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ASYMPTOTIC LIMITS OF SOLUTIONS OF CONDITIONALLY CONVERGENT DIFFERENTIAL EQUATIONS*

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This paper is dedicated to Professor C. Corduneanu on his recent retirement

1. Introduction

We are interested in studying the relationship between the asymptotic properties of the solutions of

(E) $$y' = g(t,y)$$

as $t \to \infty$ and the asymptotic behavior of the solutions of the limiting equation of (E) given by

(L) $$x' = f(t,x)$$

as $t \to \infty$. We will generally assume $f, g \in C(I \times D, R^n)$ where I is either $[0,\infty)$ or $(-\infty,\infty)$ and D is an open set in R^n. We say (L) is a limiting equation of (E) or f is in the limit set of g relative to a given topology on $I \times D$ if there is a sequence $t_n \to \infty$ such that the $\{t_n\}$ translates of g, $g(t+t_n, y)$ converges to $f(t,y)$ as $t_n \to \infty$ relative to a given topology (to be made precise in Section 3). In this case we will say $g(t,y)$ diminishes to $f(t,y)$ along the sequence $\{t_n\}$. If $g(t,y)$ diminishes to $f(t,y)$ along every sequence $\{t_n\}$ then we say $g(t,y)$ diminishes to $f(t,y)$ as $t \to \infty$. It will be convenient to consider (E) as a perturbation of (L) as follows: Let $g(t,y) = F(t,y) + h(t,y)$ so (E) becomes

(P) $$y' = F(t,y) + h(t,y),$$

and we will assume along some sequence $\{t_n\}$ both $F(t,y)$ diminishes to $f(t,y)$ and $h(t,y)$ diminishes to zero. Roughly speaking our main result shows that the positive limit set of any bounded solution $y(t)$ of (P), which we denote by $\Gamma^+y(t)$, is invariant with respect to solutions of (L) if solutions of the initial value problem of (P) and (L) are unique and $\Gamma^+y(t)$ is semi-invariant with respect to solutions of (L) when the uniqueness assumption is not imposed. We then apply this result (Theorem 3.1) to an investigation of the asymptotic behavior of solutions of the perturbed equation (P) when the asymptotic behavior of the solutions of the unperturbed equation (L) is known.

We now give a brief history of this problem: Markus [3] assumed $F(t,y) \equiv F^*(y)$ and that $h(t,y)$ diminishes to zero in the compact open topology, namely:

$$h(t,y) \to 0 \text{ as } t \to \infty \text{ uniformly on compact subsets of } D. \tag{1.1}$$

In this case (L) is given by $x' = F^*(x)$. Opial [5] also assumed $F(t,y) = F^*(y)$ where now $h(t,y)$ diminishes to zero as $t \to \infty$ in the "L^1 topology" given by: for each bounded set $B \subset D$

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$$\sup_{y(\cdot)\in B} \int_0^\infty ||h(t,y(t)\,||\,dt < \infty, \qquad (1.2)$$

where $y \in C[[0,\infty), B]$. Both Markus [3] and Opial [5] assumed uniqueness of solutions of initial value problems of (P) and (L). Yoshizawa [8] without the uniqueness assumption, assumed $F(t,y)$ diminishes to $F^*(y)$ and that $h(t,y) = h_1(t,y) + h_2(t,y)$ where $h_1(t,y)$ diminishes to zero in the sense of (1.1) and $h_2(t,y)$ diminishes to zero in the sense of (1.2) as $t \to \infty$. In all three cases the limiting equation (L) is the autonomous equation $x' = F^*(x)$ and $F(t,x) + h(t,x)$ diminishes to $F^*(x)$ as $t \to \infty$ in any topology containing (1.1) and (1.2). In such topologies therefore the asymptotic limit set of $F(t,x) + h(t,x)$ as $t \to \infty$ is the singleton $F^*(x)$. Miller [4] weakened this restriction as he assumed $F(t,y)$ is almost periodic in t for each $y \in D$, and for each compact set $D^* \subset D, F(t,y)$ is uniformly continuous on $R \times D^*$. This implies that for any sequence $t_n \to \infty$ there exists a subsequence $t_{n_k} \to \infty$ and an almost periodic function $\overline{F}(t,y)$ such that

$$F(t + t_{n_k}, y) \to \overline{F}(t,y) \text{ as } k \to \infty \qquad (1.3)$$

uniformly for $t \in R$ and x on compact subsets of D. This is a compactness condition on $F(t,y)$ and is automatically satisfied in the previous three cases. In this case the limiting function $f(t,x)$ is not a singleton but consists of the closure of the translates of f in the sense of (1.3). Miller [4] assumed that $h(t,y)$ satisfied the same conditions as Yoshizawa [8] and also that solutions are not unique. Strauss and Yorke [7] assumed that $F(t,y) \equiv F^*(y)$ and that $h(t,y)$ diminishes to zero as $t \to \infty$ in a very general sense which included (1.1) and (1.2). Namely they assumed for each bounded set $B \subset D$

