LIBERTAS MATHEMATICA, VOL. 17, 1997

CONNECTORS OF BIFURCATING EQUILIBRIA OF PLANAR HOMOGENEOUS DIFFERENTIAL SYSTEMS*

S. R. Bernfeld

Dedicated to Professor Junji Kato on his Sixtieth Birthday

1. Introduction.

We wish to analyze the properties of connectors of tori and cycles bifurcating from the origin of the four dimensional system of differential equations

$$(1.1)_\epsilon \qquad \begin{aligned} \dot{u}_i &= \epsilon\, u_i - w_i v_i + X_i(\epsilon,u,v) \\ \dot{v}_i &= \epsilon\, v_i + w_i v_i + Y_i(\epsilon,u,v), \end{aligned} \quad i = 1,2$$

where $u = (u_1, u_2)$, $v = (v_1, v_2)$, $X_i, Y_i \in C^\infty([0,\bar{\epsilon}\,] \times R^4, R)$ for some $\bar{\epsilon} > 0$, $X_i, Y_i = o(|u|^2 + |v|^2)$ and $\frac{w_1}{w_2}$ is not a rational number. Our work was originally motivated by [7] and expanded subsequently in [1],[2],[3], and [4]. In particular, we assume the origin of $(1.1)_0$ is h-asymptotically stable [5] that is the asymptotic stability of the origin $(1.1)_0$ is preserved under perturbations of

---

* This work has been partially supported by the Italian C.N.R. (National Council of Research)

order greater than h in (u, v) but there are perturbations of order h which do not preserve the asymptotic stability. When ϵ increases through zero, we find the origin of $(1.1)_\epsilon$ is completely unstable and this exchange of stability of the origin gives rise to the existence of bifurcating tori and cycles.

For physical motivation of the above phenomenon consider two springs with rationally independent frequencies w_1 and w_2. Without the effects of the nonlinearities the system $(1.1)_0$ is conservative and the uncoupled springs oscillate with constant energy. The introduction of the nonlinearities, reflecting a nonlinear Hooke's law in $(1.1)_0$ dampens the oscillations. In fact, the amplitude of oscillation of one spring will dominate over that of the other spring. When energy is introduced as measured by $\epsilon > 0$ then the oscillating springs approach one of the following equilibria of $(1.1)_\epsilon$: (i) the two springs oscillate with a single frequency (which is either w_1 or w_2) and we refer to this equilibrium as a cycle; (ii) the two coupled springs oscillate with almost periodic motion and thus lie on a two dimensional torus in four dimensional space.

In order to obtain the qualitative properties of the bifurcating tori and cycles of $(1.1)_\epsilon$ as well as the properties of the connections between these equilibria, we shall continue to develop a Reflection Principle, (see [1], [4]), which roughly says that many of the properties of the bifurcating set for $(1.1)_\epsilon$ can be obtained by an analysis of the behavior of the solutions in a neighborhood of the origin of $(1.1)_0$. We need the following [5]:

Definition 1.1. The origin of $(1.1)_0$ is h-asymptotically stable, $h \geq 3$ if (i) for every perturbation $\xi_i,\ \eta_i \in C^\infty(R^4, R)$, $i = 1, 2$ of order greater than h the origin of

$$\begin{aligned} \dot{u}_i &= -w_i v_i + X_i(0,u,v) + \xi_i(u,v), \\ \dot{v}_i &= w_i u_i + Y_i(0,u,v) + \eta_i(u,v)) \end{aligned}$$

is asymptotically stable; (ii) property (i) is not satisfied when h is replaced by any $j = 1, \cdots, h-1$.

