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DISTRIBUTIONS ADMITTING A LOCAL BASIS OF HOMOGENEOUS POLYNOMIALS

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To our friend, Professor and Academician Radu Rochas-Rosca.

Abstract. *A survey of several results by the authors on the characterization of homogeneous algebraic distributions in the vertical bundle of a vector bundle, is presented.*

1. INTRODUCTION

The purpose of this survey is to give an account of results by the authors, the main of them being a characterization of homogeneous algebraic distributions on vector bundles. As it is well-known, a vector field X on $\mathbb{R}^m$ is homogeneous algebraic of degree d if and only if $[\chi, X] = (d-1)X$, χ being the Liouville vector field. If one wants to involve exclusively the module structure spanned by a vector field X in characterizing algebraic vector fields, then one is led to study the equation $[\chi, X] = fX$. In this case, a first result (*cf.* Proposition 4.4 below) states that the function f should be constant along the zero section of the vector bundle $p\colon E \to M$ on which X is defined and this constant should be an integer ≥ -1. Our main result is a generalization of the above statement to distributions of arbitrary rank: it is stated that a vertical distribution locally spanned by $X_1, \ldots, X_r$ is homogeneous algebraic of degree d if and only if an $r \times r$ matrix $A = (a_{ij})$, $a_{ij} \in C^\infty(E)$, exists which is equal to $d-1$ times the identity matrix along the zero section of E, and such that $[\chi, X_j] = \sum_{i=1}^r a_{ij} X_i$, for $j = 1, \ldots, r$ (*cf.* Theorem 4.3 below).

Algebraic distributions play a role in several fields of real and complex geometry such as singularities of vector fields, the moduli problem for differential forms, calculation of differential invariants of a Lie group action, etc. (*e.g.*, see [6,10,14,16,22]). Thus it seems interesting to obtain a characterization of these differential systems. Linear representations of families of Lie groups on vector bundles give rise to such distributions in a natural way. Families of Lie groups and specially Lie group fibre bundles naturally appear in the field theory and gauge theories as well as in differential geometry. For example, the gauge group of a principal bundle $P \to M$ can be obtained as the global sections of the adjoint bundle of P (*cf.* [1]; also see [2, 3, 4, 9, 17, 18, 19]). This bundle is endowed with a natural structure of Lie group fibre bundle. Similarly, the gauge algebra of P is obtained by taking global sections in a Lie algebra bundle attached to P. For Lie algebra bundles this is

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indeed an extension to non-trivial bundles of the well-known process of gauging a Lie algebra $\mathfrak{g}$ with functions of a manifold M, which corresponds to consider the infinite Lie algebra $C^\infty(M)\otimes\mathfrak{g}$ with the bracket $[f\otimes A, g\otimes B] = fg\otimes[A,B]$, $f,g\in C^\infty(M)$, $A,B\in\mathfrak{g}$, as $C^\infty(M)\otimes\mathfrak{g}$ may be viewed as the global sections of the trivial bundle $\mathrm{pr}_1\colon M\times\mathfrak{g}\to M$. These techniques also apply to Supersymmetry ([15]). In §5 we introduce the general definition of these structures showing the usual settings in which they are commonly found and we characterize (see Example 5.1 below) the distributions induced by a linear representation of a family of Lie groups.

We define an algebraic distribution as a sheaf of submodules of the sheaf of germs of p-vertical vector fields on a vector bundle $p\colon E\to M$, which is locally spanned by a finite number of algebraic vector fields. Note that according to this definition, distributions may be singular and they usually are in the algebraic case but in any case the rank of the distribution is kept to be locally constant on a dense open subset. This seems to be a suitable general setting in order to introduce algebraic distributions as a vector bundle is endowed with a canonical structure of algebraic scheme over M, independently of the class of functions (C^∞, C^ω or complex-analytic) that we consider on the base manifold, although we formulate our results in the C^∞ category for the sake of simplicity.