$$\sup_{y(\cdot)\in B}\ \sup_{0\le u\le 1} \left| \int_t^{t+u} |h(s,y(s)|ds \to 0 \text{ as } t \to \infty. \qquad (1.4)$$

where $y \in C(I,B)$. This allowed them to consider for example, perturbations $h(t,y) \equiv \overline{h}(t)$ which are conditionally convergent. They did not assume uniqueness of solutions but still (P) is restricted to $y' = F^*(y) + h(t,y)$ so that the limiting equations (L) in the topology generated by (1.4) is the single equation $x' = F^*(x)$. Sell [6] developed a theory of dynamical systems that included non-autonomous systems. The state space for this dynamical systems is the cross product of the solutions of the non-autonomous system with the time translates of the right hand side of the differential equation (in an appropriate topology). He developed a theory of limiting equations and limit sets within the dynamical system. In our context Sell's [6] work allowed for the case where the limit set of $g(t,y)$ (which is quite general) is generated by topologies that include the compact open topology and the local $\mathcal{L}_p(I \times D, R^n)$ topology (see Sell [6]) but does not seem to allow for convergence in the sense of (1.4). Also because of the dynamical system structure he needs to assume a strong uniqueness assumption on (E) or (P). Namely that $g(t,y)$ essentially satisfies a Lipschitz condition in y. He thus includes the work of Miller [4] under the restriction of the Lipschitz condition. Of course the dynamic system approach lays a foundation for the study of integral, functional and difference equations to name just a few applications of this broad theory.

The work in [1] extended all of the above results in that we use a convergence that is the same as that in (1.4), but in contrast to Strauss-Yorke [7], who assumed $h(t,y)$ diminishes to zero as $t \to \infty$, we consider cases where $h(t,y)$ may only diminish to zero along some sequences and diminish to other functions along other sequences. Moreover we do not restrict our attention to the case $F(t,y) \equiv F^*(y)$ but consider a more general limit set of $F(t,y)$ which includes that of all the previous mentioned work. Finally we do not require uniqueness of solutions. Our compactness conditions on the time translates of $F(t,y)$ was given in the topology generated by (1.4), whereas, Sell [6] often assumes the compactness condition in the stronger compact open topology or local $\mathcal{L}_p$ topology.

In this paper we extend the results in [1] by assuming the perturbation term $h(t,y)$ diminish along sequences. We develop some results that include the examples in [1], Section 4 as well as provide a vehicle for generating limiting classes of differential equations generated by perturbations whose integral is bounded but not convergent.

2. Preliminaries

Let D be an open set in R^n and let $I = [0,\infty)$. We shall assume all solutions of (L) have domain R and all solutions of (P) have domain I. A solution of the differential equation (P) satisfying $y(t_0) = y_0, t_0 \in I$, is given by $y(t,t_0,y_0)$. The positive limit set of a solution $y(t)$ of (P), denoted by $\Gamma^+(y(t))$, is the set of points $z \in D$ such that there is a sequence $t_n \to \infty$ with $y(t_n) \to z$. If in addition $y(t)$ is bounded, $\Gamma^+(y(t))$ is compact and connected. For solutions of (P), whose domain is I a set $S \subset D$ is said to be positively semi invariant for (P) if for each $y_o \in S$ there exists a solution of (P), $y(t,t_0,y_0)$ such that $y(t,t_0,y_0) \in S$ for $t \geq 0$. If solutions of (P) are uniquely determined by initial conditions (t_0,y_0) with $t_0 \in I$, $y_0 \in S$ then S is positively invariant. For solutions of (L) (which are defined on R) then the set S is said to be semi-invariant if $y_0 \in S$ implies there exists a solution $y(t,t_0,y_0) \in S$ for $t \in R$ and invariant if we also assume uniqueness of solutions determined by initial conditions.

We first define a class of functions that include, for example, functions in $C(D,R^n)$ or time almost periodic functions described in the introduction, or functions satisfying (1.2). It also includes the measurable functions contained in $L_p(I \times D, R^n)$ (see Sell [6]) are continuous.

Definition 2.1

We say $k(t,y) \in C(R \times D, R^n)(C(I \times D, R^n))$, satisfies condition (UC) if for each $\epsilon > 0$, for each compact set $J \subset R(I)$, for each compact set $D^* \subset D$, there exists $T(\epsilon, D^*, J)$, $\delta(\epsilon, D^*)$ such that for $\tau \geq T$, $x,y \in D^*$, $\| x - y \| < \delta$ then $\sup_{t \in J} \int_0^t |k(s+\tau,x) - k(s+\tau,y)|\, ds < \epsilon$.

Definition 2.2.

We say $k(t,y)$ satisfies condition (EC) if for each bounded, continuous sequence $\{y_n(t)\}, y_n(t) \subset D$ for $t \in R(I)$, if for each sequence $t_n \to \infty$, and for each compact interval $\tilde{I} \subset R(I)$ there exists a continuous function $H(t), t \in \tilde{I}$, such that

$$\int_0^t k(s+t_n, y_n(s))\,ds \to H(t) \text{ as } t_n \to \infty \tag{2.1}$$

uniformly for t in $\tilde{I}$.

