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Hyperbolic functional differential inclusion in Hilbert spaces

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Abstract

In this paper we examine an existence result of solutions for a class of nonlinear functional differential inclusion of hyperbolic type.

1 Introduction

This paper presents existence result for the following second order functional differential inclusion in Hilbert spaces .

$$(P)\quad \begin{cases} u''(t) + Au(t) \in F(t, u_t) - \partial\psi(u'(t)) \quad a.e.\ t \in (0,T) \\ u_0 = \varphi \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

where A is a linear operator, $\partial\psi$ is the subdifferential of convex function ψ and F is a multifunction.

These equations are widely used in many questions in applied mathematics,such that those in control theory,electronics,chemistry,ecology and biology.....

2.Preliminaries.

Let H and V two real Hilbert spaces such that $V \subset H$ and the inclusion mapping of V into H is continuous and densely defined ;V' denotes the dual

space of V and H is identified with its own dual H$'$.We denote by (,) the duality of V and V$'$ as well as the inner produit on H,by $\|.\|$ and $|.|$ the norm in V and H respectively. Moreover,we assume in this paper that the inclusion mapping of V into H is also compact.
For a fixed real number $T > 0$,we introduce the following spaces $\mathcal{V} = L^2(0,T;V)$ and $\Re = L^2(0,T;H)$ we denote by $P_{fc}(V)$ the family of all nonempty closed and convex subsets of V.Let X be a Banach space,the symbol w-X (resp.s-X) is used to indicate the weak (resp. strong) topology. $C(0,T;X)$ denote the space of continuous functions from the time interval $[0,T]$ into X.
Let $A : V \to V'$ is linear,symmetric and strongly monotone, i.e.,$(u,Au) \geq w\|u\|^2$for all u $\in$V,where $w > 0$.

Let $\psi : V \to (-\infty, +\infty]$ is lower- semicontinuous proper and convex function.

We shall assume also that , there is $h \in H$ such that for every $\varepsilon \in (0,1)$ and any $v \in D(\psi) := \{v \in V : \psi(v) < +\infty\}$ the solution v_ε of equation $v_\varepsilon + \varepsilon A v_\varepsilon = v + \varepsilon h$ belongs to $D(\psi)$ and satisfies $\psi(v_\varepsilon) \leq \psi(v)$.

At last let $F : [0,T] \times L^2(-r,0;V) \to P_{fc}(V)$ is a multifunction such that :

$F_{1)}$ for every $\varphi \in L^2(-r,0;V)$ the multifunction $F(.;\varphi)$ admits a measurable selection ;

$F_{2)}$ for almost all $t \in [0,T]$ the multiapplication $F(t;.)$ is upper semicontinuous (u.s.c.) (see for example [1] and [3]) ;

$F_{3)}$ for all $\varphi \in L^2(-r,0;V)$ and almost all $t \in [0,T]$, there exists a function $a \in L^2_+(0,T)$ such that : $\|F(t,\varphi)\| := Sup\{\|y\| : y \in F(t,\varphi)\} \leq a(t)$.

3.Existence theorem .

For a fixed real number $r \succ 0$, $u \in L^2(-r,T;V)$ and $t \in [0,T]$ we denote by u_t the function of $L^2(-r,0;V)$ defined as $u_t(\theta) = u(t+\theta)$ for almost all $\theta \in [-r,0]$.

We consider the following problem for differential inclusion :

$$(P) \qquad \begin{cases} u''(t) + Au(t) \in F(t,u_t) - \partial\psi(u'(t)) \quad a.e.\ t \in (0,T) \\ u_0 = \varphi \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

where $\varphi \in L^2(-r,0;V)$; $\alpha, \beta \in V$ such that $A\alpha \in H$ and $\beta \in D(\psi)$.

By U we denote the set of functions $u \in L^2(-r,T;V)$ such that $u_{|[0,T]} \in C(0,T;V)$; $u'_{|[0,T]} \in L^\infty(0,T;V) \cap C(0,T;H)$ and $u''_{|[0,T]} \in \Re$,where the derivative is understood in the sens of vector valued distributions .

Definition

A function $u \in U$ is called a solution of problem (P) if

$$\begin{cases} u''(t) + Au(t) + \partial\psi(u'(t)) \ni f(t) \ a.e. \ t \in (0,T) \\ u_0 = \varphi \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

where $f \in S^1_{F(.,u.)} := \left\{ f \in L^1(0,T;V) \ : f(t) \in F(t,u_t) \, a.e. \, t \in (0,T) \right\}$

Theorem

Suppose that the conditions enumerated in the above paragraph are fulfilled ;then the problem (P) admits a solution .

Proof.

