LIBERTAS MATHEMATICA, VOL. XVII (1997)

Bounded Solutions of Systems with Impulses

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Abstract

In this paper, assuming the existence of x_0, a bounded on all of R solution of the ordinary differential equation $x' = f(t, x)$, it is proven the existence of y_0 a solution of the impulsive system $x' = f(t, x)$, $x(t_k^+) = g(t_k, x(t_k))$, $t \in R$, $k \in Z$, $\lim_{k \to \pm\infty} t_k = \pm\infty$, satisfying $|x_0(t_j^+) - y_0(t_j^+)| \leq \rho$ for all impulsive time t_j and some nonnegative constant ρ. Under suitable conditions the solution y_0 results to be a bounded solution of the impulsive system.

1 Introduction

The modelling of phenomena in ecology, social behavior, economy, electronic etc. is frequently accomplished by the ordinary differential equation

$$x'(t) = F(t, x(t)). \tag{1}$$

In general, the model (1) does not consider the perturbations of different nature that the solutions of this equation may undergo. One of these possibilities consists of the impulsive effects on the solutions of Eq. (1). This situation started to be analyzed by Mil'man and Myshkis in [4], who considered the effect of stochastic impulses on the solutions of Eq. (1). Lately, these problems began to be studied by means of differential equations with impusive effect [2, 5]:

$$\left.\begin{array}{rcl} y'(t) &=& F(t, y(t)), \quad t \neq t_k, \quad t \in R, \\ y(t_k^+) &=& G(t_k, y(t_k)), \quad k \in Z. \end{array}\right\} \tag{2}$$

In this paper we study the effect of the impulsive perturbations on the bounded solutions of (1). Concretely, we are interested in the following problem: Assuming the existence of x_0, a bounded solution of (1), and given the operator of impulses G we ask whether the impulsive equation (2) has a bounded solution y_0 "close" to solution x_0. In order to make precise statements let us introduce the following definitions.

Definition 1 *We shall say that a solution y_0 of equation (2) escorts a solution x_0 of equation (1) on all of R with distance $\rho \geq 0$ iff*

$$|x_0(t^+) - y_0(t^+)| \leq \rho, \quad \forall t \in R. \tag{3}$$

where $x(t^+)$ denotes the right hand side limit of function x at t.

Definition 2 *We shall say that a solution y_0 of equation (2) weakly escorts a solution x_0 of equation (1) with distance $r \geq 0$ iff*

$$|x_0(t_j^+) - y_0(t_j^+)| \leq r, \quad \forall j \in Z. \tag{4}$$

Since the solutions of (1) are continuous functions, while those of (2) are not, we do not expect a small number ρ in the definition of tube (3).

We will give conditions on Eq. (2) in order to prove the existence of escorting solutions for the bounded solutions of Eq. (1). Our method relies on the ideas of topological principle of Ważewski, in the form used by Pliss in the proof of existence of solutions with the property of convergence [3].

2 Notations

V will denote the space R^n or C^n. For a vector $x \in V$, $|x|$ is a fixed norm. If A is an $n \times n$ matrix, $|A|$ is the corresponding matrix norm. For a function $f : R \to \mathbf{V}$ we will denote $|f|_\infty = \sup \{|f(t)| : t \in R\}$.

Regarding Eq. (2), we will assume that $F, G : R \times V \to V$ are continuous functions, of class C^1 with respect to variable x.

The sequence of impulsive times $\{t_j\}_{j \in Z}$, satisfies $t_j < t_{j+1}$, $\forall j$ and

$$\lim_{j \to \pm\infty} t_j = \pm\infty. \tag{5}$$

The solutions of the impulsive equation (2) are C^1 functions on each interval $(t_j, t_{j+1}]$ possessing right hand side limit at t_j. We assume that all solutions of equation (2) can be forward and backward continued. The symbol $y(t; s, \xi)$ will be used to denote the solution of (2) satisfying $y(s^+; s, \xi) = \xi$.

The notation V_j will denote the hypersurface in $R \times V$, defined as

$$V_j = \{(t,x) : t = t_j, x \in V\}.$$

Definition 3 *We shall say that an ordered pair of vectors* $(\xi,\eta) \in V \times V$ *are* (i,j)*-linked iff* $i \leq j$ *and*

$$\eta = y(t_j^+; t_i, \xi).$$

These linked vectors can be described in the following manner: To each integer j we attach the operator

$$\mathcal{T}_j : V_j \to V_{j+1},$$

defined by $\mathcal{T}_j(t_j;\ \xi_j) = (t_{j+1},\ y(t_{j+1}^+;\ t_j,\ \xi_j))$. If (ξ,η) are (i,j)-linked, then

$$(t_j,\ \eta) = \mathcal{T}_{j-1} o \ldots o \mathcal{T}_{i+1} o \mathcal{T}_i(t_i,\ \xi).$$