It is convenient to put $(1.1)_\epsilon$ into a normal form in polar coordinates (ρ_i, ϕ_i) given by (see [5])

$$\begin{aligned} \dot{x} &= \epsilon x + xB_m(x,y) + xE(x,y,\phi_1,\phi_2,\epsilon) \\ \dot{y} &= \epsilon y + yC_n(x,y) + yF(x,y,\phi_1,\phi_2,\epsilon) \\ \dot{\phi}_1 &= w_1 + G(x,y,\phi_1,\phi_2,\epsilon) \\ \dot{\phi}_2 &= w_2 + H(x,y,\phi_1,\phi_2,\epsilon), \end{aligned} \tag{1.2$_\epsilon$}$$

where $x = \rho_1^2 = u_1^2 + u_2^2$ and $y = \rho_2^2 = v_1^2 + v_2^2$, $B_m(x,y)$, $C_n(x,y)$ are polynomials in (x,y) of degree m and n respectively and E and F are respectively $o(x^2+y^2)^{m/2}$ and $o(x^2+y^2)^{n/2}$ as $(x,y) \to (0,0)$ and are C^∞ in all its variables. Since the x and y axes are invariant we have $B_m(x,0) < 0$ and $C_n(0,y) < 0$. Finally we assume

(AR) $\qquad G(x,y,\phi_1,\phi_2,\epsilon),\ H(x,y,\phi_1,\phi_2,\epsilon) = o(x^2+y^2)^{\frac{h-1}{2}}$

as $(x,y) \to (0,0)$. Condition (AR) implies the radial component of the flow given by $(1.2)_\epsilon$ is stronger than the angular component. This allows for the existence and persistence of tori and cycles under higher order perturbations. We shall always assume (AR) without alluding to it.

2. Connections Between Bifurcating Sets.

The first two equations of $(1.2)_\epsilon$ are the amplitude equations and we shall concentrate on the truncated equations

$$(A_\epsilon) \qquad \begin{aligned} &\dot{x}=\epsilon x+xB_m(x,y) \\ &\dot{y}=\epsilon y+yC_n(x,y), \quad x,y\geq 0. \end{aligned}$$

We observe that $(1.1)_0$ is h-asymptotically stable if and only if (A_0) is $k=\frac{h+1}{2}$ – asymptotically stable. We shall look at the special case in which $h=3$. In this case $m=n=1$, that is (A_0) is homogeneous of degree 2. (We shall at the conclusion address the more general case when h is arbitrary). So let us write (A_ϵ) in the case $h=3$. We obtain

$$(T_\epsilon) \qquad \begin{aligned} &\dot{x}=\epsilon x-ax^2-cxy, \\ &\dot{y}=\epsilon y-dy^2-bxy. \end{aligned}$$

We shall assume $a=d=1$, which is not a restriction of our results as we shall see. So we finally arrive at the equation that we will investigate:

$$(S_\epsilon) \qquad \begin{aligned} &\dot{x}=\epsilon x-x^2-cxy, \\ &\dot{y}=\epsilon y-y^2-bxy, \quad x,y\geq 0, \end{aligned}$$

in which the origin of (S_0) is $\frac{h+1}{2}=2$ asymptotically stable (recall $h=3$).

We know [2], [6] that the bifurcating set, B_ϵ, of (S_ϵ) consist of two equilibria lying on the x and y axis respectively (cycles) and possibly an equilibrium point in the interior of the first quadrant (torus), as well as the connectors between the equilibria (see Proposition 2.2).

In our previous work [2], [3], and [4], we introduced and developed some aspects of the reflection principle, which in the special case of (S_ϵ) provided an analysis of the stability properties of the bifurcating (tori) in terms of the stability of the origin of (S$_0$). To this end we use (r,θ) to denote the polar coordinates and write (S$_\epsilon$) as

$$(\mathrm{PS}_\epsilon) \qquad \begin{aligned} &\dot{r} = \epsilon r - r^2 z(\theta) \\ &\dot{\theta} = r^2 N(\theta), \end{aligned}$$

where $$z(\theta) = \Big((\cos\theta - c\sin\theta)\cos^2\theta - (\sin\theta - b\cos\theta)\sin^2\theta\Big)$$

and $$N(\theta) = \Big((\sin\theta - b\cos\theta) \; - (\cos\theta - c\sin\theta)\Big)\sin\theta\cos\theta.$$

Either $N(\theta)$ has no zeros in the interior of the first quadrant or has one zero, $\theta = \theta_1$. The ray $\theta = \theta_1$ is invariant and will be referred to as the normal ray. Since (S$_\epsilon$) is quadratic in (x,y) $\mathrm{N}'(\theta_1) \neq 0$. The corresponding equilibrium, $p_1 = p_1(\epsilon)$ of (S$_\epsilon$) gives rise to an invariant torus of $(1.1)_\epsilon$ and lies on the ray $\theta = \theta_1$.

