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ULTIMATELY POSITIVE SOLUTIONS OF LIÉNARD TYPE EQUATIONS

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ABSTRACT. This paper concers with existence problems of ultimately positive solutions of a generalization of the Liénard equation.

1. INTRODUCTION

One of the most studied differential equation is the Liénard equation, [10],

$$x'' + f(x)x' + g(x) = 0, \tag{1.1}$$

where it is assumed that the functions f and g are, at least, continuous. A wide literature may by found in conection with different qualitative properties of this equation, begining with [1], later [13] and more recent [16], [8], [9], [3], [4], [15] and [12].

In the Liénard plane (1.1) may be written as

$$\begin{cases} x'(t) = y(t) - F(x(t)), \\ y'(t) = -g(x(t)), \end{cases} \tag{1.2}$$

where $F(x) = \int_0^x f(\xi)\,d\xi$. Starting from the form (1.2) of the equation (1.1) we consider, for the begining, the following generalization of it

$$\begin{cases} x'(t) = \varphi(x(t), y(t)), \\ y'(t) = -g(x(t)). \end{cases} \tag{1.3}$$

Later we will discuss shortly a result on the following non-autonomous case

$$\begin{cases} x'(t) = \varphi(x(t), y(t), t), \\ y'(t) = -g(x(t)). \end{cases} \tag{1.4}$$

In connection with (1.3) we will use the following hypothesis:

(h_1) $g \in C(\mathbb{R}, \mathbb{R})$, $xg(x) > 0$, for all $x \neq 0$;

(h_2) $\varphi \in C^1(\mathbb{R}^2, \mathbb{R})$, $\varphi(0,0) = 0$ and there exist a positive constant K and a function $h \in C(\mathbb{R}, \mathbb{R})$ such that

$$K \leq \varphi'_y(x,y) \leq h(y), \text{ for all } (x,y) \in \mathbb{R}^2.$$

Let us observe that from (h_1) and (h_2) it follows that the origin is the unique singular point of the system (1.3). As usually we denote $G(x) = \int_0^x g(\xi)\,d\xi$. We suppose, also, that each initial value problem coresponding to (1.3) has a unique solution.

We say that a solution (x, y) of the system (1.3) (or (1.4)) is *ultimately positive* if there exists a $t_0 \in \mathbb{R}$ such that $x(t) > 0$, $y(t) > 0$, for all $t \geq t_0$. Results on ultimately positive solutions for (1.2) may be found in [5], but our definition does not coincide with that in [5]. In [5] and [6] the negative solution case is discussed, too. The *characteristic curve* of the system (1.3) is defined as the set $\{(x,y) \mid \varphi(x,y) = 0\}$. We denote $J = [t_0, +\infty)$.

Obviously, the above definition assumes that the solution (x, y) is defined on the whole interval J. Also, under the assumption of a unique singular point an oscillatory solution is not ultimately positive and vice versa. On the oscillatory properties of the solutions of the system (1.3) some results may be found in [11].

This topic is in connection with the case when the attractivity does not imply stability, [14, 1.2.7], [2].

In the case of the system (1.3) we denote

$$\varphi_{\pm}(x,0) = \max\{0, \pm\varphi(x,0)\}$$
$$\Gamma_{\pm}(x) = \int_0^x \frac{g(\xi)}{1 + \varphi_{\pm}(\xi,0)}\,d\xi,$$

while in the case of the system (1.4) we denote

$$\varphi_{\pm}(x,0,0) = \max\{0, \pm\varphi(x,0,0)\}$$
$$\Gamma_{\pm}(x) = \int_0^x \frac{g(\xi)}{1 + \varphi_{\pm}(\xi,0,0)}\,d\xi.$$

This paper is concerned with the existence of ultimately positive solutions for (1.3) and (1.4).

2. Main results

Our first result is the following:

2.1. THEOREM. *Consider a solution* (x, y) *of the system (1.3) and suppose that:*

(i) *there are satisfied the assumptions* (h_1) *and* (h_2)*;*
(ii) $x(\cdot)$ *is strictly positive on the interval* J*;*
(iii) $y(t_0) > 0$*;*
(iv) $\varphi(x, 0) < 0$ *for all* $x > 0$.

