LIBERTAS MATHEMATICA, VOL. XV (1995)

Some existence results for Stokes and Oseen flows past rigid obstacles

Mirela Kohr-Ile

Abstract

In this paper we find some existence and uniqueness results of solutions for the steady flow of a two-dimensional viscous incompressible fluid past rigid obstacles.The Reynolds number of the flow is supposed to be small. The solutions are represented using the simple layer potentials.

1. INTRODUCTION

In some previously papers [5], [6], the author study the problem of a viscous flow past an arbitrary obstacle, using the method of matched inner and outher expansions of the corresponding solution.

Let N be the number of the rigid obstacles denoted by $\Omega_i, i = \overline{1, N}$ and $\Omega = R^2 \setminus \bigcup_{i=1}^{N} \Omega_i$. The fluid flow is characterized by the velocity $\overrightarrow{u}$ and the pressure p.

Using the dimensionless quantities: $x' = \frac{x}{l}$, $\overrightarrow{u'} = \frac{\overrightarrow{u}}{U}$, $p' = \frac{l(p - p_\infty)}{\mu U}$,

$Re = \rho\frac{lU}{\mu}$ the Reynolds number supposed to be small, where l is a characteristic length, μ the dynamic viscosity, ρ the density, U and p_∞ the characteristics of flow at infinity, it result that $\overrightarrow{u'}$ and p' satisfy the Narier-Stokes equations

$$\text{(1)} \qquad \begin{cases} \triangle\overrightarrow{u} - \nabla p = Re(\overrightarrow{u}\cdot\nabla)\overrightarrow{u} \\ \nabla\cdot\overrightarrow{u} = 0 \end{cases} \qquad \text{in } \Omega$$

with the boundary conditions

$$\text{(2)} \qquad \overrightarrow{u} = \overrightarrow{f^i} \quad \text{on} \quad C^i = \partial\Omega_i, \ \ i = \overline{1,N}$$

where C^i is the boundary of Ω_i supposed to be smooth contour, $i = \overline{1,N}$. The symbols $\triangle$ and ∇ mean the Laplace and gradient operator, respectively.

At infinity we have

$$\text{(3)} \qquad \overrightarrow{u} \to \overrightarrow{i}, \ p \to 0, \qquad \text{as } |x| \to \infty$$

where $x = x_1\overrightarrow{i} + x_2\overrightarrow{j}$ in respect with a Cartesian coordinates,

$$|x| = \sqrt{x_1^2 + x_2^2}$$

In the above equations the symbol "$'$" is neglected.

We require that the given velocities $\overrightarrow{f^i}, i = \overline{1,N}$, satisfy the conditions:

(4) $$\int_{C^i} \overrightarrow{f^i} \cdot \overrightarrow{n^i} = 0,$$

where $\overrightarrow{n^i}$ is the unit outward normal vector to $\Omega_i, i = \overline{1,N}$.

Also we suppose that the function $f^i \in C^1(C^i), i = \overline{1,N}$.

$f^i \in C^{1+\gamma_i}(C^i), 0 \leq \gamma_i \leq 1, i = \overline{1,N}$

An important kind of linearization for the equations (1), are the Stokes equations

(5) $$\begin{cases} \triangle \overrightarrow{u} - \nabla p = 0 \\ \nabla \overrightarrow{u} = 0 \end{cases}$$

together with the same conditions as in (2) and (3). For the case $N = 1$, Stokes problem $(5), (2), (3)$ has no solution, in view of the Stokes paradox. But, it is possible to obtain a solution, if the condition at infinity is replaced with:

(6) $$\overrightarrow{u} = \overrightarrow{A} \ln |x| + o(1), \qquad \text{as } |x| \to \infty$$

for any given constant $\overrightarrow{A}$ (see for example [5])

2. INTEGRAL REPRESENTATION OF STOKES SOLUTION

From continuity equation $\nabla \overrightarrow{u} = 0$, there exist a stream function Ψ such that

(7) $$\overrightarrow{u} = (\nabla \Psi)^{\perp}$$

where $\overrightarrow{V^1} = -V_2\overrightarrow{i} + V_1\overrightarrow{j}$, means the vector obtained by rotating the vector $\overrightarrow{V} = V_1\overrightarrow{i} + V_2\overrightarrow{j}$ with $\pi/2$ counterclockwise.

