LIBERTAS MATHEMATICA, VOL. XV (1995)

EXISTENCE THEOREMS FOR SOME INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper we establish global existence theorems for some integrodifferential equations. Our theorems generalize Wintner's celebrated theorem on the global existence of solutions for ordinary differential equations. The main tool employed in our analysis is based on a simple and classical application of the Leray-Schauder alternative.

1. Introduction

This paper is concerned with the global existence of solutions for initial value problems for integrodifferential equations of the forms:

(P_1) $$x' = f(t,x,Sx), x(0) = x_0$$

(P_2) $$x' = A(t)x + f(t,x,Sx), x(0) = x_0,$$

(P_3) $$[x' - g(t,x)]' = f(t,x,Sx), x(0) = x_0, x'(0) = x_1.$$

Here x, f and g are the elements of R^n, an n-dimential Euclidean space, x_0, x_1 are constants and primes will denote differentiation with respect to t. Let $I = [0,T], T > 0$ is a constant. We shall assume that $f \in C[I \times R^n \times R^n, R^n]$, $g \in C[I \times R^n, R^n]$, and that S is a continuous operator which maps R^n into R^n, $A(t)$ is a continuous $n \times n$ matrix for $t \in I$. The symbol $|.|$ will be used to denote any convenient norm in R^n as well as a corresponding consistent matrix norm. We define $B = C(I, R^n)$ to be the Banach space of all continuous functions from I into R^n endowed with sup-norm $\| \times \| = sup\{|x(t)| : t \in I\}$.

In a paper published in 1945, A. Wintner [16] proved a remarkable theorem on the nonlocal existence of solutions for ordinary differential equations. Recently, in [1,6,7,9-11] the authors have obtained some interesting extensions of the Wintner's work by using topological transversality arguments. The problems considered in $(P_1)-(P_3)$ are in the general spirit of the investigations in [12-14], and in fact such equations can serve as models of many physical phenomena in applied mathematics, control engineering and problems of biophysis and economics. In particular, if we define the operator S suitably, then the equations considered in $(P_1)-(P_3)$ have a great diversity. For example, the operators we have in mind are of the forms:

$$Sx(t) = \int_0^t k[t,s,h(s,x(s))]ds, \tag{1.1}$$

$$Sx(t) = \sigma(t) + \int_0^t k(t,s)h(s,x(s))ds, \tag{1.2}$$

and so on. The typical example of the model of the operators of the type (1.2) arise for example when a second order differential equation of the form

$$y'' = f(t,y,y'), y(0) = y_0, y'(0) = y_1, \tag{1.3}$$

is transformed into a first order differential equation. Setting $x(t) = y'(t)$, we obtain $y(t) = y_0 + \int_0^t x(s)ds$ and the problem (1.3) is equivalent to

$$x'(t) = f(t, y_0 + \int_0^t x(s)ds, x(t)), x(0) = y_1, \tag{1.4}$$

which is of the form (P_1), with $Sx(t) = y_0 + \int_0^t x(s)ds$.

The main purpose of this paper is to study the global existence of solutions of $(P_1)-(P_3)$ by using a simple and classical application of the topological transversality theorem of Granas [3, p.61], known as Leray-Schauder alternative. Further extensions to higher order integrodifferential equations are also given. An interesting feature of this method, is that this yields simultaneously the existence of a solution and the maximal interval of existence. The primary motivation for this study comes from the earlier work of Wintner [16] and its extensions recently given by various investigators in [1,6,7,9-11] by using topological arguments based on the Leray-Schauder alternative.

2. Statement of Results

Our existence theorems are based on the application of the following theorem, which is a version of the topological transversality theorem given by Granas in [3, p.61].

Theorem G :

Let B be a convex subset of a normed linear space E and assume $O \in B$. Let $F: B \to B$ be a completely continuous operator and let

$$U(F) = \{x \in B : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}.$$

Then eigher $U(F)$ is bounded or F has a fixed point.

We list the following hypotheses used in our discussion.

(H_1) there exists a continuous function $p: I \to R_+ = [0, \infty)$ such that

$$|f(t,x,z)| \le p(t)(H(|x|) + H(|z|)),$$

for every $t \in I$ and $x, z \in R^n$, where $H: R_+ \to (0, \infty)$ is a continuous nondecreasing function.