We now define the limiting set $f(t,y)$ of $g(t,y)$.

Definition 2.3

Let $g(t,y) \in C(I \times D, R^n)$. Then the limit set of g denoted by Ω_g is the set $\{f(t,y) \in C(R \times D, R^n)\}$ such that there exists $t_n \to \infty$ such that for each bounded set $B \subset D$, for each compact interval $J \subset R$, and for each continuous function $y: J \to R^n$

$$\sup_{y(\cdot) \in B} \sup_{t \in J} \int_0^t g(s + t_n, y(s)) - f(s, y(s))ds \to 0 \text{ as } t_n \to \infty \quad (2.2)$$

We say g diminishes to f along the sequence t_n when (2.2) is satisfied. If g diminishes to f along every sequence $t_n \to \infty$ we say g diminishes to f as $t \to \infty$.

If g is independent of y, that is $g(t,y) \equiv s(t)$ then we have

Definition 2.4

Let $s(t) \in C(I, R^n)$. Then the limit set of $s(t)$, denoted by Ω_s, is the set $\{\bar{s}(t) \in C(R, R^n)\}$ such that there exists a sequence $t_n \to \infty$ so that for each compact interval J in R.

$$\sup_{t \in J} \int_0^t (s(\xi + t_n) - \bar{s}(\xi))d\xi \to 0 \text{ as } t_n \to \infty. \quad (2.3)$$

And accordingly we say $s(t)$ diminishes to $\bar{s}(t)$ along the sequence t_n. If $s(t)$ diminishes to $\bar{s}(t)$ along every sequence $t_n \to \infty$ then $s(t)$ diminishes to $\bar{s}(t)$ as $t \to \infty$. Notice $s(t)$ satisfies (EC) with $H(t) \equiv \bar{s}(t)$ and also satisfies (UC).

As we have alluded to in the introduction this convergence in (2.2) and (2.3) reduces to that in (1.4) in the case when g or s diminishes to zero as $t \to \infty$. We now provide a compactness condition on $g(t,y)$ that is more general than Sell's [6] condition on the compactness of the time translates of $g(t,y)$.

Definition 2.5.

Let $g(t,y) \in C(I \times D, R^n)$. We say $g(t,y)$ satisfies (C) if for each sequence $t_n \to \infty$ there exists a subsequence $t_{n_i} \to \infty$ and a function $f \in \Omega_g$ such that (2.2) holds with t_n replace by t_{n_i}.

Condition (C) is satisfied when $g(t,y)$ satisfies the almost period conditions given in Miller [4] and whenever Ω_g is a singleton as assumed in Markus [3], Opial [5], Strauss and Yorke [7] and Yoshizawa [8].

3. Invariance of Limiting Equation

We now state and prove a slight generalization of the main result in [1] where we compare solutions of (P) with those of (L).

Theorem (3.1)

Assume we can write $g(t,y) = F(t,y) + h(t,y)$, where F satisfies (UC), (EC) and (C) and $h(t,y)$ diminishes to zero as $t \to \infty$. Let $y(t)$ be a bounded solution of (P). Then:

(i) for each $z \in \Gamma^+(y(t))$ there exists $f(t,x) \in \Omega_g = \Omega_F$, a sequence $\tau_n \to \infty$ and a solution $x(t,0,z)$ of (L) such that

$$y(t + \tau_n) \to x(t) \text{ as } \tau_n \to \infty \quad (3.1)$$

uniformly for t on compact intervals of R. Moreover $F(t,y)$ diminishes to $f(t,y)$ along the sequence $\{\tau_n\}$.

(ii) (a) $\Gamma^+(y(t))$ is a semi-invariant set of (L) and thus consists of the union of trajectories of (L).

(b) If solutions of (L) are unique, then $\Gamma^+ y(t)$ is an invariant set of (L).

(c) Assume F does not necessarily satisfy (C), Ω_F is nonempty and that. the remaining conditions of Theorem 3.1 are satisfied. Then there exists $z \in \Gamma^+(y(t))$, an $f(t,x) \in \Omega_g = \Omega_F$, a sequence $\tau_n \to \infty$ and a solution $x(t,0,z)$ of (L) such that (3.1) is satisfied.

Proof:

Since $y(t)$ is bounded there is a compact set $D^* \subset D$ such that for $t \geq 0, y(t) \in D^*$. Now there exists a sequence $t_n \to \infty$ such that $y(t_n) \to z$ as $t_n \to \infty$. By condition (C) there exists a subsequence of (t_n) which we again index by n and a function $f \in \Omega_F$ such that F diminishes to f along t_n. Since $h(t,y)$ diminishes to zero along t_n we have $f \in \Omega_g$. Pick any $T > 0$ and define for $t \in [-T,T]$, $y_n(t) = y(t+t_n)$. Clearly $y_n(t) \in D^*$ and hence the sequence $\{y_n(t)\}$ is uniformly bounded for $t \in [-T,T]$. Since $y(t)$, for $t \in [-T,T]$, satisfies $y(t) = y(t_n) + \int_{t_n}^{t} F(s,y(s))ds + \int_{t_n}^{t} h(s,y(s))ds$, then

$$y_n(t) = y(t_n) + \int_0^t F(s+t_n, y_n(s))ds + \int_0^t h(s+t_n, y_n(s))ds. \tag{3.2}$$