For a $r \geq 0$ and $\theta \in V$ we define

$$B_j[\theta, r] = \{(t_j, x) \in V_j : |x - \theta| \leq r\}.$$

Definition 4 *Let* $\{\theta_j\}_{j \in Z}$ *denote a sequence of vectors in* V. *We will say that* $\eta \in B_j[\theta_j,\ r]$ *has the property (C) iff for any* $k \leq j$, *there exists a* $\xi \in B_k[\theta_k,\ r]$ *such that pair* (ξ,η) *is* (k,j)*-linked.*

3 Existence of weakly escorting solutions

Lemma 1 *Let us assume that for some positive number* r *the operators* $\mathcal{T}_j$ *have the property:*

$$\mathcal{T}_j : B_j[\theta_j,\ r] \to B_{j+1}[\theta_{j+1},\ r],\ \forall j \in Z, \tag{6}$$

then there exists a solution y *of equation (2) such that*

$$(t_j,\ y(t_j^+)) \in B_j[\theta_j,\ r],\ \forall j.$$

Proof Let j be a nonpositive integer. In virtue of condition (6), for any $\xi \in B_j[\theta_j, r]$ we will have

$$(t_0, y(t_0^+; t_j, \xi)) \in B_j[\theta_j, r],\ \forall j.$$

Our proof will be accomplished if we prove the existence of a sequence $\{\xi_j\}_{j\leq 0}$, $\xi_j \in B_j[\theta_j, r]$, such that for any integers $k \leq j \leq 0$, the vectors (ξ_k, ξ_j) are (k, j)-linked. For $j \leq 0$ we define the closed set

$$\mathcal{K}_j := \{(t_0, y(t_0^+; t_j, \xi_j)) : (t_j,\ \xi_j) \in B_j[\theta_j, r]\}.$$

From the condition (6) it follows

$$\ldots \subset \mathcal{K}_{-2} \subset \mathcal{K}_{-1} \subset \mathcal{K}_0 = B_0[\theta_0,\ r]. \tag{7}$$

By a known result of analysis, there exists a point ξ_0 such that

$$\xi_0 \in \bigcap_{j\leq 0} \mathcal{K}_j. \tag{8}$$

Thus we have proven the existence of a vector $\xi_0 \in B[\theta_0, r]$ satisfying the property (C).

We will prove now, the existence of $(t_{-1}, \xi_{-1}) \in B_{-1}[\theta_{-1}, r]$ with the same property, such that (ξ_{-1}, ξ_0) is $(-1, 0)$ linked. Let us define the set

$$\mathcal{H}_{-1} = \{(t_{-1}, \xi) \in B_{-1}[\theta_{-1}, r] : \mathcal{T}_{-1}(t_{-1}, \xi) = (t_0,\ \xi_0)\}.$$

By this definition $\mathcal{H}_{-1}$ is the set of vectors ξ contained in the ball $B_{-1}[\theta_{-1}, r]$, such that (ξ, ξ_0) are $(-1, 0)$-linked. The set $\mathcal{H}_{-1}$ is not empty since ξ_0 has the property (C). Further for $j \leq -1$ we define

$$\mathcal{H}_j := \{(t_j, \xi) \in B_j[\theta_j, r] : (t_{-1}, y(t_{-1}^+; t_j, \xi)) \in \mathcal{H}_{-1}\}.$$

Each set $\mathcal{H}_j$ is closed and from the condition (6) is not empty. Using again condition (8) we obtain

$$\ldots \subset \mathcal{H}_{-3} \subset \mathcal{H}_{-2} \subset \mathcal{H}_{-1}.$$

Let

$$\xi_{-1} \in \bigcap_{j\leq -1} \mathcal{H}_j.$$

Clearly the pair (ξ_{-1}, ξ_0) is $(-1, 0)$-linked and ξ_{-1} has the property (C).

Repeating this reasoning, we obtain a sequence $\{\xi_j\}_{j\leq 0}$ with the following properties: Each ξ_j is contained in $B_j[\theta_j, r]$, $j \leq 0$, the vectors (ξ_{j-1}, ξ_j) are $(j-1, j)$-linked.