Using (1) and (2) we deduce the Stokes problem for the stream function Ψ :

$$\Delta^2\Psi = 0 \qquad \text{in } \Omega \tag{8}$$

$$\nabla\Psi(x) = \overrightarrow{j} - \overrightarrow{f^i}^{\perp}(x), \qquad x \in C^i, i = \overline{1,N} \tag{9}$$

We show that there exist a real constant vector $\overrightarrow{A}$ such that the problem (8) with the asymptotic conditions at infinity

$$\nabla\Psi(x) = \overrightarrow{A}\ln|x| + O(1), \qquad \text{as } |x| \to \infty \tag{10}$$

has a solution.

For this aim, we represent the stream function Ψ under the form:

$$\Psi(x) = \sum_{i=1}^{N}\int_{C^i}\nabla_y F(x,y)\cdot\overrightarrow{\Phi^i}(y)ds_y^i, \quad x \in \Omega \cup (\bigcup_{i=1}^{N} C^i), \tag{11}$$

where s_y^i is the arc-length along $C^i, i = \overline{1,N}$, and F represents a fundamental solution of biharmonious equation (8) , given by :

$$F(x,y) = \frac{1}{8\pi}|x-y|^2[\ln|x-y| - 1] \tag{12}$$

It is immediately to prove that Ψ given by (11) satisfies the equation (8) and will be a solution of (9) , if the function $\Phi : \bigcup_{i=1}^{N} C^i \to R^2, \quad \Phi(x) = \overrightarrow{\Phi^i}(x)$, for $x \in C^i, i = \overline{1,N}$, satisfy the following integral equation of the first kind :

(13) $$\sum_{i=1}^{N} \int_{C^i} \nabla_x \nabla_y F(x^j, y) \overrightarrow{\Phi^i}(y) ds_y^i = \overrightarrow{g^j}(x^j),\ x^j \in C^j, j = \overline{1, N}$$

where

(14) $$\overrightarrow{g^k} = \overrightarrow{j} - \overrightarrow{f^k}^{\perp},\ k = \overline{1, N}$$

Differentiating (13) with respect to the arc-length $s_x^j, j = \overline{1, N}$, we obtain the integral system with a Cauchy - type singularity.

(15) $$\sum_{i=1}^{N} \int_{C^i} \frac{\partial}{\partial s_x^j} \nabla_x \nabla_y F(x^j, y) \overrightarrow{\Phi}^i(y) ds_y^i = \frac{d}{ds_x^j} \overrightarrow{g}^j(x^j),\ j = \overline{1, N}$$

Because the function F depend only of $|x - y|$, it results that the adjoint homogeneous system of (15), has the form:

(16) $$\sum_{i=1}^{N} \int_{C^i} \frac{\partial}{\partial s_x} \nabla_x \nabla_y F(x, y^j) V^i(x) ds_x = 0,\ y^j \in C^j, j = \overline{1, N}$$

The function $S^i : \bigcup_{j=1}^{N} C^j \to R^2$, given by

(17) $$S^i(x) = \begin{cases} 0, & x \in C^j, j \neq i \\ a^i x + \overrightarrow{b}^i, x \in C^i \end{cases}$$

with a^i, $\overrightarrow{b}^i$ arbitrary constants, are the solution of the system (16) . The dimension of the linear space determined by these functions is $3N$. Hence the dimension of solution's space corresponding to the homogeneous system (15) is at least $3N$, because this system and system (16) have the same number of linearly independent solutions ([7]).

Theorem 1 The homogeneous system (15) has at most $3N$ linearly independent solutions.