We follow the ideas used in [4] and [5].We know (see [2] $270-279$) that for all $f \in \mathcal{V}$, the problem

$$\begin{cases} u''(t) + Au(t) + \partial\psi(u'(t)) \ni f(t) \ a.e. \ t \in (0,T) \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

admits a unique solution $u \in C(0,T;V)$ which satisfies $u' \in L^\infty(0,T;V) \cap C(0,T;H)$ and $u'' \in \Re$.

We define :

$K = \{ f \in \mathcal{V} \ : \|f(t)\| \leq a(t) \ \ a.e. \ t \in (0,T) \}$;

$\nu \ : K \to U$ defined by $\nu(f) = u$,where u is the unique solution to the problem $P(f)$:

$$\begin{cases} u''(t) + Au(t) + \partial\psi(u'(t)) \ni f(t) \ a.e. \ t \in (0,T) \\ u_0 = \varphi \\ u(0) = \alpha \\ u'(0) = \beta \end{cases}$$

G the multifunction defined on K by

$G(f) = S^1_{F(.,\nu(f).)} = \{ g \in L^1(0,T;V) \ : g(t) \in F(t,\nu(f)_t) \ a.e. \ t \in (0,T) \}$.

From conditions $F_1) - F_3)$ it follows that $G(f) \neq \emptyset$ (see [6]) .Moreover , since F is $P_{f.c}(V)$- valued and by $F_3)$, the multifunction G has closed and convex values and $G : K \to P_{f.c}(K)$.We claim that G is u .s .c . on K for the weak topologies ; but K is weakly compact in $\mathcal{V}$ it suffies to prove that graph (G) is weakly closed in $K \times K$ (see for example [3] p .300) .The main step of this proof in given by the following lemmas .

Lemma 3.1 .

Suppose that $u \in U$ is the solution of $P(f)$, then

$$\begin{aligned} &\|u\|_{L^2(-r,T;V)} \leq const \ , && (1)\\ &\|u'\|_{\Re} \leq const \ , && (2)\\ &| Au(t) | \leq const \ for \ \text{all } t \in [0,T] && (3)\\ &\|u"\|_{\Re} \leq const \ . && (4) \end{aligned}$$

Proof.

Using 1.29 (see[2] p.272) we obtain (1) and (2) .

The proof of (3) is a variant of that given before for lemma 1.1 (see [2] p.271).Using (3) and a variant of proof of theorem 2.1(see [2] p.190) , we obtain (4) .

Lemma 3.2.

Let (f_n) , f be given satisfying $f_n \in K$, $f \in K$ and $f = w- \lim\limits_{n\to+\infty} (f_n)$ in $\mathcal{V}$; then $\nu(f)_{|[0,T]} = s- \lim\limits_{n\to+\infty} \nu(f_n)_{|[0,T]}$in $C\,(0,T;V)$

Proof.

Denote by $u_n = \nu(f_n)$ the unique solution of problem $P(f_n)$ .Using lemma 3.1 and passing if necessary to a subsequence we can assume that (see for example [7] p. 419) ,

$$\begin{aligned} u_{n|[0,T]} &\rightharpoonup u \ in \ w - \mathcal{V}\\ u'_{n|[0,T]} &\rightharpoonup u' \ in \ w - \Re\\ u''_{n|[0,T]} &\rightharpoonup u'' \ in \ w - \Re . \end{aligned}$$

We shall prove that

$$\lim_{n\to+\infty} u_n\,(t) = u(t) \text{ uniformly on } [0,T] \text{ in } V$$
$$\lim_{n\to+\infty} u_n'\,(t) = u'(t) \text{ uniformly on } [0,T] \text{ in } H$$

Using the fact that $\partial\psi$ is monotone in $V \times V'$, we obtain for all $n, m \in IN$ and a .e. $t \in (0,T)$

$\left(f_n(t) - Au_n(t) - u_n''(t) - f_m(t) + Au_m(t) + u_m''(t) \; , \; u_n'(t) - u_m'(t)\right) \geq 0$.
Hence ,

$$\tfrac{d}{dt}\left[\mid u_n'(t) - u_m'(t) \mid^2 + (A(u_n(t) - u_m(t)), u_n(t) - u_m(t))\right] \leq$$

$$2\left(f_n(t) - f_m(t), u_n'(t) - u_m'(t)\right)$$

Integrating over $(0,t)$ and using the fact that A is strongly monotone we obtain

$$\mid u_n'(t) - u_m'(t) \mid^2 + w \parallel u_n(t) - u_m(t) \parallel^2 \leq$$

$$\textstyle\int_0^t \mid f_n(s) - f_m(s) \mid^2 ds + \int_0^t \mid u_n'(s) - u_m'(s) \mid^2 ds.$$

Using Gronwal' s inequality and the inclusion mapping of $\mathcal{V}$ into $\Re$ is compact we may conclude that

$$\lim_{n\to+\infty} u_n(t) = u(t) \text{ uniformly on } [0,T] \text{ in } V$$
$$\lim_{n\to+\infty} u_n'(t) = u'(t) \text{ uniformly on } [0,T] \text{ in } H$$

Now , it remains to show that the function defined by

$$\hat{u}(t) = \begin{cases} \varphi(t) \text{ if } t \in (-r,0) \\ u(t) \text{ if } t \in [0,T] \end{cases}$$

is a solution of $P(f)$.