2. ALGEBRAIC MORPHISMS OF VECTOR BUNDLES

Let $p\colon E\to M$ be a vector bundle over a C^∞ manifold M. Assume $U\subseteq M$ is an open subset such that $E|_U$ is trivial and let $s_1,\dots,s_m$ be a basis of sections of E over U. Each vector $e\in p^{-1}(x)$, $x\in U$, can be uniquely written as $e=\lambda_1 s_1(x)+\dots+\lambda_m s_m(x)$, for some scalars $\lambda_i\in\mathbb{R}$, $1\le i\le m$. Hence we can define m functions $y_i\colon p^{-1}(U)\to\mathbb{R}$, $y_i(e)=\lambda_i$. In addition, assume that U is a coordinate domain with coordinates $(x_1,\dots,x_n)$, with $n=\dim M$. Then, it is not difficult to prove that $(x_j;y_i)$, $1\le i\le m$, $1\le j\le n$, is a coordinate system on $p^{-1}(U)$ for the manifold E. Such systems of coordinates will be called *vector bundle charts.*

Let $p\colon E\to M$, $p'\colon E'\to M'$ be two vector bundles over the C^∞ manifolds M, M', respectively. A differentiable mapping $\Phi\colon E\to E'$ is said to be an *algebraic morphism* if Φ satisfies the following two conditions:

(1) Φ is a bundle map over M.

(2) For every $x\in M$, there exist open coordinate neighbourhoods U, U' of x, $\phi(x)$, respectively over which $E|_U$ and $E'|_{U'}$ are trivial, and vector bundle charts $(U;x_j,y_i)$, $(U';x'_{j'},y'_{i'})$, $1\le j\le n$, $1\le i\le m$, $1\le j'\le n'$, $1\le i'\le m'$, for E, E', respectively such that $\phi(U)\subseteq U'$, and

$$y'_{i'}(\Phi(v)) = P_{i'}(\phi(p(v)))(y_1(v),\dots,y_m(v)),\quad 1\le i'\le m',\ \forall v\in p^{-1}(U),$$

where $P_{i'}$ are polynomials in the fibre variables with coefficients in the ring of differentiable functions of U'; *i.e.*, $P_{i'}\in C^\infty(U')[t_1,\dots,t_m]$, t_i being m indeterminates.

The morphism Φ is said to be *homogeneous of degree* $(r_1,\dots,r_{m'})$ if $P_{i'}$ is a homogeneous polynomial of degree $r_{i'}$, $1\le i'\le m'$. If all polynomials $P_{i'}$ have the same degree r, we shall say that Φ is homogeneous of degree r. If $M=M'$ and ϕ is the identity map, then Φ is said to be a *vertical morphism.* Note that the definition makes sense as it does not depend on the vector bundle charts chosen. In fact, if $(U;\bar{x}_j,\bar{y}_i)$ is another vector bundle chart of E on U, then there exists an invertible $m \times m$ matrix $(a_{ij}(x))_{i,j=1}^{m}$, $a_{ij} \in C^\infty(U)$, such that $\bar{y}_i = \sum_{j=1}^{m} a_{ij}(x) y_j$, $1 \leq i \leq m$, and hence if $P(\bar{y}_1, ..., \bar{y}_m)$ is a polynomial of degree r in $C^\infty(U')[t_1, ..., t_m]$, then $P(\phi(x))(\sum_{j=1}^{m} a_{1j}(x)y_j, ..., \sum_{j=1}^{m} a_{mj}(x)y_j)$ is also a polynomial in $C^\infty(U')[t_1, ..., t_m]$ of the same degree, and similarly in changing vector bundle charts in the bundle E'.

Given a vector bundle $p: E \to M$, we denote by $S^r(E)$ the vector bundle of its r-th symmetric power and $S^\bullet(E) = \oplus_{r=0}^{\infty} S^r(E)$ stands for the direct sum of such bundles endowed with the standard $\mathbb{Z}$-graded algebra structure. We have

Proposition 1 *The vertical homogeneous algebraic morphisms of degree r from $p: E \to M$ into $p': E' \to M$ can be identified with the global sections of the bundle $S^r(E^*) \otimes E' \to M$.*

3. ALGEBRAIC VECTOR FIELDS

Let $p: E \to M$ be a vector bundle. We set $\mathcal{S}^\bullet(E^*) = \Gamma(M, S^\bullet(E^*))$. We have a natural inclusion of $\mathcal{S}^\bullet(E^*)$ as a subalgebra $\mathcal{S}^\bullet(E^*) \subset C^\infty(E)$; that is, *the sections of the symmetric algebra of E^* can be interpreted as differentiable functions on the manifold E*, by simply setting $e \in E \mapsto \sigma(e, ..., e) \in \mathbb{R}$, $\forall e \in E$, $\forall \sigma \in \mathcal{S}^r(E^*)$. Accordingly, a C^∞ vector bundle $p: E \to M$ is endowed with a natural compatible algebraic structure since the elements of $\mathcal{S}^\bullet(E^*)$ are understood to be polynomial functions on E. In fact, Spec $\mathcal{S}^\bullet(E^*)$ is a $C^\infty(M)$-scheme in the sense of [8, I. Definition 2.6.1] (also see [8, I.9.4]). Moreover, from the Stone-Weierstrass-Nachbin theorem (see [21]) it follows that $\mathcal{S}^\bullet(E^*)$ is a dense subalgebra of $C^\infty(E)$ with respect to the natural C^∞ topology in such a way that the ring of the algebraic functions on E approximates that of differentiable functions.

