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EXISTENCE OF SOLUTIONS OF STOCHASTIC INTEGRAL EQUATIONS USING INTEGRAL CONTRACTORS

R.Subramaniam and K.Balachandran

Abstract: The aim of this paper is to prove the existence of solutions of nonlinear stochastic integral equations. The method of integral contractor is used to establish the results.

1. INTRODUCTION

The concept of contractors by Altman [1] has been extended to random operators in Banach spaces by Lee and Padgett [3]. Lee and Padgett [4] and Padgett and Rao [5] used the techniques of random contractors to study random integral equations of mixed type. Here our aim is to use the concept of contractor to a more general stochastic integral equation of the form

$$x(t;w) = h(t;w) + \int_0^t f_1(t,s,x(s;w);w)\,ds + \int_0^\infty f_2(t,s,x(s;w);w)\,ds$$

$$+ \int_0^t f_3(t,s,x(s;w);w)\,d\beta(s;w) \qquad (1)$$

where $t \geq 0$, and $\beta(t;w)$ is a Brownian process. The deterministic version of (1) and other types of integral equations have been considered by many authors (see [2]). The results obtained in this paper generalize the results of [4,5].

2. PRELIMINARIES

A complete probability space is denoted by (Ω, A, P). Assume that there exist a set of sub σ-algebras A_t, $t \in R^+$ in A such that $A_s \subset A_t$ if $s < t$. Also we assume that the process $\beta(t;w)$ in (1) is a standard Brownian motion process adapted to A_t such that $\beta(t+h;w) - \beta(t;w)$, $h>0$ is independent of A_t.

We shall now define some specific function spaces which are needed for the present work. Let $x(t;w)$ be a second order stochastic process adapted to A_t. Denote

$$\|x(t;w)\|_{L_2} = (E[x(t;w)]^2)^{1/2} \qquad (2)$$

where $E(.)$ is the usual expectation with respect to (Ω, A, P).

Definition 1: Let $C \equiv C(R^+, L_2)$ denote the space of all functions $x(t;w)$ adapted to A_t such that (i) $\|x(t;w)\|_{L_2} < \infty$ and (ii) for each $t \in R^+$, the map $t \dashrightarrow \|x(t;w)\|_{L_2}$ is continuous. We shall induce a topology on C by the family of semi-norms

$$\|x(t;w)\|_n = \sup_{t \in [0,n]} \|x(t;w)\|_{L_2} \qquad (3)$$

It is known that this topology is metrizable and the resulting metric space is a Frechet space.

Definition 2: Let $C_1 \equiv C_1(\Delta, L_2)$ denote the space of all real functions $x(t,s;w)$ defined for $0 \le s \le t < \infty$, $w \in \Omega$, such that (i) $x(t,s;w)$ is adapted to A_t and (ii) for each $(t,s) \in \Delta = \{(t,s): 0 \le s \le t < \infty\}$, the mapping $t \dashrightarrow \|x(t,s;w)\|_{L_2}$ is continuous.

We define a topology on C_1 by the family of semi-norms

$$\|x(t,s;w)\|_C = \sup_{0 \le s \le t < \infty} \|x(t,s;w)\|_{L_2}$$

Definition 3: (Tsokos and Padgett [5]) Let C_u denote the space of all functions in C that satisfy the condition

$$\|x(t;w)\|_{L_2} \le K\ u(t)$$

where $u(t) > 0$ is a given continuous function and K is some positive constant. It is known that C_u is a Banach space with the norm defined by

$$\|x(t;w)\|_{C_u} = \sup\ \{\ \|x(t;w)\|_{L_2}/u(t)\ :\ t \ge 0\ \}.$$

Definition 4: If $u(t) \equiv 1$ in Definition 3, we shall denote the corresponding C_u by C_b.

Definition 5: Let $C_{1,u}$ denote the space of all functions $x(t,s;w)$ in C_1 such that

$$\|x(t,s;w)\|_{L_2} \le K\ u(t)\ u(s)$$

for some constant $K > 0$ and bounded positive continuous function $u(t)$. It is easily shown that $C_{1,u}$ is a Banach space with norm $\|\cdot\|_{C_{1,u}}$ defined by

$$\|x(t,s;w)\|_{C_{1,u}} = \sup\ \{\ \|x(t,s;w)\|_{L_2}/\ u(t)u(s)\ :\ 0 \le s \le t < \infty\ \}.$$

Definition 6: Let B and D be Banach spaces in C. The pair (B,D) is said to be admissible with respect to the operator T if and only if $T(B) \subset D$.