The proof of this point is similar to that of ([2] p .278) it is included here for completness.Let $s \in (0,T)$ be such that $u''(s)$ exists and let $(x,y) \in \partial\psi$ we have a .e .$t \in (0,T)$

$$\left(u_n''(t) + Au_n(t) - f_n(t) + y, u_n'(t) - x\right) \geq 0$$

so that

$$\frac{1}{2}\frac{d}{dt}\left|u_n'(t) - x\right|^2 + (Au_n(t) \; , \; u_n'(t) - x) \geq (f_n(t) - y, u_n'(t) - x)$$

integrating over (s,t), we obtain

$$\left|u_n'(t) - x\right|^2 - \left|u_n'(s) - x\right|^2 + (Au_n(t) \; , \; u_n(t)) - (Au_n(s) \; , \; u_n(s)) \geq$$

$$2\textstyle\int_s^t (f_n(\tau) - y, u_n'(\tau) - x)d\tau + 2\int_s^t (Au_n(\tau) \; , \; x)d\tau$$

we pass to the limit in this inequality we get

$$\left|u'(t)-x\right|^2-\left|u'(s)-x\right|^2+(Au(t)\ ,\ u(t))-(Au(s)\ ,\ u(s))\geq$$

$$2\int_s^t(f(\tau)-y,u'(\tau)-x)d\tau+2\int_s^t(Au(\tau)\ ,\ x)d\tau\,.$$

Extracting a subsequence if necessary , we may assume that $Au_n \rightharpoonup Au$ in w-$L^\infty(0,T;H)$ and we use

$$(Au(t)\,,\,u(t))-(Au(s)\,,\,u(s))=2\int_s^t(Au(\tau)\ ,\ u'(\tau))d\tau$$

$$\left|u'(t)-x\right|^2-\left|u'(s)-x\right|^2\leq 2\left(u'(t)-u'(s)\ ,u'(t)-x\right)$$

we obtain that

$$\int_s^t(f(\tau)-y,u'(\tau)-x)d\tau+\int_s^t(Au(\tau)\ ,\ x)d\tau\ \leq$$

$$(u'(t)-u'(s)\,,u'(t)-x)+\int_s^t(Au(\tau)\ ,\ u'(\tau))d\tau.$$

We further suppose that s is a Lebesgue point of the functions $Au \in L^\infty(0,T;H)$ and $u' \in L^\infty(0,T;V)$ ($u' \in L^\infty(0,T;V)$ because (u_n') is bounded in $L^\infty(0,T;V)$ taking a weakly convergent subsequence we may therefore infer that $u' \in L^\infty(0,T;V)$) .Dividing by $t-s$ and letting $t \to s$ in the last inequality we obtain .

$$\left(u''(s)+Au(s)-f(s)+y\,,u'(s)-x\right)\geq 0\,.$$

We concluede that $u''(s)+Au(s)+\partial\psi(u'(s)) \ni f(s)$.

Let us now consider two sequences (f_n), (g_n) of $\mathcal{V}$ such that $g_n \in G(f_n)$, $w-\lim_{n\to+\infty}(f_n)=f$, $w-\lim_{n\to+\infty}(g_n)=g$ in $\mathcal{V}$; we shall show that $g \in G(f)$.

Since K is closed for the weak topology then $f \in K$ and g $\in K$, it suffices to prove that $g(t) \in F(t,\nu(f)_t)$ a .e. $t \in (0,T)$

We have s$-\lim_{n\to+\infty}(g_n)=g$ in $\Re$; passing if necessary to a subsequence we can assume that (g_n) converges to g almost everywhere on $(0,T)$. From condition F_2) and lemma 3.2 it follows that for almost all $t \in (0,T)$, for a given $\varepsilon > 0$,there exists a integer $n_o := n_o(\varepsilon,t)$ such that for all $n > n_0$

$$F(t,\nu(f_n)_t)\subset F(t,\nu(f)_t)+B_\varepsilon:=\{y+z\ :y\in F(t,\nu(f)_t)\ \ ,\|z\|\leq\varepsilon\}\,,$$

then $g_n(t) \in F(t, \nu(f)_t) + B_\varepsilon$, for n > n_0 so that $g(t) \in F(t, \nu(f)_t)$ a . e .$t \in (0, T)$.
By Kakutani -Ky Fan fixed point theorem for set -valued mapping (see [1]) we deduce that there exists $f \in K$ such that $f \in G(f)$. The corresponding solution $u = \nu(f)$ is a solution to the problem (P) .

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