A vector field $X \in \mathfrak{X}(E)$ is said to be *algebraic* if it is p-vertical and leaves invariant the subalgebra $\mathcal{S}^\bullet(E^*) \subset C^\infty(E)$; *i.e.*, $X(\mathcal{S}^\bullet(E^*)) \subseteq \mathcal{S}^\bullet(E^*)$. An algebraic vector field X is said to be *homogeneous of degree r* if $X(\mathcal{S}^k(E^*)) \subseteq \mathcal{S}^{k+r-1}(E^*)$, $\forall k \in \mathbb{N}$.

It is not difficult to prove that algebraic vector fields on E can be identified with $\mathrm{Der}_{C^\infty(M)}(\mathcal{S}^\bullet(E^*))$.

Taking into account the Definition in [24, V. 6.Appendice] one can prove the parts 1) and 2) of the following

Proposition 2 1) *A vector field $X \in \mathfrak{X}(E)$ is algebraic if and only if on every vector bundle chart $(U; x_j, y_i)$ we have $X|_U = \sum_{i=1}^{m} P_i \partial/\partial y_i$, $P_i \in C^\infty(U)[y_1, ..., y_m]$.*

2) *X is homogeneous of degree r if and only if $P_1, ..., P_m$ are homogeneous polynomials of the same degree r.*

3) *There is a canonical isomorphism of $C^\infty(M)$-modules,*

$$\lambda_E: \mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E) \to \mathrm{Der}_{C^\infty(M)}(\mathcal{S}^\bullet(E^*)).$$

4) *For every $x \in M$ there is a unique algebra estructure $[\ ,\ ]$ on $S^\bullet(E_x^*) \otimes E_x$ such that*

$$[t_k \otimes e, t'_{k'} \otimes e'] = (\mathrm{i}_{e'} t_k) \cdot t'_{k'} \otimes e - (\mathrm{i}_e t'_{k'}) \cdot t_k \otimes e', \tag{*}$$

for every $t_k \in S^k(E_x^*)$, $t'_{k'} \in S^{k'}(E_x^*)$, $e, e' \in E_x$ *with respect to wich* λ_E *is an anti-isomorphism, where* i_e *stands for the interior product by* e *on the symmetric algebra* (*cf.* [5, III.11.6]).

The first homogeneous component of $S^\bullet(E^*) \otimes E$ has been introduced by P. Lecomte in studying the Lie algebra of infinitesimal automorphisms of a vector bundle (see [11] and references therein).

The bracket defined by the formula (*) on $\mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E)$ is similar to the Richardson-Nijenhuis bracket (*cf.* [23, II.c.Remarque 1] , [12]) although our bracket does not take into account the graded structure of the algebra $\mathcal{S}^\bullet(E^*) \otimes \mathcal{S}^1(E)$, as we wish λ_E to be an anti-isomorphism.

4. ALGEBRAIC DISTRIBUTIONS

We denote by $\mathfrak{X}_M$ the sheaf of germs of vector fields on M. Let $p\colon E \to M$ be a fibred manifold; *i.e.*, p is a surjective submersion. We denote by $\mathfrak{X}_E^v$ the subsheaf of p-vertical vector fields in $\mathfrak{X}_E$.

A *vertical distribution* over a fibered manifold $p\colon E \to M$ is a subsheaf of C_E^∞-modules $\mathcal{D} \subset \mathfrak{X}_E^v$. A vertical distribution is said to be *locally finitely generated* if for every $e \in E$ there exist an open neighbourhood U of $x = p(e)$ in M and vector fields $X_1, \dots, X_r \in \mathfrak{X}_E^v(p^{-1}(U))$ such that $\mathcal{D}(p^{-1}(U))$ is spanned as a $C^\infty(p^{-1}(U))$-module by $X_1, \dots, X_r$. If $p\colon E \to M$ is a vector bundle and the generators are homogeneous algebraic vector fields (resp. homogeneous algebraic vector fields of the same degree), we say that $\mathcal{D}$ is an *algebraic distribution* (resp. a *homogeneous algebraic distribution*).