We want to show that $\{y_n(t)\}$ is equicontinuous on $[-T,T]$ which would then imply by Ascoli's Theorem that $\{y_n(t)\}$ is compact. Since $h(t,y)$ diminishes to zero then letting $B = D^*, J = [-T,T]$ in (2.2) we have the existence of a sequence (ϵ_n) where $\epsilon_n \to 0$ as $n \to \infty$ such that

$$\sup_{t \in [-T,T]} \left| \int_0^t h(s+t_n, y_n(s))ds \right| < \epsilon_n. \tag{3.3}$$

Since $F(t,y) \in K$, letting $\tilde{I} = [-T,T]$ in (2.1), we conclude there is a sequence $\{\bar{\epsilon}_n\}$ where $\bar{\epsilon}_n \to 0$ as $n \to \infty$ such that for $t \in [-T,T]$

$$\left| H(t) - \int_0^t F(s+t_n, y_n(s))ds \right| \leq \bar{\epsilon}_n.$$

Hence for any $t_1, t_2 \in [-T,T]$ we have using (2.1) and the above inequality

$$\left| \int_{t_1}^{t_2} F(s+t_n, y_n(s)ds \right| \leq | H(t_2) - H(t_1) | + 2\bar{\epsilon}_n. \tag{3.4}$$

Thus

$$y_n(t_2) - y_n(t_1) = \int_{t_1}^{t_2} F(s+t_n, y_n(s))ds + \int_{t_1}^{t_2} h(s+t_n, y_n(s)ds;$$

and this implies using (3.3) and (3.4) that

$$| y_n(t_2) - y_n(t_1) | \leq | H(t_2) - H(t_1) | + 2(\epsilon_n + \overline{\epsilon}_n)$$

Since $H(t)$ is uniformly continuous on $[-T, T]$ and $\epsilon_n + \overline{\epsilon}_n \to 0$ as $n \to \infty$, we see $\{y_n(t)\}$ is equicontinuous. Hence there exists a subsequence of $\{y_n(t)\}$ which we again index by n and a function $x(t)$ such that

$$| y(t + t_n) - x(t) | = | y_n(t) - x(t) | \to 0 \text{ as } n \to \infty \tag{3.5}$$

uniformly for $t \in [-T, T]$. Since $F(t, y)$ satisfies (C) we have the existence of a subsequence $\{\tau_n\}$ of $\{t_n\}$ and a function $f(t, y)$ such that $F(t, y)$ diminishes to $f(t, y)$ along τ_n. Applying (UC) and (2.2) to $F(t, y)$ we find

$$\int_0^t F(s + \tau_n, y(s + \tau_n))ds \to \int_0^t f(s, x(s))ds \text{ as } \tau_n \to \infty \tag{3.6}$$

uniformly for $t \in [-T, T]$. Taking the limit of both sides of (3.2) with τ_n replacing t_n we have from (3.3) and (3.5) that $x(t)$ satisfies for $t \in [-T, T]$

$$x(t) = z + \int_0^t f(s, x(s))ds;$$

that is $x(t)$ is a solution of (L) and satisfies $x(0) = z$. Using a standard diagonalization process we find that $x(t)$ may be defined on R, satisfies (3.5) on all compact intervals of R, and is a solution of (L) on R. This proves (i); and (3.1) implies the semi-invariance and invariance of $\Gamma^+(y(t))$ relative to (L). Thus (ii) is proved.

The proof of (c) is similar to (i) in that there is a function $f \in \Omega_F$ and a sequence $\tau_n \to \infty$ such that (2.2) holds along the sequence $\{\tau_n\}$. Since $h(t, y)$ diminishes to zero we have $f \in \Omega_g$. The remaining proof of (c) then follows from (i).

Example 3.1.

Assume in (P), $F(t, y)$ is almost periodic in $t \in R$ for each fixed y and uniformly continuous on $R \times D^*$ for each compact set $D^* \subset D$. Let $h(t, y) \equiv s(t)$ where $\int_t^{t+1} s\{\xi\}d\xi \to 0$ as $t \to \infty$. Then for each bounded solution $y(t)$ of (P) there exists an almost periodic function $f^*(t, x) \in \Omega_F$, a solution $x(t)$ of $x' = f^*(t, x)$ and a sequence $\tau_n \to \infty$ such that (3.1) holds. If $\Gamma^+(y(t)) \equiv \overline{y}$ then $f^*(t, \overline{y}) \equiv 0$. This result follows from Theorem 3.1 since $F(t, y)$ satisfies (UC) and $s(t)$ satisfies (2.3) with $\overline{s}(t) = 0$. It extends the result of Miller [4] since he assumed either $s(t)$ is contained in $L^1[0, \infty)$ or approaches zero as $t \to \infty$. It also extends the result of Strauss and Yorke [7] who assumed $F(t, y) \equiv \overline{F}(y)$.