Proof First, we observe that the functions $\tau^i : \bigcup_{j=1}^{N} C^j \to R^2$,

$$\tau^i(x) = \begin{cases} 0, & x \in C^j, j \neq i, i = \overline{1,N} \\ \overrightarrow{\tau}^i(x), & x \in C^i \end{cases} \tag{18}$$

where $\overrightarrow{\tau}^i$ denotes the unit tangent vector to C^i, are N linearly independent solutions of the homogeneous system (15) .

Let $\varphi^i : \bigcup_{j=1}^{n} C^j \to R^2$, $i = \overline{1, 2N+1}$ be any $2N+1$ solutions of the homogeneous system (15), and $\Psi^i = \Psi^i(\varphi^i)$, $i = \overline{1, 2N+1}$, the corresponding stream functions, as in (11). Then Ψ^i satisfy:

$$\begin{cases} \Delta^2 \Psi^i = 0 \quad \text{in } \Omega \\ \nabla \Psi^i(x) = \overrightarrow{c}_i^j \quad , j = \overline{1,N} \\ \nabla \Psi^i(x) = \overrightarrow{A}^i \ln |x| + \theta(1) \;, \text{ as } |x| \to \infty \end{cases} \tag{19}$$

where $\overrightarrow{c}_i^j$ is a constant vector,

$\overrightarrow{A}^i = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{C^j} \overrightarrow{\varphi}_j^i(y)\, ds_y^j$, and $\varphi^i(x) = \overrightarrow{\varphi}_j^i(x)$, for $x \in C^j$, $j = \overline{1,N}$.

We can choose the real constants $\alpha_1, ..., \alpha_{2N+1}$, no all equal with zero, and the vector $\overrightarrow{c}(c_1, c_2)$, such that:

$$\begin{cases} \sum_{i=1}^{2N+1} \alpha_i \overrightarrow{c}_i^j - \overrightarrow{c} = 0 \quad , j = \overline{1,N} \\ \sum_{i=1}^{2N+1} \alpha_i \overrightarrow{A}^i = 0 \end{cases} \tag{20}$$

Let the function Φ_0 and Ψ_0 given by:

$$(21) \qquad \Phi_0 = \sum_{i=1}^{2N+1} \alpha_i \overrightarrow{\varphi}^i \quad , \quad \Psi_0 = \sum_{i=1}^{2N+1} \alpha_i \Psi^i,$$

and Ψ_0 satisfy:

$$(22) \qquad \begin{cases} \Delta^2 \Psi_0 = 0 & \text{in } \Omega \\ \nabla \Psi_0 (x) = \overrightarrow{c} & , x \in C^j, \ j = \overline{1, N} \\ \nabla \Psi_0 (x) = \theta (1) & , \text{ as } |x| \to \infty \end{cases}$$

The function $\tilde{\Psi}_0 (x) = c_1 x_1 + c_2 x_2$, where $x = x_1 \overrightarrow{i} + x_2 \overrightarrow{j}$ is a solution for (22). We show that the exterior Stokes problem has a uniqueness solution, so $\tilde{\Psi}_0$ is the uniqueness solution of (22). But the function Ψ_0 given by (21) is also biharmonious in each inner domain Ω_i, and is continuous together with their first derivatives on C^i, $i = \overline{1, N}$. From the uniqueness result of the inner Stokes problem [5], it results that Ψ_0 has linear form in Ω_i, $i = \overline{1, N}$.

Using [4], it is easy to show that on each contour C^j, $j = \overline{1, N}$, the stream function Ψ under the form (10) has the properties:

$$(23) \quad \lim_{x \in \Omega \to x^0 \in C^j} \Delta \Psi (x) - \lim_{x \in \Omega_j \to x^0 \in C^j} \Delta \Psi (x) = \lambda^j \left(\overrightarrow{n}^j \cdot \overrightarrow{\Phi}^j \right) (x^0) , x^0 \in C^j$$

$$(24) \qquad \lim_{x \in \Omega \to x^0 \in C^j} \left(\frac{\partial}{\partial n^j} \Delta \Psi (x) \right) - \lim_{x \in \Omega_j \to x^0 \in C^j} \left(\frac{\partial}{\partial n^j} \Delta \Psi (x) \right) =$$

$$= \lambda^j \frac{d}{ds^j} \left(\overrightarrow{\tau}^j \cdot \overrightarrow{\Phi}^j \right) (x^0),$$

where $\dfrac{\partial}{\partial n^j}$ is the normal derivative on $C^j, j = \overline{1, N}$.