Note that according to the above definition, a distribution may be singular; *i.e.*, the rank of the vector space $\mathcal{D}_e = \{X_e \mid X \in \mathcal{D}(V)\}, e \in V$, needs not be locally constant. In fact, this is usually the case for algebraic distributions. The above notion of a distribution corresponds to that of a generalized distribution given in [13, Appendix 3] .

We shall repeatedly use the fact that if $X_1, \dots, X_r$ span $\mathcal{D}(V)$, then they also span $\mathcal{D}(W)$ for each open subset $W \subset V$. This is an easy consequence of the fact that $\mathfrak{X}_E^v$ is a quasi-flasque sheaf (*cf.* [24]). We have

Proposition 3 *For each algebraic distribution* $\mathcal{D}$ *on* $p\colon E \to M$, *there exists a dense open subset* $O \subset E$ *such that* $\mathcal{D}|_O$ *is homogeneous.*

Let $p\colon E \to M$ be a vector bundle. The *Liouville vector field* on E is the infinitesimal generator $\chi \in \mathfrak{X}^v(E)$ of the one-parameter group of homotheties, $\varphi_t\colon E \to E$, $\varphi_t(e) = \exp(t) \cdot e$, $\forall t \in \mathbb{R}$, $\forall e \in E$.

It is easy to see that on each vector bundle chart $(U; x_j, y_i)$, $1 \le j \le n$, $1 \le i \le m$, we have $\chi_{|U} = \sum_{i=1}^m y_i \partial/\partial y_i$. We have

Lemma 1 *A vector field* $X \in \mathfrak{X}^v(E)$ *is a homogeneous algebraic vector field of degree* d *if and only if* $[\chi, X] = (d-1)X$.

Applying the lemma we prove

Theorem 1 *Let $\mathcal{D}$ be a vertical distribution over a vector bundle $p\colon E \to M$, and let $X_1, \dots, X_r$ be p-vertical vector fields on $p^{-1}(U)$ spanning $\mathcal{D}|_{p^{-1}(U)}$, and assume that $E|_U$ is trivial. Then, a matrix $C \in GL(r; C^\infty(p^{-1}U))$, $C = (c_{ij})_{i,j=1}^{r}$, exists so that the vector fields $Y_j = \sum_{i=1}^{r} c^{ij} X_i$, $1 \leq j \leq r$, are homogeneous algebraic of degree d, with $(c^{ij}) = C^{-1}$, if and only if a $r \times r$ matrix $A = (a_{ij})$, $a_{ij} \in C^\infty(p^{-1}U)$, exists such that,*

(1) *$A(0_x) = (d-1)I$, $\forall x \in U$, where I stands for the $r \times r$ identity matrix,*
(2) *$[\chi, X_j] = \sum_{i=1}^{r} a_{ij} X_i$, $1 \leq j \leq r$.*

As an easy consequence of the above theorem we can conclude that if f is a first integral of an algebraic distribution $\mathcal{D}$, then $\chi(f)$ also is a first integral of $\mathcal{D}$, but unfortunately this method does not provide new algebraic first integrals as we have $\chi(f) = d \cdot f$, if f is a homogeneous polynomial of degree d.

For rank-1 distributions, the condition 1 in the above theorem can be weakened. More precisely, we prove:

Proposition 4 *Let X be a p-vertical vector field on $p\colon E \to M$, such that* support $X = E$. *Furthermore, assume that there exists a function $f \in C^\infty(E)$, such that $[\chi, X] = fX$. If M is connected, then there exist an integer $d \geq 0$ and an invertible function $g \in C^\infty(E)$ such that,*

(1) *$f(0_x) = d - 1, \forall x \in M$.*
(2) *$g^{-1}X$ is a homogeneous algebraic vector field of degree d.*

5. EXAMPLES AND APPLICATIONS

Example 1 *The distribution killing a finite number of analytic functions. Let $f_1, \dots, f_r$ be real analytic functions on $\mathbb{R}^m$, and let $\mathcal{D} \subset \mathfrak{X}_{\mathbb{R}^m}$ be the involutive distribution of all vector fields X in $\mathbb{R}^m$ such that $Xf_i = 0$, $1 \leq i \leq r$. The distribution $\mathcal{D}$ is finitely generated. In fact, let $\mathcal{O}_{\mathbb{R}^m}$ be the sheaf of germs of real analytic functions and let $\mathfrak{X}^\omega_{\mathbb{R}^m}$ be the sheaf of germs of analytic vector fields. We have an exact sequence of sheaves of $\mathcal{O}_{\mathbb{R}^m}$-modules*