(H_2) the operator S is bounded, i.e. there exists a constant $a > 0$ such that $|Sx| \le a|x|$ for any $x \in R^n$.

(H_3) there exist nonnegative constants c_1, c_2 such that

$$|g(t,x)| \le c_1|x| + c_2$$

for every $t \in I$ and $x \in R^n$. Our main results are given in the following theorems.

Theorem 1:

Suppose that the hypotheses $(H_1)-(H_2)$ are satisfied. Then the problem (P_1) has a solution x defined on I provided T satisfies

$$(2.1) \qquad 2\int_0^T p(s)ds < \int_c^\infty \frac{ds}{H(s)},$$

where $c = |x_0|$.

Theorem 2:

Suppose that the hypotheses $(H_1)-(H_2)$ are satisfied. Then the problem (P_2) has a solution x defined on I provided T satisfies

$$(2.2) \qquad 2M\int_0^T p(s)ds < \int_c^\infty \frac{ds}{H(s)},$$

where $M = sup\{|Y(t)Y^{-1}(s)|: t,s \in I\}, c = M|x_0|$, and Y is the fundamential matrix of the system $x' = A(t)x, x(0) = x_0, t \in I$ such that $Y(0) = I_0$, the unit matrix.

Theorem 3:

Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied. Then the problem (P_3) has a solution x defined on I provided T satisfies

$$(2.3) \qquad \int_0^T q(s)ds < \int_c^\infty \frac{ds}{s+2H(s)},$$

where $c = [1 + c_1T]|x_0| + [|x_1| + 2c_2]T$ and $q(t) = max\{c_1, p(t)\}, t \in I$.

Remark 1.

We note that our result given in Theorem 1 extends the well known theorem of Wintner [16] on the global existence of solutions of ordinary differential equation to the problem (P_1). In Wintner's theorem the function f involved in (P_1) is continuous and is of the form $f(t,x), p(t) = 1$ in hypothesis (H_1) and the integral on the right side in (2.1) is assumed to diverge. Thus a solution exists for ever $T < \infty$. The results given in Theorems 2 and 3 can be considered as further extensions of the Wintner's theorem given in [16] to the problems $(P_2)-(P_3)$.

3. Proofs of Theorems 1-3

Since the proofs of Theorems 1-3 resemble one another, we give the details for Theorems 2 and 3 only, the proof of Theorem 1 can be completed by following the proofs of Theorems 2 and 3.

In order to prove Theorem 2 we apply Theorem G. First we establish the priori bounds for the solutions of the problem $(P_2)_\lambda, \lambda \in (0,1)$, where

$$(P_2)_\lambda \quad x' = \lambda A(t)x + \lambda f(t, x, Sx), x(0) = x_0$$

If $x(t)$ is a solution of $(P_2)_\lambda$, then it satisfies the equivalent integral equations

$$x(t) = \lambda Y(t)Y^{-1}(0)x_0 + \lambda \int_0^t Y(t)Y^{-1}(s)f(s, x(s), Sx(s))ds. \tag{3.1}$$

From (3.1) and using hypotheses $(H_1)-(H_2)$ we have

$$|x(t)| \le M|x_0| + M\int_0^t P(s)(H(|x(s)|) + H(a|x(s)|))ds. \tag{3.2}$$

Define a function $z(t)$ by the right side of (3.2), then $|x(t)| \le z(t), z(0) = M|x_0| = c$ and

$$\begin{aligned} z'(t) &\le Mp(t)(H(z(t)) + H(az(t))) \\ &\le 2Mp(t)H(z(t)), (0 < a \le 1), \\ &\le 2Mp(t)H(az(t)), (1 \le a < \infty). \end{aligned} \tag{3.3}$$