Definition 7: Let B be a Banach space in a Frechet space F. B is said to be stronger than F if every sequence that converges in the norm of B also converges in the topology of F.

LEMMA 1: (Tsokos and Padgett [5]) Let T be a continuous linear operator from C_1 to C. Let B and D be Banach spaces stronger than C_1 and C respectively. Then T is continuous from B into D, provided (B,D) is admissible with respect to T.

We shall now introduce the concept of a bounded integral vector contractor of three functions. Let B and D be Banach spaces stronger than C_1 and C respectively. Define the integral operators T_1, T_2 and T_3 on B by

$$(T_1x)(t;w) = \int_0^t x(t,s;w)\,ds \tag{4}$$

$$(T_2x)(t;w) = \int_0^\infty x(t,s;w)\,ds \tag{5}$$

$$(T_3x)(t;w) = \int_0^t x(t,s;w)\,d\beta(s;w) \tag{6}$$

LEMMA 2: The operators T_i, $i = 1,2,3$ are continuous.

Let us assume that (B,D) is admissible with respect to T_i, $i = 1,2,3$ and $F_i(t,s,x;w)$, $i=1,2,3$ be real valued functions defined for $(t,s) \in \Delta$, $x \in R$ and $w \in \Omega$ such that $F_i(t,s,;w) \in B$ whenever $x(s;w) \in D$.

Definition 8: The vector of functions (F_1, F_2, F_3) is said to have a bounded integral vector contractors $(\Gamma_1, \Gamma_2, \Gamma_3)$ with respect to (B,D) if

(i) for each $(t,s) \in \Delta$, $x(s;w) \in D$, there exist bounded linear operators Γ_i, $i = 1,2,3$ from D to B such that $\|\Gamma_i\|$, $i=1,2,3$ are continuous and

$$\sum_{i=1}^{3} \|\Gamma_i(t,s,x(s;w))\| \leq Q(t)$$

where Q(t) is a bounded continuous function;

(ii) for each $x(s;w)$, $y(s;w) \in D$, $(t,s) \in \Delta$ there exist constants $\alpha_i > 0$, $i=1,2,3$ such that

$$\|F_i(t,s,x(s;w)+y(s;w) + \sum_{i=1}^{3} (T_i\Gamma_i(s,\tau,x(\tau;w))y)(s;w) ;w)$$

$$- F_i(t,s,x(s;w);w) - \Gamma_i(t,s,x(s;w))y(t,s;w)\|_B \le \alpha_i\|y\|_D$$

The vector $(\alpha_1, \alpha_2, \alpha_3)$ will be called vector of contractor constants.

If B is stronger than C_1 and D is stronger than C such that (B,D) is admissible with respect to T_i, i=1,2,3 it follows that T_i, i = 1,2,3 are bounded linear operators from B into D. Then there exist constants K_i such that

$$\|T_ix\|_D \le K_i \|x\|_B, \quad i = 1,2,3. \tag{7}$$

Assume that the functions f_i, i = 1,2,3 in (1) map $\Delta \times R \times \Omega$ into R in such a way that $x(s;w) \in C$ implies that $f_i(t,s,x(s;w);w)$, i = 1,2,3 are in C. Under this assumption, the integrals in (1) are well defined.

3.MAIN THEOREMS

THEOREM 1: Assume that

(i) $h(t,w) \in D$

(ii) the pair (B,D) is admissible with respect to the operators T_i, i = 1,2,3 and

(iii) the vector of functions (f_1, f_2, f_3) has a bounded integral vector contractor $(\Gamma_1, \Gamma_2, \Gamma_3)$ with the vector of constants $(\alpha_1, \alpha_2, \alpha_3)$.

Then there exists a solution of (1), if

$$\sum_{i=1}^{3} K_i \, \alpha_i < 1$$

where K_i are constants as given in (7).