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathfrak{X}^\omega_{\mathbb{R}^m} \xrightarrow{f^\omega} (\mathcal{O}_{\mathbb{R}^m})^r,$$

*where $\mathcal{K} = \ker f^\omega$ and $f^\omega(X) = (Xf_1, \dots, Xf_r)$. From the Noetherian properties of stalks of $\mathcal{O}_{\mathbb{R}^m}$ and $\mathfrak{X}^\omega_{\mathbb{R}^m}$ (*e.g., *see* [24, II.Thèorème 1.5] *and remark that $\mathfrak{X}^\omega_{\mathbb{R}^m} \cong (\mathcal{O}_{\mathbb{R}^m})^m$) it follows that $\mathcal{K}$ is a finitely generated sheaf of $\mathcal{O}_{\mathbb{R}^m}$-modules. Moreover, we have a natural isomorphism of sheaves of $C^\infty_{\mathbb{R}^m}$-modules, $C^\infty_{\mathbb{R}^m} \otimes \mathfrak{X}^\omega_{\mathbb{R}^m} \cong \mathfrak{X}_{\mathbb{R}^m}$. Tensoring the above exact sequence over $C^\infty_{\mathbb{R}^m}$ we obtain an exact sequence of sheaves of $C^\infty_{\mathbb{R}^m}$-modules,*

$$0 \longrightarrow C^\infty_{\mathbb{R}^m} \otimes_{\mathcal{O}_{\mathbb{R}^m}} \mathcal{K} \longrightarrow \mathfrak{X}_{\mathbb{R}^m} \xrightarrow{f^\infty} (C^\infty_{\mathbb{R}^m})^r,$$

where f^∞ is given by $f^\infty(X) = (Xf_1, \dots, Xf_r)$, for every $X \in \mathfrak{X}(\mathbb{R}^m)$, as follows from Malgrange's division theorem taking into account that $\mathcal{O}_{\mathbb{R}^m} \hookrightarrow C^\infty_{\mathbb{R}^m}$ is a flat ring extension (see [24, VI.Corollaire 1.3] *). Accordingly, from the very definition of $\mathcal{D}$, we have a canonical isomorphism $\mathcal{D} \cong C^\infty_{\mathbb{R}^m} \otimes_{\mathcal{O}_{\mathbb{R}^m}} \mathcal{K}$. Hence $\mathcal{D}$ is locally generated by a number of analytic vector fields and, recalling that $\mathcal{D}$ is a quasi-flasque sheaf by using* [24, V.Proposition

6.4] *and a partition of unity, we conclude that $\mathcal{D}$ is finitely generated.*

Moreover, if $f_1, \dots, f_r$ are homogeneous polynomials of common degree d, then it can be proved that $\mathcal{D}$ is a homogeneous algebraic distribution of degree $(d-1)r$, where we further assume that at least one of the $r \times r$ determinants of the Jacobian matrix $(\partial f_i / \partial y_j)$, $1 \leq i \leq r$, $1 \leq j \leq m$, does not vanish identically. Also remark that the above result can be generalized to the vertical bundle of an arbitrary analytic morphism between C^ω manifolds.

Next, we confine ourselves to the case of homogeneous algebraic distributions of degree 1, which correspond to linear representations of Lie group families. Precisely,

A *family of Lie groups* on a manifold M is a surjective submersion $\pi\colon \mathcal{G} \to M$ endowed with two differentiable mappings $\mu\colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $\iota\colon \mathcal{G} \to \mathcal{G}$, such that for every $x \in M$, the mappings $\mu_x\colon \mathcal{G}_x \times \mathcal{G}_x \to \mathcal{G}_x$, $\iota_x\colon \mathcal{G}_x \to \mathcal{G}_x$ induced on the fibre, define on $\mathcal{G}_x$ a structure of Lie group.

An *operation of a family of Lie groups* $\pi\colon \mathcal{G} \to M$ on a fibred manifold $p\colon E \to M$ is a differentiable map $\lambda\colon \mathcal{G} \times_M E \to E$ such that,

1. $\lambda(1_x, y) = y$, $\forall x \in M$, $\forall y \in \pi^{-1}(x)$,
2. $\lambda(g', \lambda(g'', y)) = \lambda(g'g'', y)$, $\forall x \in M$, $\forall g', g'' \in \pi^{-1}(x)$, $\forall y \in p^{-1}(x)$.

