other hand, by (2.10) and (2.14) one gets

$$\nabla\xi = 2f\,dp + \lambda dp^{\perp} - \eta \otimes \xi \tag{2.20}$$

and one finds at once $\nabla_{\xi}\xi = \lambda\xi$, i.e. ξ is an *affine geodesic*.

Operating on (2.20) by d^{∇} and taking account of (2.14), one derives

$$d^{\nabla}(\nabla\xi) = \nabla^2\xi = (f + c)\eta \wedge dp + \lambda\eta \wedge dp^{\perp}\,. \tag{2.21}$$

Therefore, since $\nabla^2 X(Z, Z') = R(Z, Z')X$; $X, Z, Z' \in \Gamma TM$, where R denotes the curvature tensor, one finds

$$R(Z, Z')\xi = 2c(Z' \wedge Z)\xi\,. \tag{2.22}$$

Hence, by reference to [BKP], the above equation proves that ξ satisfies the *nullity condition*.

Also, it follows at once by (2.22) that the *Jacobi operator* R_{ξ} defined by ξ is given by

$$R_{\xi} = 2c(\xi \wedge Z)\xi\,. \tag{2.23}$$

On the other hand, if we take the Lie derivatives with respect to ξ of the Reeb covectors η^r, one gets

$$\mathcal{L}_{\xi}\eta^r = (c + f)\eta^r - f_r\eta\,, \tag{2.24}$$

and by exterior differentiation we obtain

$$d(\mathcal{L}_\xi \eta^r) = (\lambda + 2c)\eta \wedge \eta^r \,, \tag{2.25}$$

which may be written as $d^{-a\eta}(\mathcal{L}_\xi \eta^r) = 0$, where $a = -\frac{\lambda+2c}{\lambda-2c}$.

Therefore, one may say that the Lie derivatives $\mathcal{L}_\xi \eta^r$ are $d^{-a\eta}$-closed.

Further, we have

$$d^\nabla \Pi_r = \frac{4c-\lambda}{2c-\lambda}\eta \wedge \Pi_r \,, \tag{2.26}$$

which shows that Π_r are *exterior recurrent* [D] vector-valued 2-forms having $\frac{4c-\lambda}{2c-\lambda}\eta$ as recurrence form.

Finally, we will outline some properties of the immersion $x : M^\top \to M$. Denote like usual by γ^r_{ab} the connection coefficients corresponding to x. Then, the *mean curvature* vector field H is expressed by

$$H = \frac{\gamma^r_{aa}\xi_r}{\dim M^\top} \,.$$

Then, on behalf of (2.10), we quickly find $H = -\xi$.

Next, recalling that the second and third fundamental forms II and III associated with x are given by

$$II = -< dp^\top, \nabla\xi > , \quad III = -\Sigma(\theta^r_a)^2 \,,$$

one derives on behalf of (2.10)

$$II = -2fg^\top \,, \quad III = 2fg^\top \,.$$

Therefore, since II is conformal to the metric tensor $g^\top$ of $M^\top$, the immersion x is *pseudo-umbilical* and since III is also conformal to $g^\top$, the *Gauss map* (in the sense of M. Obata [O]) is conformal.

Summing up, we have the following.

Theorem 2.1. Let $M(\phi, \Omega, \xi_r, \eta^r; f_r, g)$ be a framed $(2m+3)$-dimensional almost K-manifold endowed with a vertical ξ-parallel connection. Let ξ (resp. ξ_r) be the principal vertical vector field of M (resp. the Reeb vector fields) and let f_r be the structure scalars of M.

Any such manifold may be viewed as the local Riemannian product

$$M = M^\top \times M^\perp \,,$$

such that:

i) $M^\perp$ is a 3-dimensional submanifold of M, tangent to $\{\xi_r\}$, which is a space form of curvature c;

ii) $M^\top$ is a $2m$-dimensional submanifold of M, tangent to $\{\xi_r\}^\perp$, having $-\xi$ as mean curvature vector field.

In addition, the immersion $x : M^\top \to M$ is pseudo-umbilical and the Gauss map associated with x is conformal.

Moreover, M enjoys the following properties:

a) Ω defines a locally cosymplectic 3-structure on M, having $\eta = \xi^\flat$ as covector of Lee;

b) the structure scalars f_r are eigenfunctions of Δ and the spectrum of Δ on $\wedge^1 M$ is $(2m+3)\lambda, (\lambda = c - ||\xi||^2)$, and f_r also define an isoparametric system on M;

c) ξ satisfies the nullity condition, that is

$$R(Z,Z')\xi = 2c(Z' \wedge Z)\xi\,, \quad Z, Z' \in \Gamma TM\,;$$

d) the Lie derivatives $\mathcal{L}_\xi \eta^r$ are $d^{-a\eta}$-closed, where $a = -\frac{\lambda+2c}{\lambda-2c}$;

e) the transversal curvature forms define 3 vector-valued 2-forms Π_r, which are d^∇-exterior recurrent.

Recall now that the torsion tensor field S of a framed structure is the vector-valued 2-form defined by

$$S = N_\phi + 2\Sigma_r d\eta^r \otimes \xi_r\,,$$

where

$$N_\phi(Z,Z') = [\phi Z, \phi Z'] + \phi^2[Z,Z'] - \phi[Z,\phi Z'] - \phi[\phi Z, Z']\,,$$

is the Nijenhuis tensor field of ϕ, and

$$S^\perp = 2\Sigma_r d\eta^r \otimes \xi_r$$

is the vertical component of S [R3].