Since Ψ_0 has a linear form in Ω and Ω_j respectively, from (23) and (24), there exists the constant $\gamma^j \in R$, such that

$$(25) \qquad \overrightarrow{\Phi_0}^j (x) = \gamma^j \overrightarrow{\tau}^j (x), \qquad x \in C^i$$

with $\Phi_0(x) = \overrightarrow{\Phi_0}^j(x)$, for $x \in C^j, j = \overline{1,N}$.

From the above arguments we obtain that:

$$\sum_{i=1}^{2N+1} \alpha_i \varphi^i(x) - \sum_{j=1}^{N} \gamma^j \tau^j(x) = 0, \qquad x \in \bigcup_{j=1} C^j, \tag{26}$$

so the functions $\varphi^i, i = \overline{1, 2N+1}$ and $\tau^j, j = \overline{1,N}$, are linearly dependent.

We proved that the dimension of solutions space for the adjoint system (16) is exactly $3N$, and each solution V has the linear form:

$$V(x) = \alpha_i x + \overrightarrow{\beta_i}, \qquad x \in C^i, \quad i = \overline{1,N}. \tag{27}$$

where α_i, $\overrightarrow{\beta_i}$ are constants.

Using the Fredholm alternative [7] , the integral system (15) has solutions if and only if

$$\sum_{i=1}^{N} \int_{C^i} \frac{d}{ds_x^i} \overrightarrow{g}^i(x) \cdot \overrightarrow{S}^i(x)\, ds_x^i = 0, \tag{28}$$

where $S : \bigcup_{j=1}^{N} C^i \rightarrow R^2, S(x) = \overrightarrow{S}^j(x), x \in C^j$, is a solution of (16) . But, from (27) , (14) and (4) the condition (28) is clear.

Now let Φ^0 be a solution of system (15), with $\Phi^0(x) = \overrightarrow{\Phi_j}^0(x)$, $x \in C^i, j = \overline{1,N}$. The stream function $\Psi^0 = \Psi^0(\Phi^0)$ satisfy

$$\begin{cases} \Delta^2 \Psi^0(x) = 0 \qquad \text{in } \Omega \\ \nabla \Psi^0(x) = \overrightarrow{g}^i(x) + \overrightarrow{k}^i, \quad x \in C^i, i = \overline{1,N} \\ \nabla \Psi^0(x) = \overrightarrow{A^0} \ln|x| + O(1), \quad \text{as } |x| \rightarrow \infty \end{cases} \tag{29}$$

where $\overrightarrow{k}^i$ are the constants vectors, $i = \overline{1,N}$ and

$$\overrightarrow{A}^0 = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{C^j} \overrightarrow{\Phi_j}^0(x)\, ds_x^j.$$

We define the function $k^0 : \bigcup_{j=1}^{N} C^j \to R^2$, by

$$k^0(x) = \overrightarrow{k^i}, \text{ for } x \in C^i, i = \overline{1,N}.$$

Also let $\Phi^i, i = \overline{1,2N}$ and $\tau^j, j = \overline{1,N}$, be the $3N$ linearly independent solutions of the homogeneous system (15). Then the stream function $\Psi^i = \Psi^i(\Phi^i)$ satisfy:

$$\text{(30)} \qquad \begin{cases} \Delta^2 \Psi^i = 0 \qquad \text{in } \Omega \\ \nabla \Psi^i(x) = \overrightarrow{k}_j^i, \quad x \in C^j, j = \overline{1,N}, i = \overline{1,2N} \\ \nabla \Psi^i(x) = \overrightarrow{A^i} \ln|x| + O(1), \quad \text{as } |x| \to \infty \end{cases}$$

with $\overrightarrow{A}^i = \frac{1}{4\pi} \sum_{j=1}^{N} \int_{C^j} \overrightarrow{\varphi_j}^i(x)\, ds_x^j$, $\quad \Phi^i(x) = \overrightarrow{\varphi_j}^i(x)$, $\quad x \in C^j$, and $\overrightarrow{k_j}^i$, $j = \overline{1,N}$ are the constant vectors.