Example 2 *Fundamental vector fields of a Lie group family action. Let us consider an operation of a family of Lie groups $\pi\colon \mathcal{G} \to M$ on a fibred manifold $p\colon E \to M$. A vector bundle of Lie algebras $\bar{\pi}\colon \mathfrak{G} \to M$ is defined by setting $\bar{\pi}^{-1}(x) = \mathfrak{G}_x$ = Lie algebra of $\pi^{-1}(x) = \mathcal{G}_x$. Each section $\xi\colon U \to \mathfrak{G}$ of $\bar{\pi}$ induces a p-vertical vector field $\tilde{\xi} \in \mathfrak{X}^v_E(p^{-1}(U))$ whose flow is given by $\tau_t(e) = \lambda(\exp(t\xi(p(e))), e)$, $e \in p^{-1}(U)$. Then, the given operation λ induces a p-vertical distribution $\mathcal{D}$ on E defined as follows. For every open subset $V \subseteq E$, $\mathcal{D}(V)$ is the $C^\infty(V)$-module spanned by the vector fields $\tilde{\xi}$, where $\xi \in \Gamma(p(V), \mathfrak{G})$. Note that $\mathcal{D}$ is involutive as we have $[\xi_1, \xi_2]^\sim = [\tilde{\xi}_1, \tilde{\xi}_2]$, for every ξ_1, $\xi_2 \in \Gamma(\mathfrak{G})$.*

Example 3 *Linear representations of Lie group families. In the previous example assume that $p\colon E \to M$ is a vector bundle and also that for every $x \in M$, the operation induced on the fibre $\lambda_x\colon \mathcal{G}_x \times E_x \to E_x$ is a linear representation. Then, for every $\xi \in \Gamma(\mathfrak{G})$, we have $[\chi, \tilde{\xi}] = 0$, and hence the associated distribution is algebraic of degree 1. In fact, $\tilde{\xi}$ is linear as its flow is a one-parameter group of linear automorphisms of the vector bundle E.*

Lemma 2 *With the above hypotheses and notations, let $\mathcal{D}$ be the p-vertical distribution on E defined by $\lambda\colon \mathcal{G} \times_M E \to E$. Assume that on an open subset $U \subseteq M$ the following holds true: $\dim \mathfrak{G}_x = \max_{e \in E_x}(\operatorname{rk} \mathcal{D}_e)$, $\forall x \in U$. If $\xi_1, \dots, \xi_r$ is a basis of sections of $\bar{\pi}\colon \mathfrak{G} \to M$ over U, then $\tilde{\xi}_1, \dots, \tilde{\xi}_r$ are linearly independent on a dense open subset in $p^{-1}(U)$.*

By applying Lemmas 4.2 and 5.1, and Example 5.3, we prove

Proposition 5 *If $\lambda\colon \mathcal{G} \times_M E \to E$ is a linear representation of $\pi\colon \mathcal{G} \to M$ on a vector bundle $p\colon E \to M$ and $\dim \mathfrak{G}_x = \max_{e \in E_x}(\operatorname{rk} \mathcal{D}_e) = r$, $\forall x \in M$, then a linear vector field $X \in \mathfrak{X}^v(E)$ belongs to the distribution $\mathcal{D}$ induced from λ if and only if a vector field $\xi \in \Gamma(M, \mathfrak{G})$ exists such that $X = \tilde{\xi}$.*

By using 3) in Proposition 3.1, we prove

Proposition 6 *Let $\mathcal{D}$ be a homogeneous algebraic involutive distribution of degree* 1 *over a vector bundle $p\colon E \to M$. Then, a family of Lie groups $\pi\colon \mathcal{G} \to M$ and a linear representation $\lambda\colon \mathcal{G} \times ME \to E$ exist such that $\mathcal{D}$ is the distribution induced by λ.*

Families of Lie groups appear in a natural way in the framework of differential geometry and the field theory. Let us describe some examples of such a structure which are closely related to our present work.