Then $y(\cdot)$ *is strictly positive on* J.

Proof. From the assumption (iv) we have that the characteristic curve is in the first quadrat for $x > 0$.

Let us suppose that $y(\cdot)$ is not strictly positive on J. It follows that there exists a $t_1 > t_0$ with $y(t_1) = 0$. We have

$$\begin{cases} y'(t_1) = -g(x(t_1)) < 0, \\ x'(t_1) = \varphi(x(t_1), y(t_1)) = \varphi(x(t_1), 0) < 0. \end{cases}$$

It means that the trajectory enters in the fourth quadrat. In this quadrat we have $y' < 0$.

There are two cases:

(a) there exists a $t_C > t_1$ such that at the moment t_C the trajectory intersects the vertical axis. In this cas we have $x(t_C) = 0$, but this it violates the hypothesis (i).
(b) there exists a vertical asymptote to which the trajectory tends. But from the proof of Lemma 2.1 in [11] it follows that this is impossible.

Hence for all $t \in J$ it results that $y(t) > 0$ and the Theorem is proved. ■

COROLLARY. *Let* $\varphi(x, 0) < 0$ *for all* $x > 0$. *Suppose that we have a solution* (x, y) *of system (1.3) defined on* J *having the following property: there exists a* t_0 *such that* $x(t_0) > 0$, $y(t_0) > 0$ *and there exists a* $t_1 > t_0$ *with* $y(t_1) = 0$. *Then* $x(\cdot)$ *has a zero for a* $t > t_0$.

Remark. Theorem 2.1 is a generalization of Theorem 3.1 from [5], where the Liénard equation under the (1.2) form is considered.

2.2. THEOREM. *Let* (x, y) *be a solution of the system (1.3) defined on the interval* J *such that:*

(i) *there are satisfied the assumptions* (h_1) *and* (h_2)*;*
(ii) $x(\cdot)$ *is strictly positive on* J*;*
(iii) $\varphi(x, 0) < 0$ *for any* $x > 0$*;*
(iv) $y(t_0) > 0$*;*

(v)

$$\limsup_{x\to+\infty}[\Gamma_+(x)-\varphi(x,0)]=0. \tag{2.1}$$

Then there exists a $t_2 > t_0$ *such that:*

(a) $x'(t) < 0$ *for any* $t > t_2$*;*

(b) *the set* $\{x(t) \mid t > t_2\}$ *is bounded;*

(c)

$$\lim_{t\to+\infty} x(t) = \lim_{t\to+\infty} y(t) = 0.$$

Proof. The function $x(\cdot)$ being strictly positive on the interval J it follows that $y'(t) < 0$ for any $t \in J$, hence $y(\cdot)$ is strictly decreasing on J. By Theorem 2.1 we have that $y(t) > 0$ for all $t \in J$. From the assumption (iii) it follows that the first inequality of (3.1) in [11] is satisfied and based on Lemma 3.1 in [11], the trajectory intersects the characteristic curve, provided $(x(t_0), y(t_0)) \in D_1$. Then, if the trajectory intersects the characteristic curve we take t_2 as the moment of this intersection, otherwise $t_2 = t_0$. Hence we may suppose that the trajectory lies in the first quadrat but under the characteristic curve. Anyway, for all $t > t_2$ $x'(t) < 0$. On the other side, by (ii), $0 < x(t)$ for all $t \in J$. Hence the set $\{x(t) \mid t > t_2\}$ is bounded.

The last sentence, (c), follows from the following properties: on $[t_2, +\infty)$ the functions $x(\cdot)$ and $y(\cdot)$ are strictly positive, strictly decreasing, and the unique singular point lies in the origin. ■

Remark. From the proof of the above Theorem it is clear the use of the assumption (v): to make sure that the trajectory will enter in the region from the first quadrat, but under the characteristic curve. A similar result, for the equation (1.2), appears in [5] (Theorem 3.3). In that paper instead of (v) it is assumed that

$$\liminf_{x\to+\infty} g(x) > 0.$$

But this is a very strong condition, since from $\liminf_{x\to+\infty} g(x) > 0$ it follows $G(+\infty) = +\infty$, the converse implication is not true. In fact, for the intersection of the trajectory with the characteristic curve it is sufficient if the condition $G(+\infty) = +\infty$ is satisfied. This is realized in Theorem 3.4 [5]. Hence Theorem 2.2 generalizes Theorems 3.3 and 3.4 from [5].