Hence, following a known definition, the framed structure under consideration is $D^\top$-*normal* and consequently $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ is a *framed CR* manifold (in the sense of [Be]).

Further, by (2.13) one quickly finds $S^\perp = 2\eta \wedge dp^\perp$, and one derives $d^\nabla S^\perp = 0$, which shows that $S^\perp$ is, as dp, a d^∇-closed vector-valued form.

Since one has $i_\xi S^\perp = 4f\,dp^\perp$ and

$$\mathcal{L}_\xi S^\perp = 4(\lambda - f)\eta \wedge (dp^\perp - dp^\top)\,,$$

then one finally deduces

$$d^\nabla(\mathcal{L}_\xi S^\perp) = 0$$

Hence $S^\perp$ is a *relatively integral invariant* (see also [A]) of ξ.

We may state the

Theorem 2.2. Any framed almost K-manifold endowed with a vertical ξ-parallel connection is a framed CR manifold and the vertical component $S^\perp$ of the torsion tensor field associated with the almost K-structure of M is a relatively integral invariant of the principal vector field ξ of M.

3. SKEW SYMMETRIC KILLING VECTOR FIELDS.

Let now $V = V^a e_a \in D^\top$ be any horizontal vector field of the framed manifold $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ under discussion, i.e. $\eta^r(V) = 0$.

Then, taking the covariant differential of V, one has

$$\nabla V = (\lambda V^a + V^b \theta^a_b) \otimes e_a - V^b \otimes \xi \tag{3.1}$$

Next, by reference to [R2], [MRV], one finds by (3.1) that the necessary and sufficient condition in order that V be a *skew-symmetric* Killing vector field having ξ as generative, i.e.

$$\nabla V = V \wedge \xi\,, \tag{3.2}$$

is expressed by

$$dV^\flat = 2\eta \wedge V^\flat \,. \tag{3.3}$$

The above equation proves that the dual form $V^\flat$ of V is an exterior recurrent 1-form [D].

If we set $2l = ||V||^2$, one finds from (3.2)

$$\frac{dl}{2l} = \eta \,, \tag{3.4}$$

which is coherent with (2.14).

Therefore, we may say that the existence of V is determined by a closed Pfaff system.

We notice that since η is an exact form, then in terms of d^ω-cohomology we may write (3.3) as $d^{-2\eta}(V^\flat) = 0$ and say that $V^\flat$ is a $d^{-2\eta}$-exact form.

Furthermore, by the structure equation (2.19), one easily finds by (3.2)

$$\begin{cases} \nabla \phi V = \phi V \wedge \xi \,, \\ d(\phi V)^\flat = 2\eta \wedge (\phi V)^\flat \,. \end{cases} \tag{3.5}$$

Then, the property of skew-symmetric vector field is invariant under the action of ϕ (as in the case of standard Kenmotsu manifolds).

On the other hand, we have seen that the Jacobi operator $R_\xi = R(\cdot, \xi)\xi$ is expressed by

$$R_\xi(Z) = 2c(\eta \odot Z - Z^\flat \otimes \xi)\xi \,.$$

Then, putting $Z = V$, one derives

$$R_\xi(V) = c||\xi||^2 V \,,$$

which shows that the action of R_ξ on V is a conformal application.

By (3.3) and (3.5) it is also seen that one has

$$[V, \xi] = 0 \,, \quad [\phi V, \xi] = 0 \,, \quad [V, \phi V] = 0 \,.$$

Therefore, the vector fields ξ, V and ϕV define a *commutative triple.*

We also notice that by (3.3) and the second equation (3.5), one finds

$$\mathcal{L}_V(V^\flat) = 0 \,, \quad \mathcal{L}_{\phi V}(\phi V)^\flat = 0 \,,$$

which proves that V and ϕV are *self invariant* vector fields.

Recall now that the (1.1)-operator L of Weyl [We] associated with a 2-form Ω is the application $L : u \mapsto u \wedge \Omega$, $L^q u = u_q : u \mapsto u \wedge \Omega^q$, $u \in \wedge^1 M$.

Comming back to the case under discussion, one derives by (3.3) and (3.5)

$$\mathcal{L}_\xi(V^\flat)_q = 2(1+q)(V^\flat)_q \,,$$
$$\mathcal{L}_\xi(\phi V)^\flat_q = 2(1+q)(\phi V)^\flat_q \,.$$

Hence one may state that the principal vertical vector field ξ defines an infinitesimal conformal transformation of all the $(2q+1)$-forms $(V^\flat)_q$ and $(\phi V)^\flat_q$.

Summarizing, we have the following.

Theorem 3.1. The existence on the framed manifold $M(\phi, \Omega, \xi_r, \eta^r, f_r, g)$ of a skew-symmetric Killing vector field V having ξ as generative is assured by a closed Pfaff system, and this property is invariant under the action of ϕ.

In this case ξ, V and ϕV define a commutative triple and the action of the Jacobi operator R_ξ on V and ϕV is a conformal application.

Moreover, the principal vertical vector field ξ defines an infinitesimal conformal transformation of the $(2q+1)$-forms $(V^\flat)_q$ and $(\phi V)^\flat_q$.

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