Example 4 *The Lie group family of automorphisms of a vector bundle. Each vector bundle $p\colon E \to M$ defines a Lie group fibre bundle, precisely, the bundle of its fibred automorphisms $\pi\colon \mathrm{Aut}(E) \to M$, whose standard fibre is $Aut(E_x) \cong GL(m;\mathbf{R})$, $m = \mathrm{rk}\, E$. Several Lie group subfamilies in $Aut(E)$ are also interesting when different geometric structures on E are considered. Also note that $Aut(E)$ admits a natural linear representation on E.*

Example 5 *Higher order linear group bundles acting on jets of metrics. Given a manifold M, let $\pi_r\colon G^r(M) \to M$ be the bundle of Lie groups whose fibre over $x \in M$ is the group of r-jets at x of differentiable mappings $f\colon M \to M$ such that $f(x) = x$ and $f_*\colon T_xM \to T_xM$ is an isomorphism. Then, $G^r(M)$ is a Lie group fibre bundle whose standard fibre is $G^r(n)$, $n = \dim M$, the r-th order linear group (see* [10, 12.6] *). Let $p\colon \mathcal{M} \to M$ be the bundle of metrics of a prescribed signature. An operation of the Lie group fibre bundle $G^{r+1}(M)$ on $J^r(\mathcal{M})$ is defined by setting $j_x^{r+1}f \cdot j_x^r g = j_{f(x)}^r(\bar{f} \circ g \circ f^{-1})$, for every $j_x^{r+1}f \in G^{r+1}(M)$ and every local section g of $p\colon \mathcal{M} \to M$, where $\bar{f}\colon \mathcal{M} \to \mathcal{M}$ stands for the natural lifting of f to the metric bundle;* i.e., *$\bar{f} \cdot g_x = \left(f^{-1}\right)^* g_x$, for $g_x \in \mathcal{M}_x = p^{-1}(x)$. Note that $\mathcal{M}$ is a convex open subset in $S^2T^*(M)$ and the above operation is the restriction of the natural linear representation of $\pi_{r+1} : G^{r+1}(M) \to M$ on $p_r\colon J^r\left(S^2T^*(M)\right) \to M$. It can be proved (*cf. [20] *) that the determination of the r-th order $Diff(M)$-invariant Lagrangians on the metric bundle is reduced to calculate C^∞ functions on the r-jet bundle which are invariant under the above operation of the $(r+1)$-th order linear group fibre bundle.*

Example 6 *Associated bundles admitting a Lie group bundle structure. The basic groups in the field theory are the group of diffeomorphisms of a manifold and the group of vertical automorphisms of a principal bundle. The previous example shows how $Diff(M)$ gives rise to natural operations of a family of Lie groups. Next, we consider the gauge groups. Let $\pi\colon P \to M$ be a principal bundle with structure group G_0, and let us consider an operation of G_0 on another Lie group F by acting on the left by automorphisms of F;* i.e., *$g \cdot (f_1 f_2) = (g \cdot f_1)(g \cdot f_2)$, $\forall g \in G_0$, $\forall f_1, f_2 \in F$. Then, the associated bundle $\pi_F\colon \mathcal{G} = P \times^{G_0} F \to M$ is endowed with a natural structure of Lie group fibre bundle, uniquely determined by imposing $[u, f_1] \cdot [u, f_2] = [u, f_1 f_2]$, for every $u \in P$, $f_1, f_2 \in F$, where $[u, f]$ stands for the coset defined by the pair $(u, f) \in P \times F$ in the quotient manifold $\mathcal{G} = (P \times F)/G_0$.*

Example 7 *The case of the gauge group. The above situation is obtained when* G_0 *acts onto itsef by conjugation;* i.e., $g \cdot f = gfg^{-1}$, $\forall f, g \in G_0$. *In this case the associated bundle is called the* adjoint bundle of P *and denoted by* $\pi_{G_0} \colon \mathrm{Ad}\, P \to M$. *Its sections can be identified with the gauge group:* $\Gamma(M, \mathrm{Ad}\, P) = \mathrm{Gau}\, P$. *Similarly, if we consider the adjoint representation of* G_0 *onto its Lie algebra* $\mathfrak{g}_0$*, then the associated fibre bundle* $\bar{\pi} = \pi_{\mathfrak{g}_0} \colon \mathfrak{G} = \mathrm{ad}\, P = P \times^{G_0} \mathfrak{g}_0 \to M$ *is endowed with a natural structure of Lie algebra fibre bundle given by* $[[u, \xi_1], [u, \xi_2]] = [u, [\xi_1, \xi_2]]$*, for* $u \in P$, $\xi_1, \xi_2 \in \mathfrak{g}_0$. *With such a structure,* $\mathfrak{G} = \mathrm{ad}\, P$ *can be identified with the Lie algebra bundle of* $\mathcal{G} = \mathrm{Ad}\, P$ *in the sense of Example 5.2. Also note that the global sections of* $\bar{\pi}$ *can be identified with the gauge algebra of* P *(see* [4, 9] *). Let* $p \colon C(P) \to M$ *be the bundle of connections on* P *(here, a connection is understood to be an splitting of the Atiyah sequence;* cf. [1, 17] *). According to Utiyama's theorem (see* [3, 4] *) the determination of the gauge invariant Lagrangians on* $J^1(\mathcal{C}(P))$ *can be reduced to calculate* C^∞ *functions on the "curvature bundle"* $\bigwedge^2 T^*(M) \otimes \mathrm{ad}\, P$ *which are invariant under the natural representation of the adjoint bundle* $\pi_{G_0} \colon \mathrm{Ad}\, P \to M$ *on* $\bigwedge^2 T^*(M) \otimes \mathrm{ad}\, P$.