Remark. In Theorems 2.1 and 2.2 we have supposed that $x(\cdot)$ is strictly positive on J. Now the ultimately positivity of a solution of (1.3) which starts in a point belonging to the characteristic curve will be obtained by an analytic condition.

2.3 THEOREM. *Let (x, y) be a solution of the system (1.3) defined on the interval J such that:*

(i) *there are satisfied the assumptions (h_1) and (h_2);*
(ii) *$\varphi(x,0) < 0$ for all $x > 0$;*
(iii) *the point $B = (x_0, y_0)$ has the following property: $\varphi(x_0, y_0) = 0$ and $x_0 > 0$;*
(iv) *the function $h(\cdot)$ from (h_2) is a constant denoted by h;*
(v) *for any $x > 0$ we have*

$$\frac{1}{\varphi(x,0)} \int_{0+}^{x} \frac{g(\xi)}{\varphi(\xi,0)}\, d\xi \leq \frac{1}{4h}. \tag{2.2}$$

Then the solution of the system (1.3) which passes through the point B is ultimately positive.

Proof. The vector field shows that, strating from the point B at $t_0 = 0$, the trajectory enters in the region bounded by the characteristic curve, the horizontal axis and the line $x = x_0$. There are two possibilities:

Case (a). The trajectory will lie in this region forever. In this case it will converge to the unique singular point, and the solution is ultimately positive.

Case (b). There exists a $t_\tau > t_0$ such that the trajectory which passes through the point B intersects the positive semiaxis x at the moment t_τ, that is $y_\tau = y(t_\tau) = 0$, $x_\tau = x(t_\tau) > 0$. Hence $y(t) > 0$ for any $t \in [t_0, t_\tau)$.

It will be shown that the assumption that the trajectoria touches the horizontal axis for a finite time leads to a contradiction. Therefore we construct a sequence $(y_n(\cdot))_n$, as in [2], defined on the interval $[x_\tau, x_0]$ by the following reccurence relation

$$y_{n+1}(x) = \int_{x_\tau}^{x} \frac{-g(\xi)}{\varphi(\xi, y_n(\xi))}\, d\xi, \quad y_1(x) = 0, \text{ for } x \in [x_\tau, x_0],\ n \in \mathbb{N}, \tag{2.3}$$

having the properties

$$\begin{cases} 0 \leq y_n(x) < y_{n+1}(x) < -c_n \varphi(x,0), & n \in \mathbb{N}, \\ c_{n+1} = \dfrac{1}{4h}\dfrac{1}{1 - hc_n}, \quad c_1 = \dfrac{1}{4h}, & n \in \mathbb{N}, \\ y_n(x_\tau) = 0, \quad n \geq 2. \end{cases} \tag{2.4}$$

For $n = 1$ from (2.3) we have, using (2.2)

$$\begin{aligned} y_2(x) &= \int_{x_\tau}^{x} \frac{-g(\xi)}{\varphi(\xi, y_1(\xi))}\, d\xi = \int_{x_\tau}^{x} \frac{-g(\xi)}{\varphi(\xi,0)}\, d\xi = \int_{0+}^{x} \frac{-g(\xi)}{\varphi(\xi,0)}\, d\xi - \int_{0+}^{x_\tau} \frac{-g(\xi)}{\varphi(\xi,0)}\, d\xi \\ &< \int_{0+}^{x} \frac{-g(\xi)}{\varphi(\xi,0)}\, d\xi \leq -\frac{1}{4h}\varphi(x,0). \end{aligned}$$