Let us define the following solution of Eq. (2)

$$y(t) = \begin{cases} y(t; t_0, \xi_0), & t > 0, \\ y(t; t_j, \xi_j), & t \in (t_j, t_{j+1}], j < 0. \end{cases}$$

According to condition (6), $(t_j, y(t_j^+)) \in B_j[\theta_j, r]$ for all $j \geq 0$. By the construction of the sequence $\{\xi_j\}_{j\leq 0}$ we have

$$y(t_j^+) = y(t_j^+; t_j, \xi_j) = \xi_j,$$

implying that $(t_j, y(t_j^+)) \in B_j[\theta_j, r]$, $\forall j \leq 0$. Moreover

$$y(t_j^+) = G(t_j, y(t_j)),$$

and

$$y'(t) = F(t, y(t)), t \neq t_j. \qquad \square$$

For a bounded, contained in V set K we define the number

$$\mathrm{diam}(K) = \sup\{|x - y| : x, y \in K\}.$$

Theorem 1 *Let us consider x_0, a bounded solution of (1). If hypothesis (6) is satisfied, then there exists a solution of Eq. (2) weakly escorting the solution x_0. Moreover, if the following condition is satisfied*

$$diam(\mathcal{T}_j\,[B_j[x_0(t_j), r)]]) \leq \alpha diam(B_j[x_0(t_j), r]), 0 \leq \alpha < 1, \tag{9}$$

then this weak escort is unique.

Proof The first part of the theorem follows from Lemma 1. The uniqueness of this escorting solution follows from (9), since this condition implies that the point ξ contained in (8) is unique. $\square$

4 Impulsive effect on bounded solutions

Let us assume that x_0, a bounded solution of equation (1) defined on all of R undergoes the impulsive effects defined by the equation (2). Performing the change of variables $z = y - x_0$, we may reduce Eq. (2) into

$$\left.\begin{array}{rcl} z'(t) & = & A(t)z(t) + f(t, z(t)), t \neq t_j \\ z(t_j^+) & = & g(t_j, z(t_j)), \end{array}\right\} \quad (10)$$

where

$$A(t) = \frac{\partial F}{\partial x}(t, x_0(t)),$$

$$f(t, z) = F(t, z + x_0(t)) - F(t, x_0(t)) - \frac{\partial F}{\partial x}(t, x_0(t))z,$$

and

$$g(t, z) = G(t, z + x_0(t)) - x_0(t).$$

The reduction of system (2) into (10) modifies the original problem, now we seek a solution z of (10) escorting the null solution of system

$$z'(t) = A(t)z(t) + f(t, z(t)). \quad (11)$$

We will assume that:

$H1$: *The fundamental matrix* $\Phi(t)$ *of the linear system*

$$z'(t) = A(t)z(t), \quad (12)$$

has the following estimate

$$|\Phi(t)\Phi^{-1}(s)| \leq K_j h(t)p(s), \; t_j < s \leq t \leq t_{j+1}, \quad (13)$$

where h *and* p *are continuous functions on each interval* $(t_j, t_{j+1}]$ *possessing a right hand side limit at each impulsive time* t_j. $K_j \geq 1$ *is a constant depending on the interval* $(t_j, t_{j+1}]$.

$H2$: *The function* $f(t, x)$ *satisfies the following Lipschitz condition*

$$|f(t, x) - f(t, y)| \leq \lambda(t)|x - y|, \quad (14)$$

where λ *is a locally integrable function.*

$H3$: *The function* $g(t, x)$ *satisfies the following Lipschitz condition*

$$|g(t, x) - g(t, y)| \leq \ell(t)|x - y|. \quad (15)$$

$H4$: *The function* $g(t, 0)$ *is bounded.*

In what follows we will assume that $g(t, 0)$ is not identically zero, otherwise $G(t, x_0) =$ constant $= x_0$. This would imply that x_0 is solution of the ordinary equation (1) and the impulsive equation (2).

Theorem 2 *Let conditions $H1 - H4$ be satisfied. If*

$$K_j \ell(t_{j+1}) h(t_{j+1}) p(t_j^+) exp\left\{ K_j \int_{t_j}^{t_{j+1}} h(s)p(s)\lambda(s)ds \right\} \leq \alpha < 1, \ \forall j, \quad (16)$$

and r is the solution of the algebraic equation

$$\alpha r + |g(\,\cdot\,,0)|_\infty = r, \quad (17)$$

then there exists a unique solution z_0 of system (10) weakly escorting the null solution Eq. (11) with distance r. If, in addition to the above conditions, we assume

$H5$: There exists a constant L, independent of j, such that

$$K_j h(t) p(t_j^+) exp\left\{ K_j \int_{t_j}^{t_{j+1}} h(s)p(s)\lambda(s)ds \right\} \leq L, \ \forall t \in (t_j, t_{j+1}],$$

then z_0 escorts the null solution of (11) with distance $\rho = max\{r, rL\}$.