REFERENCES

[1] M.F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207.

[2] M.F. Atiyah, *Geometry and Physics: where are we going?*, Lecture Notes in Pure and Appl. Math. **184** (1997), 1–7.

[3] D. Betounes, *The geometry of gauge-particle field interaction: a generalization of Utiyama's theorem*, J. Geom. Phys. **6** (1989), 107–125.

[4] D. Bleecker, *Gauge Theory and Variational Principles*, Addison-Wesley, Reading, MA, 1981.

[5] N. Bourbaki, *Eléments de Mathématique, Algèbre*, Hermann, Paris, 1970.

[6] F. Cano, *Réduction de singularités des feuilletages holomorphes*, C. R. Acad. Sci. Paris Sér. I Math. **307** (1988), 795–798.

[7] H. Cartan, *Calcul différentiel*, Hermann, Paris, 1967.

[8] A. Grothendieck, J.A. Dieudonné, *Eléments de Géometrie Algébrique*, Springer-Verlag, Berlin-Heidelberg, 1971.

[9] V. Guillemin, S. Sternberg, *Symplectic techniques in physics*, Cambridge University Press, Cambridge, UK, 1983.

[10] I. Kolář, P.W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, Springer-Verlag, Berlin-Heidelberg, 1993.

[11] P. Lecomte, *On the infinitesimal automorphisms of a vector bundle*, J. Math. Pures Appl. **60** (1981), 229–239.

[12] P. Lecomte, P.W. Michor, H. Schicketanz, *The multigraded Nijenhuis-Richardson algebra, its universal property and applications*, J. Pure Appl. Algebra **77** (1992), 87–102.

[13] P. Libermann, Ch.M. Marle, *Symplectic Geometry and Analytical Mechanics*, D. Reidel, Dordrecht, Holland, 1987.

[14] B. Malgrange, *Frobenius avec singularités*, Invent. Math. **39** (1977), 67–89.

[15] Y.I. Manin, *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Berlin-Heidelberg, 1988.

[16] J. Martinet, J.P. Ramis, *Problèmes de modules pour des équations différentielles non linéaires du premier ordre*, Inst. Hautes Études Sci. Publ. Math. **55** (1981), 63–164.

[17] K. Michael, *About the geometric structure of symmetry-breaking*, J. Math. Phys. **32** (1991), 1065–1071.

[18] J. Mickelsson, *Current Algebras and Groups*, Plenum Press, New York, 1989.

[19] P.K. Mitter, C.M. Viallet, *On the Bundle of Connections and the Gauge Orbit Manifold in Yang-Mills Theory*, Comm. Math. Phys. **79** (1981), 457–472.

[20] J. Muñoz Masqué, A. Valdés Morales, *The number of functionally independent invariants of a pseudo-Riemannian metric*, J. Phys. A **27** (1994), 7843–7855.

[21] L. Nachbin, *Sur les algèbres denses de fonctions différentiables sur une varieté*, C. R. Acad. Sci. Paris Sér. I Math. **228** (1949), 1549-1551.

[22] P. Olver, *Equivalence, Invariants, and Symmetry*, Cambridge University Press, Cambridge, UK, 1995.

[23] C. Roger, *Algèbres de Lie graduées et quantification*, Symplectic Geometry and Mathematical Physics, Progress in Mathematics **99**, Birkhäuser, Boston, 1991, pp. 374–421.

[24] J.C. Tougeron, *Ideaux de fonctions differentiables*, Springer-Verlag, Berlin-Heidelberg, 1972.