Example 3.2.

In (P) assume for $t \geq 0$, $| F(t, y) | \leq \lambda(t)\rho(| y |)$ where $\lambda(\cdot), \rho(\cdot)$ are non-negative functions in which $\int_0^\infty \lambda(s)ds < \infty$ and $\rho(| y |)$ is a continuous function. Let $\int_t^{t+1} s(\xi)d\xi \to 0$ as $t \to \infty$. Then using Theorem 3.1 each bounded solution of (P) approaches a constant as $t \to \infty$. Indeed we find that $F(t, y)$ satisfies (UC) since

$\int_0^\infty \lambda(s)ds < \infty$ and we choose $H(t) \equiv 0$; moreover we have that $\Omega_F = \Omega_s \equiv \{0\}$. Hence (L) becomes $x' = 0$ and (3.1) gives us the above assertion. This example partially complements a result of F. Brauer which may be found in Lakshmikantham and Leela [2] where it is assumed $s(t) \equiv 0$ in his study of asymptotic equilibria.

4. Perturbations that Diminish Along Sequences

Let us again consider (P) where now we no longer assume $s(t)$ is diminishing to zero as $t \to \infty$ as it has been assumed in Examples 3.1 and 3.2 but rather assume

$$-\infty < \liminf_{t \to \infty} \int_0^t s(\xi)d\xi < \limsup_{t \to \infty} \int_0^t s(\xi)d\xi < \infty \tag{4.1}$$

This will include the periodic and almost periodic nonconstant functions. When (4.1) is satisfied $s(t)$ is not a singleton. We shall present a description of a large class of functions satisfying (4.1), characterize the limit set of such functions and apply Theorem 3.1 in this context. Our results will include the examples in [1], Section 4 as a special case. We will first present some examples motivating our result.

Example 4.1

Consider the class of perturbations consisting of a "train of waves" satisfying various properties. A typical example is given as follows:

for $t \in [0, \infty)$ define

$$U_1(t) = \begin{cases} 0 & t = 0, 1 \\ 1 & t \in (\frac{1}{4}, \frac{3}{4}) \\ \text{linear} & t \in (0, \frac{1}{4}] \cup [\frac{3}{4}, 1) \\ 0 & t \in R^+/[0, 1] \end{cases}$$

and

$$L_2(t) = \begin{cases} 0 & t = 2, 3 \\ -1 & t \in (2 + \frac{1}{4}, 3 - \frac{1}{4}) \\ \text{linear} & t \in (2, 2 + \frac{1}{4}] \cup [3 - \frac{1}{4}, 3) \\ 0 & t \in R^+/[2, 3] \end{cases}$$

We will denote the set of upper trapezoids with odd indices as $U_{2n+1}(t)$ (with domain $[0, \infty)$) and the set of lower trapezoids with even indices $L_{2n}(t)$ (with domain $[0, \infty)$): Define recursively for $n = 1, 2, 3, \cdots$

$$U_{2n+1}(t) = \begin{cases} U_1\big(t - \big(\sum_{j=1}^{2n+1} j - 1\big)\big) & t \in \left[\sum_{j=1}^{2n+1} j - 1,\ \sum_{j=1}^{2n+1} j\right] \equiv A_n \\ 0 & t \in [0,\infty)/A_n \end{cases}$$

and

$$L_{2n+2}(t) = \begin{cases} L_2\big(t - \big(\sum_{j:=1}^{2n+1} j - 3\big)\big) & t \in \left[\sum_{j=1}^{2n+2} j - 1,\ \sum_{j=1}^{2n+2} j\right] \equiv B_n \\ 0 & t \in [0,\infty)/B_n. \end{cases}$$

Define on $[0,\infty)$, $\bar{s}(t) = \sum_{n=0}^{\infty}(U_{2n+1}(t) + L_{2n+2}(t))$. Let

$$s_0(t) = \begin{cases} \bar{s}(t) & t \geq 0, \\ \bar{s}(-t) & t < 0. \end{cases}$$

Notice $s_0(t)$ satisfies (4.1) and Ω_{s_0}=$\{U_1(t+\alpha), L_2(t+\beta), 0\}$ for all $\alpha, \beta \in R$.

Now let $M_n = A_n \cup B_n \cup \tilde{A}_n \cup \tilde{B}_n$ where $\tilde{A}_n = \{-t \in R$ such that $t \in A_n\}$ and $\tilde{B}_n = \{-t \in R$ such that $t \in B_n\}$ and let $M = \bigcup_{n=1}^{\infty} M_n$. Consider a particular modification of $s_0(t)$ given by

$$s_1(t) = \begin{cases} s_0(t) & t \in M, \\ \sin t^2 & t \in R/M. \end{cases}$$

Notice R/M is the infinite union of disjoint open intervals whose lengths become unbounded as $|t| \to \infty$. Each intervals adjoins two consecutive trapezoids. We find $s_1(t)$ satisfies (4.1) and $\Omega_{s_0} = \Omega_{s_1} = \{U_1(t+\alpha), L_2(t+\beta), 0\}$ for each $\alpha, \beta \in R$ since $\int_0^\infty s_1(t)dt < \infty$. Interestingly enough in the compact open topology or in the $\mathcal{L}_p(I, R^n)$ topology the zero function is not in the limit set of $s_1(t)$ since $\sin(t+t_n)^2$ for any sequence $t_n \to \infty$ does not converge to 0 in the above mentioned topologies. Notice for any sequence t_n contained in R/M and any compact interval $I_1 \subset R$ we have $\sup_{t \in I_1} \int_0^t s_1(\xi + t_n)d\xi \to 0$.