PROOF : Consider the sequence $\{x_n\}$ in D defined by

$$x_{n+1}(t;w) = x_n(t;w) - \Big[y_n(t;w) + \int_0^t \Gamma_1(t,s,x_n(s;w))\, y_n(s;w)\, ds$$
$$+ \int_0^\infty \Gamma_2(t,s,x_n(s;w)) y_n(s;w) ds + \int_0^t \Gamma_3(t,s,x_n(s;w)) y_n(s;w)\, d\beta(s;w) \Big] \quad \ldots\ldots (8)$$

where

$$y_n(t;w) = x_n(t;w) - h(t;w) - \int_0^t f_1(t,s,x_n(t;w))\, ds$$
$$- \int_0^\infty f_2(t,s,x_n(t;w)) ds - \int_0^t f_3(t,s,x_n(t;w)) d\beta(s;w) \qquad (9)$$

and $x_o \in D$.

The admissibility of the pair (B,D) with respect to the operators T_i, $i = 1,2,3$ and assumption (i) clearly imply that x_n, $y_n \in D$, for $n = 1, 2, \ldots$. We shall now prove that $\lim_{n\to\infty} \|y_n\|_D = 0$. From (8) and (9), we have

$$y_{n+1}(t;w) = - \int_0^t f_1(t,s,x_n(s;w)-y_n(s;w)-\int_0^s \Gamma_1(s,\tau,x_n(\tau;w)) y_n(\tau;w) d\tau$$
$$-\int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w)) y_n(\tau;w) d\tau - \int_0^s \Gamma_3(s,\tau,x_n(\tau;w)) y_n(\tau;w) d\beta(\tau;w);w) ds$$
$$-\int_0^\infty f_2(t,s,x_n(s;w)-y_n(s;w) - \int_0^s \Gamma_1(s,\tau,x_n(\tau;w)) y_n(\tau;w) d\tau$$

$$-\int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau -\int_0^s \Gamma_3(s,\tau,x_n(\tau;w))y_n(\tau;w)d\beta(\tau;w);w)ds$$

$$-\int_0^s f_3(t,s,x_n(s;w)-y_n(s;w) - \int_0^s \Gamma_1(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau$$

$$-\int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau -\int_0^s \Gamma_3(s,\tau,x_n(\tau;w))y_n(\tau;w)d\beta(\tau;w);w)ds$$

$$-\int_0^t \Gamma_1(t,s,x_n(s;w))y_n(s;w)ds-\int_0^\infty \Gamma_2(t,s,x_n(s;w))y_n(s;w)ds$$

$$- \int_0^t \Gamma_3(t,s,x_n(s;w))y_n(s;w)\, d\beta(s;w) + \int_0^t f_1(t,s,x_n(s;w);w)ds$$

$$+ \int_0^\infty f_2(t,s,x_n(s;w);w)ds + \int_0^t f_3(t,s,x_n(s;w);w)d\beta(s;w).$$

Since (B,D) is admissible with respect to T_i, $i = 1,2,3$ we get

$$\|y_{n+1}(t;w)\|_D$$

$$\leq K_1\|f_1(t,s,x_n(s;w) - y_n(s;w)- \int_0^s \Gamma_1(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau$$

$$- \int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau - \int_0^s \Gamma_3(s,\tau,x_n(\tau;w))y_n(\tau;w)d\beta(\tau;w);w)$$

$$- f_1(t,s,x_n(s;w);w) - \Gamma_1(t,s,x_n(s;w))y_n(s;w)\|_B$$

$$+K_2\|f_2(t,s,x_n(s;w)-y_n(s;w) - \int_0^s \Gamma_1(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau$$

$$- \int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau - \int_0^s \Gamma_3(s,\tau,x_n(\tau;w))y_n(\tau;w)d\beta(\tau;w);w)$$

$$- f_2(t,s,x_n(s;w);w) - \Gamma_2(t,s,x_n(s;w))y_n(s;w)\|_B$$

$$+ K_3 \|f_3(t,s,x_n(s;w)-y_n(s;w) - \int_0^s \Gamma_1(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau$$

$$- \int_0^\infty \Gamma_2(s,\tau,x_n(\tau;w))y_n(\tau;w)d\tau - \int_0^s \Gamma_3(s,\tau,x_n(\tau;w))y_n(\tau;w)d\beta(\tau;w);w)$$

$$- f_3(t,s,x_n(s;w);w) - \Gamma_3(t,s,x_n(s;w))y_n(s;w)\|_B$$

Using (iii) on the functions f_i, i = 1,2,3, we obtain

$$\|y_{n+1}\|_D \le (K_1\alpha_1 + K_2\alpha_2 + K_3\alpha_3)\|y_n\|_D$$

Therefore $\lim_{n\to\infty} \|y_n\|_D = 0$, since $\sum_{i=1}^{3} K_i\alpha_i < 1$.