The following definitions are useful.

Definition 2.1. The normal ray $\theta = \theta_1$ is a hyperbolic attractor (repeller) of (PS$_0$) if $\mathrm{N}'(\theta_1) < 0 \, (\mathrm{N}'(\theta_1) > 0)$.

Definition 2.2. We say an orbit $y(x)$ of (S$_0$) is algebraic at the normal ray $\theta = \theta_1$ if $y(x)$ is tangent to $\theta = \theta_1$ as $x \to 0$ and has a formal representation in a neighborhood of $\theta = \theta_1$ given by $y - (\tan\theta_1)x = \sum_{j=1}^{\infty} a_j x^{bj}$ where a_j and b_j are real numbers. An orbit that is not algebraic at a normal ray $\theta = \theta_1$ is said to be transcendental at $\theta = \theta_1$. The normal ray is algebraic (transcendental) if there exists a neighborhood of $\theta = \theta_1$ such that all orbits are algebraic (transcendental). We have the following results [4].

Proposition 2.1. Assume the origin of (S_0) is 2-asymptotically stable and assume $\theta = \theta_1$ is a normal ray. Then $\theta = \theta_1$ is hyperbolic and algebraic. The rays $\theta = 0$ and $\theta = \frac{\pi}{2}$ are also algebraic. If there is no interior normal ray then one of the rays $\theta = 0$ or $\theta = \frac{\pi}{2}$ is algebraic and the other transcendental.

Proposition 2.2. Under the assumptions of Proposition 2.1 system (S_ϵ), for ϵ sufficiently small, has two equilibrium one on each axis and has at most one equilibrium in the interior of the first quadrant. This equilibrium point when it exists is either an attractor or a saddle point. Letting p_1 be the interior equilibrium point and p_0 and p_2 be the equilibrium points on the x and y axis respectively, then the bifurcating set B_ϵ of (S_ϵ), that is the largest compact invariant set not containing the origin consist of $\{p_0, p_1, p_2\}$ and two connectors C_{10} and C_{12} where C_{10} connects p_0 and p_1 and C_{12} connects p_1 and p_2.

In Proposition 2.2 the existence of the equilibria of (S_ϵ) can be found by inspecting the behavior of (S_0). The direction of the flow on C_{10} and C_{12} tell us whether the equilibrium p_1 is a saddle point or attractor since the radial direction is always part of the stable manifold of p_1, The flows on C_{10} and C_{12} can be found by determining the behavior of the orbits in a neighborhood of the normal ray $\theta = \theta_1$ (the angular coordinate of p_1) of (S_0). These comments are again examples of the reflection principle.

3. Smoothness of Connectors.

We are now interested in using the reflection principle to obtain the continuity properties of $C_1 \equiv C_{10} \cup C_{12}$ at p_1. Clearly if p_1 is a saddle point then C_1 is continuously differentiable (C^1) at p_1. Now if p_1 is an attractor then we will obtain one of the main results of this paper.

Theorem 3.0. If the torus given by p_1 is an attractor of (S_ϵ) then the two connectors at p_1, C_{10} and C_{12} either form (i) a cusp at p_1; (ii) a corner at p_1; or (iii) C_1 is C^1 at p_1. Conditions on the coefficients of the differential equation determine which of the above three possibilities hold. If p_1 is a saddle point of (S_ϵ) then C_1 are C^1 at p_1.

To obtain the proof of Theorem 3.0, we will provide sufficient conditions on b and c in (S_ϵ) to provide the validity of cases (i), (ii), or (iii). Letting x and y be the coordinate of the torus p_1, we find from (S_ϵ) that $x = \left(\frac{1-c}{1-cb}\right)\epsilon$ and $y = \frac{(1-b)\epsilon}{(1-cb)}$. Using the fact a torus exists if and only if $x > 0$ and $y > 0$, and that two cycles always exist if the origin of (S_0) is asymptotically stable we have the following characterization between b and c and the equilibria contained in the bifurcating set.