We now present the main result of this section which will be concerned with the limit sets of particular functions $s(t)$ satisfying (4.1). This result will include the previous examples of this section as special cases and will be used with Theorem 3.1 to obtain qualitative behavior of solution of (P).

Now define $s^+(t) = \max(s(t), 0)$ and $s^-(t) = \min(s(t), 0)$ and consider the following assumptions:

(HA) (i) Let $A = \{t \in R$ such that $s(t) > 0\}$ and assume $A = \bigcup_{i=1}^{\infty} A_i$ where $A_i = (a_i, b_i)$ such that $a_{i+1} > a_i$, $b_{i+1} > b_i$. Moreover assume there exist γ such that $\lim_{i\to\infty}(b_i - a_i) = \gamma$.

(HA) (ii) There exists a function $\omega(t)$ such that $\omega(t) = 0$ on $R/[0,\gamma]$ and there exists a sequence $t_i \to \infty$ in which $a_i < t_i < b_i$ such that $\int_0^\beta (s^+(t+t_i) - \omega(t))dt \to 0$ as $t_i \to \infty$ for each $\beta \leq \gamma$.

(HB) (i) Let $B = \{t \in R$ such that $s(t) > 0\}$ and assume $B = \bigcup_{i=1}^{\infty} B_i$ where $B_i = (c_i, d_i)$ such that $c_{i+1} > c_i$, $d_{i+1} > d_i$. Moreover assume there exist δ such that $\lim_{i\to\infty} (d_i - c_i) = \delta$.

(HB) (ii) There exists a function $v(t)$ such that $v(t) = 0$ on $R/[0, \delta]$ and there exists a sequence $\tau_i \to \infty$, $c_i < \tau_i < d_i$ such that $\int_0^\alpha (s^-(t+\tau_i) - v(t))dt \to 0$ as $\tau_i \to \infty$ for each $\alpha \leq \delta$.

(HC) (i) $c_i - b_i \to \infty$ as $i \to \infty$, $a_{i+1} - d_i \to \infty$ as $i \to \infty$

(HC) (ii) There exists m_i, $i = 1, 2$ such that $\lim_{i\to\infty} (c_i - b_i) = m_1 > 0$ and $\lim_{i\to\infty} (a_{i+1} - d_i) = m_2 > 0$

With these assumptions we have the following result:

Theorem 4.1.

Let $s(t)$ satisfy (4.1) such that $s(t)$ diminishes to $\overline{s}\,(t)$ along t_n. If

(1) (HA), (HB) and $(HC)\,(i)$ hold then $\Omega_s = \{v(t+\alpha), 0, \omega(t+\beta)$, for any $\alpha, \beta \in R\}$,

or if

(2) (HA), (HB), $HC\,(ii)$ hold then, $\Omega_s = \overline{s}\,(t)$, where $\overline{s}\,(t)$ is periodic of period $\gamma + m_1 + \delta + m_2$ and is given by

$$\overline{s}\,(t) = \begin{cases} w(t) & t \in [0, \gamma] \\ 0 & t \in [\gamma, \gamma + m_1] \\ v(t - (\gamma + m_1)) & t \in [\gamma + m_1, \gamma + m_1 + \delta] \\ 0 & t \in [\gamma + m_1 + \delta, \gamma + m_1 + \delta + m_2] \\ \overline{s}\,(t - (\gamma + m_1 + \delta + m_2)) & (t \in [\gamma + m_1 + \delta + m_2, \infty) \\ \overline{s}\,(-t) & t \in (-\infty, 0) \end{cases}$$

We now prove Theorem 4.1. Assume (HA), (HB) and $(HC)_i$ hold. Let t_n be a sequence such that $t_n \to \infty$ and $(i)\, d_n < t_n < a_{n+1}$.

Now consider the following possibilities:

(a) the sequence $a_{n+1} - t_n \to \overline{r}$ as $n \to \infty$ for some $\overline{r} \geq 0$

(b) the sequence $t_n - d_n \to r$ as $n \to \infty$ for some $r \geq 0$

(c) $a_{n+1} - t_n \to \infty$ as $n \to \infty$ and $t_n - d_n \to \infty$ as $n \to \infty$.