From (3.3) it follows that

$$\frac{z'(t)}{H(z(t))} \le 2Mp(t), (0 < a \le 1), \tag{3.4}$$

$$\frac{z'(t)}{H(az(t))} \le 2Mp(t), (1 \le a < \infty). \tag{3.5}$$

Integration of (3.4) and (3.5) from 0 to t and use of the change of variable and the condition (2.2) gives

$$\begin{aligned} \int_c^{z(t)} \frac{ds}{H(s)} &= \int_0^t \frac{z'(s)}{H(z(s))} ds \le 2M \int_0^t p(s)ds \\ &\le 2M\int_0^T p(s)ds < \int_c^\infty \frac{ds}{H(s)}, \end{aligned} \tag{3.6}$$

for $0 < \alpha \le 1$ and

$$\begin{aligned} \int_c^{z(t)} \frac{ds}{H(s)} = \int_{\alpha c}^{\alpha z(t)} \frac{ds}{aH(s)} &= \int_0^t \frac{z'(s)}{H(\alpha z(s))} ds \\ &\le 2M\int_0^t p(s)ds \le 2M\int_0^T p(s)ds < \int_c^\infty \frac{ds}{H(s)}, \end{aligned} \tag{3.7}$$

for $1 \leq \alpha < \infty$. From (3.6) and (3.7) it follows that $z(t)$ must be bounded on I, i.e. there is a positive constant Q independent of $\lambda \in (0,1)$ such that $z(t) \leq Q$ and hence $|x(t)| \leq Q$ for $t \in I$, and consequently $\|x\| \leq Q$.

In the second step we rewrite the problem (P_2) as follows. If $y \in B$ and $x(t) = y(t) + Y(t)Y^{-1}(0)x_0$, it is easy to see that y satisfies

$$y(0) = 0,$$

$$y(t) = \int_0^t Y(t)Y^{-1}(s)f(s, y(s)) + Y(s)Y^{-1}(0)x_0, S(y(s) + Y(s)Y^{-1}(0)x_0))ds$$

if and only if $x(t)$ satisfies

$$x(t) = Y(t)Y^{-1}(0)x_0 + \int_0^t Y(t)Y^{-1}(s)f(s, x(s), Sx(s))ds.$$

Define $F: B_0 \to B_0, B_0 = \{y \in B: y(0) = 0\}$ by

$$Fy(t) = \int_0^t Y(t)Y^{-1}(s)f(s, Y(s) + Y(s)Y^{-1}(0)x_0, S(y)(s) + Y(s)Y^{-1}(0)x_0))ds. \tag{3.8}$$

Then F is clearly continuous. Now we shall prove that F is completely continuous.

Let $\{w_k\}$ be a bounded sequence in B_0, i.e. $\|w_k\| \leq b$, for all k, where b is a positive constant. From (3.8) and using the hypotheses $(H_1$-$H_2)$ and letting $M^* = sup\{p(t): t \in I\}$, we have

$$|Fw_k(t)| \leq MTM^*[H(b + M|x_0|) + H(\alpha(b + M|x_0|))] = MTL, \tag{3.9}$$

where

$$L = M^*[H(b + M|x_0|) + H(\alpha(b + M + |x_0|))]. \tag{3.10}$$

Hence from (3.9) we obtain $\|Fw_k\| \leq MTL$. This means $\{Fw_k\}$ is uniformly bounded.

Now we shall show that the sequence $\{Fw_k\}$ is equicontinuous. Let $0 \leq t_1 \leq t_2 \leq T$. Then writing $f^*(w_k(s))$ for

$$f(s, w_k(s) + Y(s)Y^{-1}(0)x_0, S(w_k(s) + Y(s)Y^{-1}(0)x_0))$$

and using the hypotheses $(H_1)-(H_2)$ we have

$$(3.11)\qquad |Fw_k(t_2)-Fw_k(t_1)|$$

$$= |\int_0^{t_2} Y(t_2Y^{-1}(s)f^*(w_k(s))ds - \int_0^{t_1} Y(t_2)Y^{-1}(s)f^*(w_k(s))ds$$

$$+\int_0^{t_1} Y(t_2)Y^{-1}(s)f^*(w_k(s))ds - \int_0^{t_1} Y(t_1)Y^{-1}(s)f^*(w_k(s))ds|$$

$$= |\int_{t_1}^{t_2} Y(t_2)Y^{-1}(s)f^*(w_k(s))ds + \int_0^{t_1}[Y(t_2)-Y(t_1)]Y^{-1}(s)f^*(w_k(s))ds|$$

$$\le \int_{t_1}^{t_2} M\ Lds + \int_0^T ML|Y(t_2)-Y(t_1)|ds,$$

where L is as defined in (3.10). From (3.11) and by virtue of the continuity of $Y(t), t\in I$, we conclude that $\{Fw_k\}$ is equicontinuous and hence by Arzelà-Ascoli theorem the operator F is completely continuous.