Lemma 3.1. The $b - c$ plane can be characterized as follows:

(i) the region I given by $\{(b, c) : b > 1,\ c > 1)\}$ contains a torus which is a saddle point and two cycles;

(ii) the regions IIa, IIb, III, IV, V given by $\left\{(b,c) : \left(\frac{1}{b} \leq c \leq 1\right)\right\}$, $\left\{(b,c) : \left(0 \leq c \leq \frac{1}{b}\right)\right\}$, $\{(b,c) : (0 \leq b \leq 1),\, c \geq 0\}$, $\{(b,c) : (b \geq 1,\, c < 0)\}$, $\{(b,c) : (b \leq 0,\, c \geq 1)\}$, respectively, each contains two cycles and no tori.

(iii) the region VI given by $\left\{(b,c) : \left(b \leq \frac{1}{c},\, b < 0\right)\right\}$ has no bifurcating set

(iv) the regions VII, VIII, IX given by $\{(b,c) : (0 \leq b < 1,\, c < 0)\}$, $\left\{(b,c) : \left(\frac{1}{c} < b \leq 0,\, c < 0\right)\right\}$; $\{(b,c) : (b < 1,\, 0 \leq c < 1)\}$ respectively each contain two cycles and one torus.

We sketch a proof of Lemma 3.1. Observe $x = \left(\frac{1-c}{1-cb}\right)\epsilon$ and $y = \frac{(1-b)}{(1-cb)}\epsilon$ are coordinates of a torus whenever $x > 0$ and $y > 0$. Thus we require $\frac{1-b}{1-cb} > 0$ and $\frac{1-c}{1-cb} > 0$. We find the normal ray is given by $y = \frac{b-1}{c-1}x$ (we see that $\frac{b-1}{c-1} > 0$). Using these inequalities it is not difficult to ascertain the validity of (i) and (iv) since there will exist bifurcating cycles. Indeed, in these cases the origin of (S_0) is 2-asymptotically stable and the origin of (S_ϵ) is completely unstable. In case (iii) an easy calculation leads to the observation that the origin of (S_0) is unstable implying there is no bifurcation phenomenon. In case (ii) the above inequalities implies there is no tori. However, (S_0) is 2-asymptotically stable and thus there appear bifurcating cycles.

Thus we have exhibited another implication of the reflection principle that is a relationship between the asymptotic stability of the origin of (S_0) and the components of the bifurcating set B_ϵ.

We now address the question of the smoothness of the connector at the interior equilibrium point p_1-(torus) of (S_ϵ). Recall we have defined $C_1 = C_{10} \cup C_{12}$. In Theorem 3.0, we find the connector C_1 at p_1 may be C^1, $C^{0,1}$(corner) or $C^{0,\alpha}, \alpha < 1$ (cusp) and that we can identify what type of smoothness C_1 has by studying the connectors C_{10} and C_{12} of the perturbed flow (S_ϵ) at p_1. This we can ascertain using the behavior of the unperturbed flow (S_0) at the origin. Toward this end, we have:

Theorem 3.1. Using the description of the regions given in Lemma 3.1 we find that the continuity properties of the connector C_1 at the torus p_1 satisfies:

(a) in region I the connector is C^1 at p_1,

(b) in region VII the connector is C^1 at p_1,

(c) in region VIII the connector is $C^{0,\alpha}(\alpha < 1)$ (a cusp) at p_1,

(d) in region IX the connector is C^1 at p_1,

(e) on the one dimensional sets $\{(b,c) : b = 0,\, c < 0)\}$ and $\{(b,c) : b < 0, c = 0)\}$ the connector is $C^{0,1}$ (corner) at p_1.

These two one dimensional sets in (e) form part of the boundary of Regions VII, VIII, and IX and the continuity properties at p_1 of the connectors change as we cross these two sets.