By (8) and definition 8, we have

$$\|x_{n+1}-x_n\|_D \le \|y_n\|_D + K_1\|\Gamma_1 y_n\|_D + K_2\|\Gamma_2 y_n\|_D + K_3\|\Gamma_3 y_n\|_D$$

$$\le \sum_{i=1}^{3} (K_i)\ Q^*\ \|y_n\|_D \tag{10}$$

$$\le \sum_{i=1}^{3} (K_i)\ Q^*\ (\sum_{i=1}^{3} K_i\alpha_i)^{n+1}\ \|y_o\|_D$$

where $Q^* = \sup\{Q(t) : t\}$. Thus $\{x_n\}$ is Cauchy sequence and hence there exists an $x \in D$ such that $\lim_{n\to\infty} x_n = x$.

Proof of the theorem is now complete.

THEOREM 2: Let $u(t)>0$ be continuous such that $\int_0^\infty [u(t)+u^2(t)]dt < \infty$ Let the stochastic integral equation (1) satisfy the following conditions:

(i) $h(t;w) \in C_u$;

(ii) $\sum_{i=1}^{3} |f_i(t,s,x;w)| \le u(t)[z(s;w) + \phi(t,s)\ |x|]$ a.s., for $0\le s\le t$, where $z(s;w)$ is a second order random process in C_u and $\phi(t,s)>0$ is a bounded continuous function defined for $s\leq t$; and

(iii) for each $t \in R^+$, there exists linear operators Γ_i, $i=1,2,3$ on L_2 such that for $x(s;w)$, $y(s;w) \in C_u$ $(s\leq t)$,

$$\|\Gamma_i(t)x(s;w)\|_{L_2} \leq K\ u(t)\ \|x(s;w)\|_{L_2} \quad i=1,2,3$$

for some constant $K>0$, and

$$\|f_i(t,s,x(s;w)+y(s;w)+\int_0^s(\Gamma_1 y)(s;w)\,ds + \int_0^\infty(\Gamma_2 y)(s;w)\,ds$$

$$+\int_0^s(\Gamma_3 y)(s;w)\,d\beta(s;w);w) - f_i(t,s,x(s;w) - (\Gamma_i y)(s;w)\|_{L_2}$$

$\leq \alpha_i\ \|y(s;w)\|_{L_2}$ for $i = 1,2,3$.

Then there exists a solution $x(t;w)$ of (1) such that $E[(x(t;w))^2] \leq u^2(t)$, provided $\sum_{i=1}^{3} K_i\ \alpha_i < 1$.

PROOF : We shall now prove that hypotheses of Theorem 2 imply those that of Theorem 1 with $B = C_{1,u}$ and $D = C_u$.

Let us prove first that $(C_{1,u}, C_u)$ is admissible with respect of T_i, $i = 1,2,3$. Let $x \in C_{1,u}$. Now,

$$\|(T_1x)(t;w)\|_{L_2}/u(t) \leq \int_0^t\|x(t,s;w)\|_{L_2}[u(t)]^{-1}\,ds$$

$$\leq \sup\{\ \|x(t,s;w)\|_{L_2}/\ [\ u(t)\ u(s)\ ]:\ 0\leq s\leq t\}\int_0^t u(v)\,dv$$

$\leq M\ \|x\|_{C_{1,u}}$ for some constant $M > 0$. This implies that $T_1x \in C_u$.

$$\|(T_2x)(t;w)\|_{L_2}/u(t) \leq \int_0^\infty\|x(t,s;w)\|_{L_2}[u(t)]^{-1}\,ds$$

$\leq \sup \{ \|x(t,s;w)\|_{L_2} / [u(t) u(s)]: 0 \leq s \leq t \} \int_0^\infty u(v)\, dv$

$\leq M \|x\|_{C_{1,u}}$ for some constant $M > 0$.

Therefor $T_2x \in C_{1,u}$.

Now, consider

$[\|(T_3x)(t;w)\|_{L_2}/u(t)]^2 \leq \int_0^t E[x(t,s;w)]^2 [u(t)]^{-2}\, ds$

$\leq \sup \{ [\|x(t,s;w)\|_{L_2}]^2 / [u(t) u(s)]^2: 0 \leq s \leq t \} \int_0^t [u(v)]^2\, dv$

$\leq M \|x\|_{C_{1,u}}$ for some constant $M > 0$.