Moreover, the set $U(F)=\{y\in B_0 : y=\lambda Fy; \lambda\in(0,1)\}$ is bounded, since for every y in $U(F)$ the function $x(t)=y(t)+Y(t)Y|^{-1}(0)x_0$ is a solution of $(P_2)_\lambda$, for which we have proved that $\|x\|\le Q$ and hence $\|y\|\le Q+M|x_0|$. Now an application of Theorem G, the operator F ahs a fixed point in B_0. This means that the problem (P_2) has a solution $x(t)$ defined on I. This completes the proof of Theorem 2.

In order to prove Theorem 3, we apply Theorem G. First we establish the a priori bounds for the solutions of the problem
$(P_3)_\lambda, \lambda\in(0,1)$, where

$$(P_3)_\lambda\ \ [x'-\lambda g(t,x)]' = \lambda f(t,x,Sx), x(0)=x_0, x'(0)=x_1.$$

Let $x(t)$ be a solution of $(P_3)_\lambda$, then it satisfies the equivalent integral equation

$$(3.12)\qquad x(t)=x_0+[x_1-\lambda g(0,x_0)]t+\lambda\int_0^t g(s,x(s))ds$$
$$+\lambda\int_0^t\int_0^t f(\tau,x(\tau),Sx(\tau))d\tau ds.$$

From (3.12) and using hypotheses $(H_1)-(H_3)$ we have

$$(3.13) \qquad |x(t)| \le c + c_1 \int_0^t |x(s)|\,ds + \int_0^t \int_0^s p(\tau)(H(|x(\tau)|) + H(\alpha|x(\tau)|))\,d\tau\,ds,$$

where c is defined as in Theorem 3. Define a function $z(t)$ by the right side of (3.13), then $|x(t)| \le z(t), z(0) = c$ and

$$z'(t) \le q(t)[z(t) + \int_0^t q(\tau)(H(z(\tau)) + H(\alpha z(\tau)))d\tau],$$

where $q(t)$ is as defined in Theorem 3. Put

$$u(t) = z(t) + \int_0^t q(\tau)(H(z(\tau)) + H(\alpha z(\tau)))d\tau,$$

then $z(t) \le u(t), u(0) = z(0) = c$ and

$$(3.14) \qquad u'(t) \le q(t)(u(t) + 2H(u(t))), (0 < \alpha \le 1),$$
$$\le q(t)(\alpha u(t) + 2H(\alpha u(t))), (1 \le \alpha < \infty).$$

From (3.14) it follows that

$$(3.15) \qquad \frac{u'(t)}{u(t) + 2H(u(t))} \le q(t), (0 < \alpha \le 1),$$

$$(3.16) \qquad \frac{u'(t)}{\alpha u(t) + 2H(\alpha u(t))} \le q(t), (1 \le \alpha < \infty).$$

Now as in the proof of Theorem 2, integrations of (3.15) and (3.15) from 0 to t and use of the change of variable and condition (2.3) implies that $u(t)$ is bounded on I, i.e. there is a positive constant Q independent of $\lambda \in (0,1)$ such that $u(t) \le Q$ and hence $z(t) \le Q$ and $|x(t)| \le Q$, $t \in I$, and consequently $\|x\| \le Q$.

In the second step, we rewrite the problem (P_3) as follows. If $y \in B$ and $x(t) = y(t) + x_0, t \in I$, it is easy to see that y satisfies

$$y(0) = 0$$

$$y(t) = [x_1 - g(0, x_0)]t + \int_0^t g(s, y(s) + x_0)ds + \int_0^t \int_0^s f(\tau, y(\tau) + x_0, S(y(\tau) + x_0))d\tau ds,$$

if and only if $x(t)$ satisfies the equivalent integral equation version of problem

(P_3). Define $F: B_0 \to B_0, B_0 = \{y \in B: y(0) = 0\}$ by

(3.17) $$Fy(t) = [x_1 - g(0, x_0)]t + \int_0^t g(s, y(s) + x_0)ds + \int_0^t \int_0^s f(\tau, y(\tau) + x_0, S(y(\tau) + x_0))d\tau ds.$$

Then F is clearly continuous. Now using the same method, as in the proof of Theorem 2 with suitable modifications, we can easily prove that the operator F defined in (3.17) is completely continuous. Furthermore, by using the last arguments as in the proof of Theorem 2 with suitable changes, wew see that the problem (P_3) has a solution $x(t)$ on I. This completes the proof of Theorem 3.