Assume (i) (a) holds. Let I be any closed interval in R. Now for any $\epsilon > 0$ there exists $N_1 > 0$ such that $s^+(t + \overline{r} + t_n) = 0$ for $t \in I \cap (-\infty, -\epsilon)$ and $n \geq N_1$ since $HC(i)$ implies $t_n - d_n \to \infty$ as $n \to \infty$. Moreover there exists $N_2 > 0$ such that $s^+(t + \overline{r} + t_n) = 0$ for $t \in I \cap (\gamma + \epsilon, \infty)$ and $n > N_2$ since $t_n - b_n \to \infty$ as $n \to \infty$. Now for $w(t)$ satisfying $HA(ii)$ we have

$$\int_I (s^+(t+\overline{r}+t_n) - w(t))dt = \int_{I\cap(-\infty,-\epsilon)} (s^+(t+\overline{r}+t_n) - w(t))dt$$
$$+ \int_{-\epsilon}^{0} (s^+(t+t_n+\overline{r}) - w(t))dt + \int_{I\cap(0,\gamma)} (s^+(t+\overline{r}+t_n) - w(t))dt$$
$$+ \int_{\gamma}^{\gamma+\epsilon} (s^+(t+\overline{r}+t_n) - w(t))dt \tag{4.2}$$
$$+ \int_{I\cap(\gamma+\epsilon,\infty)} (s^+(t+\overline{r}+t_n) - w(t))dt$$

Now since $w(t) = 0$ for $t \in (-\infty, 0) \cup (\gamma, \infty)$ we have for $n > \max(N_1, N_2)$ that the first and fifth integrals on the right hand side of (4.2) approach zero as $t_n \to \infty$. Since $s^+(t)$ and $w(t)$ are bounded then the second and fourth integrals can be made arbitrarily small. Moreover $(HA)\,(ii)$ implies

$$\int_{I\cap(0,\gamma)} (s^+(t+\overline{r}+t_n) - w(t))dt \to 0.$$

Hence $s^+(t+t_n)$ diminishes to $w(t+\overline{r})$.

If we now consider $s^-(t)$ we find that $s^-(t) = 0$ on $[d_n, c_{n+1}]$. Since $t_{n+1} - d_n \to \infty$ and $c_{n+1} - t_n \to \infty$ as $n \to \infty$ we find that $s^-(t+t_n) = 0$ for t_n sufficiently large and $t \in I$. Thus $\overline{s}(t) = w(t+\overline{r})$, where $\overline{s}^-(t)$ diminishes to $\overline{s}(t)$ along t_n.

Now consider $(i)-(b)$. We briefly sketch the proof as it is very similar to the case $(i)-(a)$. Indeed, if we substitute for $s^+(t+\overline{r}+t_n)$, $w(t)$, and $[0,\gamma]$ respectively, the functions $s^-(t-r+t_n)$, $v(t)$, and the interval $[0,\delta]$, then the analysis of the expression corresponding to (4.2) proceeds exactly as in case $(i)-(a)$. We then obtain $\overline{s}(t) = v(t-r)$, where $s^+(t)$ diminishes to $\overline{s}(t)$ along t_n.

In case $(i)-(c)$ we immediately find that $\overline{s}(t) \equiv 0$ on R. Indeed for any compact interval J we have for n sufficiently large that $s(t+t_n) = 0$ for $t \in R$.

This completes the proof of case (i).

Now assume:

(ii) $a_n < t_n < b_n$ and $\lim_{n\to\infty} t_n - a_n = c$.

Once again, we can apply the techniques from case (i) and ascertain $\overline{s}^+(t) = w(t-c)$, $\overline{s}^-(t) = 0$; thus $\overline{s}^+(t) = w(t-c)$.

Now consider the complimentary condition of (i) and (ii); that is let $\{t_n\}$ be a sequence such that $t_n \to \infty$. Consider

$(i)'$ $b_n < t_n < c_n$ with the following possibilities:

- $(a)'$ $c_n - t_n \to r_0$ as $n \to \infty$ for some $r_0 \geq 0$
- $(b)'$ $t_n - b_n \to \overline{r}_0$ as $n \to \infty$ for some $\overline{r}_0 \geq 0$
- $(c)'$ $|t_n - b_n| \to \infty$ and $|t_n - c_n| \to \infty$ as $n \to \infty$

as well as

$(ii)'$ $c_n < t_n < d_n$ such that $t_n - c_n \to r_1$ as $n \to \infty$.

Once again the proof of $(i)', (a)', (c)'$ and $(ii)'$ is the same as $(i), (a), (b), (c)$ and (ii), respectively. Indeed, we find in case $(i)'(a)'$ that $\overline{s}(t) = w(t-r_0)$ and in case $(i)'(b)'$ $\overline{s}(t) = v(t-r_0^-)$ while in case $(i)'(c)'$ $\overline{s}(t) \equiv 0$. Finally, in case $(ii)'$ $\overline{s}(t) = v(t-r_1)$.