This implies that $T_3x \in C_u$. Thus $(C_{1,u}, C_u)$ is admissible with respect to T_i, $i = 1,2,3$. Condition (ii) implies that $f_i(t,s,x(s;w);w)$ are in $C_{1,u}$ whenever $x(s;w) \in C_u$. Also, condition (iii) implies the existence of a bounded integral vector contractor for the vector (f_1, f_2, f_3). In fact, the contractor is the vector $(\Gamma_1, \Gamma_2, \Gamma_3)$ with contractor constants $(\alpha_1, \alpha_2, \alpha_3)$. Thus all the hypotheses of Theorem 1 are satisfied and the proof is complete.

THEOREM 3: Let the stochastic integral equation (1) satisfy the following conditions:

(i) $h(t;w) \in C_b$;

(ii) $\sum_{i=1}^{3} |f_i(t,s,x;w)| \leq \phi(t,s) |x|$ a.s. for $0 \leq s \leq t$ where $\phi(t,s) > 0$ is such that $\sup \{ \int_0^t [\phi(t,s) + \phi^2(t,s)]\, ds : t \geq 0 \} < \infty$

(iii) same as (iii) of Theorem 2 except that ϕ_i, $i = 1,2,3$ are bounded linear operators on L_2 such that

$$\sum_{i=1}^{3} \|\Gamma_i(t)y(s;w)\|_{L_2} \le K\,\phi(t,s)\,\|y(s;w)\|_{L_2}$$

for some constant $K > 0$.

Then there exists a solution $x(t;w)$ of (1) such that $\sup\{E[x(t;w)]^2 : t \ge 0\} < \infty$.

Proof : Proof parallels that of Theorem 1 and Theorem 2.

THEOREM 4: Assume the hypotheses of Theorem 1. Also assume that

$$y(t;w) + \int_0^t \Gamma_1(t,s,x(s;w);w)y(s;w)ds + \int_0^\infty \Gamma_2(t,s,x(s;w);w)y(s;w)\,ds + \int_0^t \Gamma_3(t,s,x(s;w);w)y(s;w)d\beta(s;w) = z(t;w) \quad (11)$$

where $x(t;w)$, $z(t;w) \in D$, has a solution in D. Then the equation (1) has a unique solution.

PROOF : Let $x_1(t;w)$ and $x_2(t;w)$ be two solutions in D of the equation (1) corresponding to two functions $h_1(t;w)$ and $h_2(t;w)$. Then

$$\begin{aligned} x_1(t;w) - x_2(t;w) &= h_1(t;w) - h_2(t;w) \\ &+ \int_0^t [f_1(t,s,x_1(s;w);w) - f_1(t,s,x_2(s;w);w)]\,ds \\ &+ \int_0^\infty [f_2(t,s,x_1(s;w);w) - f_2(t,s,x_2(s;w);w)]\,ds \\ &+ \int_0^t [f_3(t,s,x_1(s;w);w) - f_3(t,s,x_2(s;w);w)]\,d\beta(s;w) \quad (12) \end{aligned}$$

Let $z(t;w) = x_1(t;w) - x_2(t;w)$.

We conclude from the fact that (11) has a solution for every z and x that there is a $y(t;w) \in D$ such that

$$x_1(t;w) - x_2(t;w) = z(t;w)$$

$$= y(t;w) + \int_0^t \Gamma_1(t,s,x_2)\, y(s;w)\, ds + \int_0^\infty \Gamma_2(t,s,x_2)\, y(s;w)\, ds$$

$$+ \int_0^t \Gamma_3(t,s,x_2)\, y(s;w)\, d\beta(s;w). \qquad (13)$$

Using (13) in (12) and after simplification, we obtain

$$\|y(t;w)\|_D \le \|h_1(t;w) - h_2(t;w)\|_D + \left(\sum_{i=1}^{3} K_i\, \alpha_i\right) \|y(t;w)\|_D$$

Then we get,

$$\|y(t;w)\|_D \le \left[1 - \left(\sum_{i=1}^{3} K_i\, \alpha_i\right)\right]^{-1} \|h_1(t;w) - h_2(t;w)\|_D. \qquad (14)$$

If $h_1(t;w) = h_2(t;w)$, then it follows that $y = 0$, which implies that $x_1(t;w) = x_2(t;w)$.

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