Hence

$$0 = y_1(x) < y_2(x) < -\frac{1}{4h}\varphi(x,0).$$

Let us denote

$$c_1 = \frac{1}{4h}.$$

For $n = 2$, taking into account the following inequalities

$$\varphi(\xi, y_2(\xi)) \le \varphi(\xi, 0) + hy_2(\xi) \le \varphi(\xi, 0) - \frac{1}{4h}h\varphi(\xi,0) = \frac{3}{4}\varphi(\xi,0)$$

we have

$$\begin{aligned} y_3(x) &= \int_{x_r}^{x} \frac{-g(\xi)}{\varphi(\xi, y_2(\xi))}\,d\xi < \int_0^x \frac{-g(\xi)}{\varphi(\xi, y_2(\xi))}\,d\xi \\ &\le \int_{0+}^{x} \frac{-g(\xi)}{\frac{3}{4}\varphi(\xi,0)}\,d\xi \le -\frac{4}{3}\frac{1}{4h}\varphi(x,0) = -\frac{1}{3h}\varphi(x,0). \end{aligned}$$

Let us denote

$$c_2 = \frac{1}{3h}.$$

We suppose that $y_{n+1}(x) < -c_n\varphi(x,0)$ and, using the reccurence relation (2.3), we search a constant c_{n+1} such that $y_{n+2}(x) < -c_{n+1}\varphi(x,0)$.

Since

$$\begin{aligned} y_{n+2}(x) &= \int_{x_r}^{x} \frac{-g(\xi)}{\varphi(\xi, y_{n+1}(\xi))}\,d\xi < \int_0^x \frac{-g(\xi)}{\varphi(\xi, y_{n+1}(\xi))}\,d\xi \\ &\le \int_{0+}^{x} \frac{-g(\xi)}{(1-hc_n)\varphi(\xi,0)}\,d\xi \le -\frac{1}{4h}\frac{1}{1-hc_n}\varphi(\xi,0), \end{aligned}$$

we take

$$c_{n+1} = \frac{1}{4h}\frac{1}{1-hc_n}. \tag{2.5}$$

Thus all the properties from from (2.4) are satisfied. It is obvious that the sequence defined by (2.5) is convergent with limit $\frac{1}{2h}$. The function sequence defined by the relation (2.3) converges uniformly to a continuous function, let it be $\bar{y}(\cdot)$. From (2.4) we have

$$\begin{cases} \bar{y}(x) \le -\dfrac{1}{2h}\varphi(x,0), \\ \bar{y}(x) = \displaystyle\int_{x_r}^{x} \frac{-g(\xi)}{\varphi(\xi, \bar{y}(\xi))}\,d\xi. \end{cases} \tag{2.6}$$

But the last relation in (2.6) shows that the function $\bar{y}(\cdot)$ is the solution of the following Cauchy problem

$$\bar{y}'(x) = \frac{-g(\xi)}{\varphi(\xi, \bar{y}(\xi))}\, d\xi, \quad \bar{y}(x_\tau) = 0. \tag{2.7}$$

The problem (2.7) and the problem considered in the hypothesis have as commun things: the system and a point which belongs to a solution of each of two problems. Hence the two solution coincide on their commun domain of definition. This means that the solution of the problem (2.7) passes through the point B.

From the inequality $\varphi'_y < h$ we have that

$$\varphi(x_0, y_0) - \varphi(x_0, y(x_\tau)) \le h(y_0 - y(x_\tau)).$$

But $y(x_\tau) = 0$ and $\varphi(x_0, y_0) = 0$, it follows that

$$-\varphi(x_0, 0) \le hy_0.$$

Together with (2.6) we have

$$2hy_0 \le -\varphi(x_0, 0) \le hy_0.$$

This is a contradiction. Hence the assumption that the solution of the system (1.3) which passes through the point B touches in a finite time the positive horizontal axis leads to a contradiction, and this proves the Theorem. ■

Remark. If we consider (1.2), then $h = 1$, and we get a result from [2].

Remark. In [11] using a very similar condition to (2.2) it is proved that if

$$\lim_{x \to +\infty} \varphi(x, 0) = +\infty,$$

there is a solution of (1.3) which does not intersect the characteristic curve, remaining in the fourth quadrat, hence it does not oscillate.

Remark. The Theorem 2.2 can be modified in such a way to get ultimately negative solutions, that is solutions which are negative starting at a moment.