Proof Let z be a solution of the Eq. (10) such that $z(t_j^+) = \xi \in B_j[0,r]$. On the interval $(t_j,\ t_{j+1}]$ we obtain

$$z(t) = \Phi(t)\Phi^{-1}(t_j)\xi + \int_{t_j}^{t} \Phi(t)\Phi^{-1}(s) f(s, z(s))ds.$$

The properties $H1 - H2$ allow to write the estimate:

$$|z(t)| \leq K_j h(t) p(t_j^+) + K \int_{t_j}^{t} h(t)p(s)\lambda(s))|z(s)|ds.$$

By the Gronwall-Bellman inequality and $H3$ we obtain

$$|z(t)| \leq K_j h(t) p(t_j^+)|\xi| \exp\left\{ K_j \int_{t_j}^{t} h(s)p(s)\lambda(s)ds \right\}. \quad (18)$$

From $z(t_{j+1}^+) = g(t_{j+1}, z(t_{j+1}))$, hypothesis $H4$ and (18) we obtain

$$\begin{aligned} |z(t_{j+1}^+)| \ &\leq \ K_j \ell(t_{j+1}) h(t_{j+1}) p(t_j^+) exp\left\{ K_j \int_{t_j}^{t_{j+1}} h(s)p(s)\lambda(s)ds \right\} |\xi| \\ &+ \ |g(t_{j+1}, 0)| \leq \alpha r + |g(\,\cdot\,,0)|_\infty. \end{aligned}$$

This last estimate implies $|z(t_{j+1})| \leq r$. Therefore the condition (6) is satisfied. From Lemma 1, there exists a solution z_0 of equation (10) such that

$$|z_0(t_j^+)| \leq r, \; \forall j. \tag{19}$$

Thus the existence of a weakly escorting solution has been proven.

Let us assume further condition $H5$. From estimate (18) we obtain

$$|z_0(t^+)| \leq \rho, \; \forall t \in J.$$

Let (t_j, ξ) and (t_j, η) be two vectors in $B_j[0, r]$. For short we abbreviate $u(t) = z(t; t_i, \xi)$ and $v(t) = z(t; t_i, \eta)$. Then for $t \in (t_i, t_{i+1}]$ we have

$$u(t) - v(t) = \Phi(t)\Phi^{-1}(t_j)(\xi - \eta) + \int_{t_j}^{t} \Phi(t)\Phi^{-1}(u)[f(s, \; u(s)) - f(u, \; v(s))]ds,$$

implying

$$|u(t) - v(t)| \leq K_j|\xi - \eta|h(t)p(t_j)\exp\left\{K_j \int_{t_j}^{t} h(s)p(s)\lambda(s)ds\right\}. \tag{20}$$

From this last estimate and

$$u(t_{j+1}^+) - v(t_{j+1}^+) = g(t_{j+1}, \; u(t_{j+1})) - g(t_{j+1}, \; v(t_{j+1})),$$

we obtain

$$|u(t_{j+1}^+) - v(t_{j+1}^+)| \leq K_j|\xi - \eta|\ell(t_{j+1})h(t_{j+1})p(t_j^+)\exp\left\{K_j \int_{t_j}^{t_{j+1}} (hp\lambda)(s)ds\right\}.$$

Therefore

$$|\mathcal{T}_i(\xi) - \mathcal{T}_i(\eta)| \leq \alpha|\xi - \eta|.$$

Thus condition (9) is satisfied implying the uniqueness of solution z_0. □

We will use the following notation $\beta_j = \sup\{|g(t, 0)| : t \geq t_j\}$.

Theorem 3 *Under hypotheses $H1 - H5$, any solution z of Eq. (10) satisfying $z(t_j^+) = \xi$, $|\xi| \leq r$ has the estimate*

$$|z(t_{j+k}^+)| \leq \alpha^k |z(t_j^+)| + \sum_{i=1}^{k} \alpha^{k-j}\beta_{j+i}, \tag{21}$$

and

$$|z(t)| \leq L\left(\alpha^k |z(t_j^+)| + \sum_{i=1}^{k} \alpha^{k-j}\beta_{j+i}.\right), \; t \in (t_{j+k}, t_{j+k+1}]. \tag{22}$$

Proof From (18) and $z(t_{j+1}^+) = g(t_{j+1}, z(t_{j+1}))$, we obtain

$$|z(t_{j+1}^+)| \leq \alpha|z(t_j^+)| + \sup_{t \geq t_{j+1}} |g(t, 0)|.$$

From this discrete inequality it follows (21). The estimate (22) follows from $H5$ and (18). □

From this result it follows

Corollary 1 *Under conditions of Theorem 3, if*

$$\lim_{t\to\infty} g(t,0) = 0, \tag{23}$$

then any solution o z of Eq. (10), starting at $z(t_j^+) = \xi$, $|\xi| \leq r$, *satisfies*

$$\lim_{t\to\infty} z(t) = 0. \tag{24}$$

The conditions $H1 - H5$ and (23) implies that the bounded solution of equation (10) given by Theorem 5 satisfies (24).

Acknowledgments. This research was supported by Comisión de Investigación de la Universidad de Oriente under grant Proyecto CI-5-025-00730/95.

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