There exist the real numbers $\alpha_1, ..., \alpha_{2N}$ such that

$$\text{(31)} \qquad \sum_{i=1}^{2N} \alpha_i k^i(x) + k^0(x) = 0, \quad x \in \bigcup_{j=1}^{N} C^i,$$

where $k^i(x) = \overrightarrow{k_j}^i$, for $x \in C^j, j = \overline{1,N}$, because the set

$$A = \left\{ \bar{k} : \bigcup_{j=1}^{N} C^j \to R^2 \,\middle|\, \bar{k}(x) = \overrightarrow{k}^j\,,\ x \in C^j,\ \overrightarrow{k}^j \text{a constant vector}, j = \overline{1,N} \right\}$$

is a $2N$ dimensional linear space and the function $k^0, k^i, i = \overline{1,2N}$, belong to A, in the next hypotesys:

The linear independent solutions Φ^i. $i = \overline{1,2N}$ satisfy the conditions:

(32) $\overrightarrow{A}^i = 0,\ i = \overline{1, 2N}$

Then it is easy to prove that the functions k^i, $i = \overline{1, 2N}$ are linearly independent, so we have (31).

From (29) – (31), the function $\Psi = \Psi^0 + \sum_{i=1}^{2N} \alpha_i \Psi^i$ is a solution of the Stokes problem (8) – (9). At infinity Ψ has the form:

(33) $\Psi(x) = \overrightarrow{A}^0 \ln|x| + O(1)$, as $|x| \to \infty$,

Hence, we obtain the following result:

<u>Theorem 2</u> Under the assumption (4) and hypotesis (32), there exist an constant vector $\overrightarrow{A}$ such that the problem (8) – (9) with the condition (33) at infinity, has a solution.

In the proof of Theorem 1, we use the uniqueness result of solution for the exterior Stokes problem. This is given by:

<u>Theorem 3</u> The Stokes problem (8) – (9) has at most one solution (up to an additive constant), if we require:

(34) $\Psi(x) = O\left(|x|^{-1}\right), D^m\Psi(x) = O\left(|x|^{-2}\right)$, as $|x| \to \infty, m \geq 1$

and

(35) $\int_{C^i} \frac{\partial \omega}{\partial n}(x)\, ds_x^i = 0, \quad i = \overline{1, N}$

where $\omega = \Delta\Psi$.

<u>Proof</u> If we suppose that there exist two solutions Ψ_1 and Ψ_2 of the problem (9) , then the function $\Psi = \Psi_1 - \Psi_2$ satisfy:

$$\Delta^2\Psi = 0 \qquad \text{in } \Omega$$

$$\nabla\Psi(x) = 0, \quad x \in C^j, j = \overline{1, N} \tag{36}$$

and the conditions (34) , (35) .

Applying the Green's formula in the domain $\Omega_R = \Omega \cap B_R$ where B_R is a large disk of radius R, and using the above conditions, we obtain

$$\int_{\Omega_R} \left[\Psi(x)\Delta^2\Psi(x) - (\Delta\Psi(x))^2\right] dx = \sum_{i=1}^{N} \int_{C^i} [\Psi(x)\frac{\partial\omega}{\partial n}(x) - \tag{37}$$

$$-\omega(x)\frac{\partial\Psi}{\partial n}]ds_x^i + \int_{\partial B_R} \left[\Psi(x)\frac{\partial\omega}{\partial n}(x) - \omega(x)\frac{\partial\Psi}{\partial n}\right] ds_x = 0, \quad \text{for } R \to \infty$$

Hence $\Delta\Psi = 0$ in Ω.