In connection with (1.4) some results are introduced in [7]. Here let us suppose that:

(h_3) $g \in C(\mathbb{R}, \mathbb{R})$ and $\exists a \ge 0$ with $xg(x) > 0$, $\forall x$, $|x| \ge a$;

(h_4) $\varphi \in C^1(\mathbb{R}^3, \mathbb{R})$ and $\exists c_5 \ge 0$ with $\varphi(x, y, t) \ge \varphi(x, y, 0) - c_5$, $\forall (x, y, t) \in \mathbb{R}^3$;

(h_5) $\liminf_{x \to +\infty} \varphi(x, 0, 0) < +\infty$ and $\exists K > 0$ with $K < \varphi'_y(x, y, 0)$, $\forall (x, y) \in \mathbb{R}^2$.

2.4. THEOREM. *Consider (1.4) together with the assumptions* (h_3)*-*(h_5)*. If*

$$\limsup_{x\to+\infty}[\Gamma_+(x)-\varphi(x,0,0)]<+\infty, \tag{2.8}$$

then (1.4) has an unbounded solution.

Proof. From the first condition in (h_5) it results that there exists a sequence $(x_n)_{n\in\mathbb{N}}$, $x_n\to+\infty$ with $\lim_{n\to\infty}\varphi(x_n,0,0)=c_1<+\infty$. From (2.8) we have that the sequence

$$\left(\int_0^{x_n}\frac{g(\xi)}{1+\varphi_+(\xi,0,0)}\,d\xi\right)_{n\in\mathbb{N}}$$

is increasing and bounded, and using (h_3) it follows that $\Gamma_+(+\infty)<+\infty$. Let us denote $c_2=\sup_{x\geq 0}\Gamma_+(x)$. From $\Gamma_+(+\infty)<+\infty$ and (2.8) we have $\liminf_{x\to+\infty}\varphi(x,0,0)>-\infty$. Therefore there exists a constant $c_3>0$ such that $\varphi(x,0,0)>-c_3$ for any $x\geq 0$. From the second condition in (h_5) it results that there exists $c_4\geq 0$ which verifies $\varphi(x,c_2+c_3+c_4,0)>\varphi(x,0,0)+c_2+c_3+c_5+1$. Now we define two planar regions:

$$R_1=\{(x,y)\mid x>a,\ y>c_2+c_3+c_4\},$$
$$R_2=\{(x,y)\mid x>a,\ y>c_3+c_4\}$$

and we desire to show that any solution of the system (1.4) which starts in a point of R_1 will remain forever in R_2. Therefore let's consider a point $(x(t_0),y(t_0))\in R_1$ and we suppose there exists a $\tau>0$ such that

$$(x(t),y(t))\in R_2 \text{ for } t\in[t_0,\tau) \text{ and } y(\tau)=c_3+c_4.$$

Then for any $t\in[t_0,\tau)$ we have

$$\begin{aligned}x'(t)&=\varphi(x(t),y(t),t)\geq\varphi(x(t),y(t),0)-c_5\\&\geq\varphi(x(t),0,0)+c_2+c_3+1>1+\varphi_+(x(t),0,0)\end{aligned}$$

and

$$\begin{aligned}y(\tau)&=y(t_0)-\int_{t_0}^{\tau}g(x(t))\,dt>c_2+c_3+c_4-\int_{x(t_0)}^{x(\tau)}\frac{g(\xi)}{1+\varphi_+(\xi,0,0)}\,d\xi\\&=c_2+c_3+c_4-[\Gamma_+(x(\tau))-\Gamma_+(x(t_0))]\geq c_3+c_4,\end{aligned}$$

which contradicts our assumption that $y(\tau)=c_3+c_4$. Therefore the solution lies in the region R_2 forever and the Theorem is proved. ■

Remark. From the proof of the above theorem it is clear that the solutions which start in R_1 are ultimately positive.

Remark. Theorem 2.4 generalizes Theorem 1 in [2], supplying a necessary condition for the boundedness of solutions of (1.4) under (h_3)-(h_5) assumptions. Let us mention that similarly it is possible to state and proof a theorem in the case when $x\to-\infty$.

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