We will indicate the proof after the ensuing discussion. One way to attack this question, not using the reflection principle, is to linearize around p_1, compute the eigenvalue along the angular direction $\theta = \theta_1$ (we refer to this direction as e_r)

and compute the eigenvalue along the other eigenvector (we refer to this direction as e_τ) which is transverse to e_r. The direction of the approach of the connector C_1 at p_1 is given by the eigenvector corresponding to the smaller negative eigenvalue. We then ascertain whether the approach of C_{10} and C_{12} is from the interior or exterior of the bifurcating set, the analysis depending on the behavior of the sets $\dot{x} = 0$ and $\dot{y} = 0$. We may assume, without loss of generality, the two eigenvalues are negative-(the radial eigenvalue is always negative [1]) for if the other eigenvalue is positive then we have that p_1 is a saddle point and the connector is tangent to the transverse direction that forms the unstable manifold at p_1 which is C^1 (this in fact proves (a) in Theorem 3.1). In regions VII, VIII, and IX, p_1 is an attractor, and for both $i = 0, 2$, C_{1i} is tangent to either e_r or e_τ. And then the smoothness of C_1 at p_1 depends on the manner in which C_{10}, C_{12} approach p_1 at either e_r or e_τ. By analyzing the flow on the line given by $\{(x, y) : \dot{y} = 0\}$ in the wedge $[\theta_1, \pi/2]$ and the flow on the line $\{(x, y) : \dot{x} = 0\}$ in the wedge $[0, \theta_1]$ we can ascertain the approach to p_1 of C_{12} and C_{10}, respectively. We would like to obtain this information on C_{10} and C_{12} at p_1, not by linearization around p_1 but by an analysis of the flow of (S_0) at the origin- that is an application of the reflection principle. First we will present a few examples to depict this.

We will obtain solutions of (S_0) in terms of $v = \frac{y}{x}$ and x or in terms of v and y [8], and relate the behavior of the order of x as a function of v at the normal ray $v = \tan\theta_1$ with the continuity of the connector C_1 at p_1. More precisely we have:

Theorem 3.2. Assume (S_0) is written in coordinates $v = y/x$ and x; we find the orbits are given as $cx = f(x)\,g(|v - \tan\theta_1|)$, where $f(v)$ and $g(|v - \tan\theta_1|)$ are algebraic. Assume p_1 is an attractor. Then

(i) If $g(|v - \tan\theta_1|)$ has order greater than one at $v = \tan\theta_1$ and $\frac{1}{f(v)}$ has order greater than one at $v = 0$, then the connector is C^1 and tangent to e_τ.

(ii) If $g(|v - \tan\theta_1|)$ has positive order less than one at $v = \tan\theta_1$ and $\frac{1}{f(v)}$ has order less than one at $v = 0$ then the connector has a cusp at p_1 and is tangent to e_τ.

(iii) If $g(|v - \tan\theta_1|)$ has positive order less than one at $v = \tan\theta_1$ and $\frac{1}{f(v)}$ has order greater than one at $v = 0$ then the connector is C^1 at p_1.

(iv) If $(i) - (iii)$ are not satisfied and $g(|v - \tan\theta_1|)$ *has positive order at* $v = \tan\theta_1$ then the connector is $C^{0,1}$ (corner) at p_1.

(v) If p_1 is a saddle point then $|g(|v - \tan\theta_1|)| \to \infty$ as $v \to \tan\theta_1$ and we know the connector is C^1 at p_1.

We now give examples of Theorem 3.2 and then briefly outline its proof.

Example 1. Let

$$\dot{x} = -x^2 - \frac{1}{4}xy$$
$$\dot{y} = -y^2 - \frac{1}{2}xy.$$

The substitution $v = \frac{y}{x}$ gives us

$$cx = \frac{|v-\frac{2}{3}|^{7/3}}{v^2}$$

for the orbits, where c is any constant So $f(v) = \frac{1}{v^2}$, $g(|v - \frac{2}{3}|) = |v - \frac{2}{3}|^{7/3}$ and $\tan\theta_1 = \frac{2}{3}$.

Then Theorem 3.2 (i) implies the connector is C^1 at the attractor $p_1 = (3\epsilon, 2\epsilon)$ and tangent to e_r.

Example 2. Let

$$\dot{x} = -x^2 + xy$$
$$\dot{y} = -y^2 + \frac{1}{2}xy.$$

In this case the orbits satisfy

$$cx = \frac{|v-\frac{3}{4}|^{1/6}}{v^{\frac{2}{3}}}$$

and thus $f(v) = \frac{1}{v^{\frac{2}{3}}}$ and $g(\left|v - \frac{3}{4}\right|^{1/6}$. Then from Theorem 3.2 (ii) we conclude the connector has a cusp at $p_1 = (4\epsilon, 3\epsilon)$ and is tangent to e_r, that is to the line $y = (\tan\theta_1)x$.