These cases then complete the description of Ω_z for if $\overline{s}\,(t)$ is any element of Ω_s then there exists a sequence $t_n \to \infty$ such that $\int_I (s(t+t_n) - \overline{s}\,(t))dt \to 0$ as $t_n \to \infty$ uniformly for t in compact intervals I. If $\{t_n\}$ does not satisfy $(i)\,(a)$, $(i)\,(b)$, $(i)\,(c)$, $(i)'\,(a)'$, $(i)'\,(b)'$, $(i)'\,(c)'$, (ii) or $(ii)'$ then there will be two subsequence of $\{t_n\}$ satisfying two different conditions among the ones given above. But then $s(t+t_n)$ does not diminish to $\overline{s}\,(t)$ along t_n. This concludes the proof of the first half of Proposition 3.4.

The proof of the second half of the proposition has many of the ingredients of the first half. We consider the following possibilities:

$(i)_0$ $a_n < t_n < b_n$ and $t_n - a_n \to \ell$ as $n \to \infty$ for some $\ell \geq 0$,
$(ii)_0$ $b_n < t_n < c_n$ and $c_n - t_n \to \overline{\ell}$ as $n \to \infty$ for some $\overline{\ell} \geq 0$,
$(iii)_0$ $c_n < t_n < d_n$ and $c_n - t_n \to \ell_0$ as $n \to \infty$ for some $\ell_0 \geq 0$,
$(iv)_0$ $d_n < t_n < a_{n+1}$ and $a_{n+1} - t_n \to \overline{\ell}_0$ as $n \to \infty$ for some $\overline{\ell}_0 \geq 0$.

If $\{t_n\}$ satisfies $(i)_0$ then we claim that Ω_s is given by $\overline{s}\,(t-\ell)$ were $\overline{s}\,(t)$ is defined in (2) and is periodic of period $\gamma + m_1 + \delta + m_2$. Indeed, let I be any interval and assume without loss of generality that $\ell = 0$. Then we find for $t \in \widehat{I} \equiv I \cap [0, \gamma + m_1 + \delta + m_2]$ that

$$\begin{aligned} \int_{\widehat{I}} (s(t+t_n) - \overline{s}\,(t))dt &= \int_{\widehat{I} \cap [0,\gamma]} (s(t+t_n) - w(t))dt + \int_{\widehat{I} \cap [\gamma, \gamma+m_1]} s(t+t_n)dt \\ &+ \int_{\widehat{I} \cap [\gamma+m_1,\, \gamma+m_1+\delta]} (s(t+t_n) - v(t-\gamma+m_1)dt \\ &+ \int_{\widehat{I} \cap [\gamma+m_1,\, \delta+m_2]} (s(t+t_n))dt. \end{aligned} \tag{4.3}$$

Now as $t_n \to \infty$ the first integral on the right hand side of (4.3) approaches zero as a consequence of $(HA)(ii)$ while the second and fourth integrals approaches zero as a consequence of $(HC)(ii)$ and finally the third integral approaches zero as a consequence of $(HB)(ii)$. From this, we can obtain the periodicity of $\overline{s}\,(t)$, and from $(i)_0, (HB), (HA)$, and $HC(ii)$ we can extend the analysis for all $t \in I$. Hence $\overline{s}\,(t) \in \Omega_s$.

Using a similar proof we find in $(ii)_0$ that $\Omega_s = \overline{s}\,(t + \overline{\ell}\,)$, in $(iii)_0$ $\Omega_s = \overline{s}\,(t+\ell_0)$ and in $(iv)_0$ $\Omega_s = \overline{s}\,(t + \overline{\ell}_0)$ where $\overline{s}\,(t)$ is defined in (2) of the statement of the theorem. As in the first half of the proof let $\{t_n\}$ be any sequence such that $s(t+t_n)$ diminishes to some function $f(t)$ along $\{t_n\}$. If $\{t_n\}$ does not satisfy $(i)_0$, $(ii)_0$, $(iii)_0$ or $(iv)_0$, then there exists two subsequences satisfying two of the above conditions. But then $s(t+t_n)$ does not diminish to $\overline{s}\,(t)$ along t_n. This concludes the proof of Theorem 4.1.

Remark: There are many directions in which Theorem 4.1 can be extended. For example in $HA(i)$ let us not assume that the $\lim_{i\to\infty} b_i - a_i$ exists but rather assume $a_{i+1} - b_i \to \infty$ as $i \to \infty$ and $0 < \liminf_{i\to\infty} (b_i - a_i) < \limsup_{i\to\infty} (b_i - a_i) < \infty$.

In addition, suppose that the limit set of $\{b_i - a_i\}_{i=1}^{\infty}$ consists of a finite sequence of points $\{\gamma_i\}_{i=1}^{N}$. If we can find N functions $w_i(t)$ defined on $[0, \gamma_i]$ such that $s^+(t)$ diminishes to $w_i(t)$ along some sequence $\{t_{n_i}\}$ then we can obtain conclusion (1) in Theorem 4.1 in which Ω_{s^+} consists of $w_i(t+\beta)$ and 0. Similar statements hold for the existence of $v_i(t+\alpha)$. Notice we do not require that $s(t)$ is bounded on R. Thus after applying the above theorem to describe the limit set Ω_s we use Theorem 3.1(c) to obtain information on the asymptotic behavior of solutions of (P) where now $F(t, y) = s(t)$ and $h(t, y)$ diminishes to zero.

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