4. Extensions to Higher Order Equations

In this section we apply the topological transversality method of Granas [3, p.61] to study the global existence of solutions of the following initial value problems for higher order integrodifferential equations of the forms:

$$(P_4) \quad \begin{aligned} &[r(t)x^{(n-1)}]' = f(t, x, Sx), \\ &x(0) = x_0, x^{(i-1)}(0) = 0, i = 2, \cdots, n; \end{aligned}$$

$$(P_5) \quad \begin{aligned} &[r(t)x']^{(n-1)} = f(t, x, Sx), \\ &x(0) = x_0, [r(0)x'(0)]^{(1-2)} = 0, i = 2, \cdots, n; \end{aligned}$$

$$(P_6) \quad \begin{aligned} &[r(t)x^{(n)}]^{(n)} = f(t, x, Sx), \\ &x(0) = x_0, x^{(i-1)}(0) = 0, i = 2, \cdots, n, \\ &[r(0)x^{(n)}(0)]^{(i-1)} = 0, i = 1, 2, \cdots, n; \end{aligned}$$

for $n \geq 1$, where $r(t)$ is a real-valued positive and sufficiently smooth function defined on I, f and S are as defined in problems $(P_1)-(P_3)$. Although, much research has been devoted to the oscillatory and asymptotic behavior of the solutions of equations of the forms in $(P_4)-(P_6)$ when $Sx=x$, (see, [4,5,15]), it seems to us, that very little is known about the existence of solutions of such equations.

We next establish the following theorems which deals with the global existence of solutions of problems $(P_4)-(P_6)$.

Theorem 4

Suppose that the hypotheses $(H_1)-(H_2)$ are satisfied. Then the problem (P_4) has a solution x defined on I provided T satisfies

$$2\int_0^T M(s)ds < \int_c^\infty \frac{ds}{H(s)}, \tag{4.1}$$

where $c=|x_0|$ and

$$M(t)=\int_0^t \int_0^{t_2} \cdots \int_0^{t_2} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n)dt_n dt_{n-1}\cdots dt_2, \tag{4.2}$$

for $t\in I$.

Theorem 5

Suppose that the hypotheses $(H_1)-(H_2)$ are satisfied. Then the problem (P_5) has a solution x defined on I proveded T satisfies

$$2\int_0^T N(s)ds < \int_c^\infty \frac{ds}{H(s)}, \tag{4.3}$$

where $c=|x_0|$ and

$$N(t)=\frac{1}{r(t)}\int_0^t \int_0^{t_2} \cdots \int_0^{t_{n-1}} p(t_n)dt_n dt_{n-1}\cdots dt_2, \tag{4.4}$$

for $t\in I$.

Theorem 6

suppose that the hypotheses $(H_1)-(H_2)$ are satisfied. Then the problem (P_6) has a solution x defined on I proveded T satisfies

$$2\int_0^T G(s)ds < \int_c^\infty \frac{ds}{H(s)}, \tag{4.5}$$

where $c = |x_0|$ and

$$G(t) = \int_0^t \int_0^{s_2} \cdots \int_0^{s_{n-1}} \frac{1}{r(s_n)} \int_0^{s_n} \int_0^{t_1} \cdots \int_0^{t_{n-1}} p(t_n)dt_n \cdots dt_1 \, .. ds_n \cdots ds_2, \tag{4.6}$$

for $t \in I$.

Remark 2

We note that our results in Theorems 4-6 extends the well known theorem of Wintner [16], to higher order differential equations of the forms in (P_4)-(P_6). In the special (4.5) and the integrals on the right sides are assumed to diverge, then the solutions of problems (P_4)-(P_6) exist for every $T < \infty$.