Example 3. Let

$$\dot{x} = -x^2 - 3xy$$
$$\dot{y} = -y^2 - 2xy.$$

Then the orbits satisfy

$$cx = v|(1-2v)|^{-5/2}$$

and thus $f(v) = v$ and $g(|1-2v|) = |(1-2v)|^{-5/2}$. According to Theorem 3.2(v), the connector is C^1 at $p_1 = (2\epsilon, \epsilon)$ and p_1 is a saddle point.

Example 4. Let

$$\begin{aligned} \dot{x} &= -x^2 + \frac{1}{2}xy \\ \dot{y} &= -y^2. \end{aligned}$$

We find

$$cx = \frac{(|\frac{1}{2}v-1|)^2}{v},$$

and so $f(v) = \frac{1}{v}$ and $g(|\frac{1}{2}v - 1|) = (\frac{1}{2}v - 1)^2$. Notice that the conditions $(i) - (iii)$ in Theorem 3.2 are not satisfied; then from (iv) we conclude that the connector is $C^{0,1}$ (a corner) at $p_1 = (\epsilon, 2\epsilon)$.

Basically to prove Theorem 3.1 we show that the smoothness properties of the connectors can be obtained by linearization around p_1 for (S_ϵ). We then show that this procedure can be made equivalent to the procedure given in Theorem 3.2 which only depends on the unperturbed system (S_0). Thus, the reflection principle allows for the determination of the structure of the connectors at p_1 based on knowledge of the stability properties of the origin of (S_0) (Theorem 3.2).

An important open question in the development of the reflection principle is whether the properties of the orbits of (S_0) at the origin, algebraic or transcendental, implies certain structures of the bifurcating set of (S_ϵ). As of now we have not been able to obtain a relationship between those properties of the orbits of (S_0) and the continuity properties of the connectors. Because there is some connection between the algebraic properties of the orbits and the transcendental properties that can be ascertained in terms of a non-resonance condition (similar to ones given for linear systems) [4] then we will in the future explore this non-resonance condition to see if it can be used further in developing the reflection principle.

Finally, it would be interesting to assume the origin of $(1.1)_0$ is 5-asymptotically stable, that is, (S_0) consists of cubic terms and we find that there exist two bifurcating cycles and at most two bifurcating tori. The analysis of the behavior of the connectors at each tori depends on the sets given by $\{(x,y) : \dot{x} = 0\}$ and $\{(x,y) : \dot{y} = 0\}$ respectively, which are quadratic.

Bibliography

1. Bernfeld, S.R., Existence and stability of bifurcating tori in a neighborhood of an equilibrium position. *Jour. Math. Anal. and Appl.* 155 (1991), 541-561.
2. Bernfeld, S.R., Exchange of stability and bifurcation of tori in four dimensions: Proc. of First World Congress of Nonlinear Analysis in Tampa, (1996), 1819-1828, Ed. V. Lakshmikantham, Walter de Gruyler Publisher, New York.

3. Bernfeld, S. R., Stability and bifurcation for planar polynomial differential systems, *Qual. Prob. for Diff. Eq. and Control Theory*, (1995), 1-11, Ed. C. Corduneanu, World Scientific Publisher, Singapore.

4. Bernfeld, S.R., Exchange of stability of an equilibrium of homogeneous planar differential systems, *J. Nonl. Anal., T.M.A.*, 25 (1995), 873-883.

5. Salvadori, L., Sulla stabilita dell' equilibria casi critici, *Ann. Mat. Pura Appl.* 69 (1965), 1-34.

6. Salvadori, L., Bifurcation and stability problems for periodic differential systems, Proc. Conf. Nonlinear Oscillations of Conservaitve Systems, A. Ambrosetti, Ed., Pitagora Editrice, Bologna (1985).

7. Samoilenko, A.M. and Polesya, I.V., Genesis of invariant sets in the neighborhood of an equilibrium position, *Differential Uravnenya* 11 (1975), 1409-1415.

8. Sansone, G. and Conti, R., *Nonlinear Differential Equations*, Macmillan, New York, 1964.