We give below the proof of Theorem 4 only, the proofs of Theorems 5 and 6 can be completed by following the same method with suitable changes.

In order to prove Theorem 4 we apply Theorem G. First we establish the priori bounds for the problem $(P_4)_\lambda, \lambda \in (0,1)$, where

$$(P_4)_\lambda \qquad [r(t)x^{(n-1)}]' = \lambda f(t, x, Sx), x(0) = x_0, x^{(i-1)}(0) = 0, i = 2, \cdots, n.$$

Let $x(t)$ be a solution of $(P_4)_\lambda$, then it satisfies the equivalent integral equation

$$x(t) = x_0 + \lambda \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} f(t_n, x(t_n), Sx(t_n))dt_n dt_{n-1} \cdots dt_1. \tag{4.7}$$

From (4.7) and using the hypotheses $(H_1)-(H_2)$ we have

$$(4.8)\qquad |x(t)| \le |x_0| + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n) \times$$

$$(H(|x(t_n)|) + H(\alpha |x(t_n)|))dt_n dt_{n-1} \cdots dt_1.$$

Define a function $z(t)$ by the right side of (4.8), then $|x(t)| \le z(t), z(0) = |x_0|$ and

$$(4.9)\, z(t) \le c + \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n) . . (H(z(t_n)) + H(\alpha z(t_n)))dt_n dt_{n-1} \cdots dt_1.$$

Since $z(t)$ is nondecreasing in t, from (4.9) we observe that

$$(4.10)\, z(t) \le c + \int_0^t (H(z(t_1) + H(\alpha z(t_1))) \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} p(t_n) dt_n dt_{n-1} \cdots dt_1.$$

Define a function $u(t)$ by the right side of (4.10) the, $z(t) \le u(t), u(0) = z(0) = c$, and

$$(4.11)\qquad \begin{aligned} u' &= M(t)(H(z(t)) + H(az(t))) \\ &\le 2M(t)H(u(t)), \ (0 < \alpha \le 1), \\ &\le 2M(t)H(au(t)), \ (1 \le \alpha < \infty). \end{aligned}$$

Now by following exactly the same arguments as in the proof of Theorem 1 given below the inequality (3.3) up to (3.7) it follows that $u(t)$ must be bounded on I, i.e. there is a positive constant Q independent of $\lambda \in (0,1)$ such that $u(t) \le Q$ and hence $z(t) \le Q$ and $|x(t) \le Q, t \in I$, and consequently $\|x\| \le Q$.

In the second step we rewrite the problem (P_4) as follows. If $y \in B$ and $x(t) = y(t) + x_0, t \in I$, it is easy to see that y satisfies

$$y(0) = 0,$$

$$y(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} f(t_n, y(t_n) + x_0, S(y(t_n) + x_0)).dt_n dt_{n-1} \cdots dt_1,$$

if and only if $x(t)$ satisfies (P_4) or its equivalent integral equation. Define $F: B_0 \to B_0, B_0 = \{y \in B: y(0) = 0\}$ by

(4.12) $$Fy(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-2}} \frac{1}{r(t_{n-1})} \int_0^{t_{n-1}} f(t_n, y(t_n) + x_0, S(y(t_n) + x_0)).$$

$$.dt_n dt_{n-1} \cdots dt_1,$$

for $t \in I$. Then F is clearly continuous. The rest of the proof can be completed by following the same arguments as in the proofs of Theorems 2 and 3 given above with suitable modifications. Here we omit the further details. This completes the proof of Theorem 4.

Remark 3

We note that the results obtained in Theorems 1-6 can be extended when the function f involved on the right sides of the equations in $(P_1)-(P_6)$ is replaced by

(4.13) $$f(t, x(t), Sx(t)) = \int_0^t k(t,s)x(s)ds + (hx)(t),$$

under some suitable hypotheses on the functions involved on the right side of (4.13). For detailed discussion on such equations, see [2,8]. We also note that one can very easily extend the ideas of this paper to the problems of the forms $(P_1)-(P_6)$ when the function f depends on the delay arguments, under appropriate initial conditions. For similar results, see [7,9,10].

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