Using again the Green's formula, we have:

$$\int_{\Omega_R} \Psi(x)\Delta\Psi(x)\,dx + \int_{\Omega_R} (\nabla\Psi(x))^2\,dx = \tag{38}$$

$$= \int_{\partial B_R} \Psi(x)\frac{\partial\Psi}{\partial n}(x)\,ds_x + \sum_{j=1}^{N} \int_{C^j} \Psi(x)\frac{\partial\Psi}{\partial n}(x)\,ds_x^i$$

and right side of (38) become equal with zero for $R \to \infty$. So $\nabla\Psi = 0$ in Ω, and hence $\Psi_1 = \Psi_2$ (up to an additive constant).

<u>Observation</u> Because we determine the stream function Ψ under the form (11), the additionally conditions (35) can be easy obtained as a consequence of Green's formula.

With the stream function we obtain the velocity $\overrightarrow{u} = (\nabla\Psi)^{\perp}$, and the pressure p which is locally harmonious conjugate with $\omega = \Delta\Psi$, because the domain Ω is not simply connected.

3. INTEGRAL REPRESENTATION OF OSEEN SOLUTION

An other important kind of linearization for the Narier-Stokes equations (1), which take account of the behavior of the flow at infinity, is obtained considering the representation

$$\overrightarrow{u} = \overrightarrow{i} + \overrightarrow{U}, \qquad p = p_\infty + P \tag{39}$$

and neglecting the nonlinear inertia term in (1). The resulting equation are called the Ossen equation:

$$\begin{cases} \Delta\overrightarrow{U} - Re\left(\overrightarrow{i}\nabla\right)\overrightarrow{U} - \nabla P = 0, & \text{in } \Omega \\ \nabla\cdot\overrightarrow{U} = 0, & \text{in } \Omega \end{cases} \tag{40}$$

with the boundary conditions and at infinity:

$$\begin{cases} \overrightarrow{U} = -\overrightarrow{i} + \overrightarrow{f}^{j} \quad \text{on } C^j, j = \overline{1,N} \\ \overrightarrow{U} \to 0, \quad P \to 0, \quad \text{as } |x| \to \infty \end{cases} \tag{41}$$

Also using the stream function Ψ, the above problem is reduced to:

$$\begin{cases} \Delta^2\Psi - Re\Delta\Psi_{x_1} = 0 \quad \text{in } \Omega \\ \nabla\Psi = \overrightarrow{j} - \overrightarrow{f}^{k\perp} \quad \text{on } C^k, k = \overline{1,N} \\ \nabla\Psi(x) \to 0, \qquad \text{as } |x| \to \infty \end{cases} \tag{42}$$

where Ψ_{x_1} denote the partial derivative in respect with x_1.
We consider the function Ψ under the form:

(43) $$\Psi(x) = \sum_{i=1}^{N} \int_{C^i} \nabla_y G(x,y) \cdot \overrightarrow{\varphi}^i(x)\, ds_y^i, x \in \Omega \cup \left(\bigcup_{i=1}^{N} C^i \right),$$

where G is the fundamental solution of the first equation (42), given by:

(44) $$G(x,y) = F(x,y) + |x-y|^2 \ln Re - (x_2 - y_2)^2 + H(x,y),$$

H has continuous third derivatives and $H = O(Re \ln Re)$ as $Re \to \infty$, uniformly for x and y in the compact sets.

The density function $\Phi : \bigcup_{i=1}^{N} C^i \to R^2, \Phi(x) = \overrightarrow{\varphi}^i(x)$ for $x \in C^i$, $i = \overline{1,N}$, will be solution for following integral system:

(45) $$\sum_{i=1}^{N} \int_{C^i} \nabla_x \nabla_y G\left(x^k, y\right) \overrightarrow{\varphi}^i(y)\, ds_y^i = \overrightarrow{g}^k\left(x^k\right), x \in C^k, k = \overline{1,N}$$

where the functions $\overrightarrow{g}^k, k = \overline{1,N}$ are given in (14).

Analogously with the previous section, we can prove the following existence result:

<u>Theorem 4</u> For any functions $\overrightarrow{f}^i$ which satisfy the conditions (4) , the integral equation system (45) has an uniqueness solution (up to an additive constant).

<u>Observation</u> The proof of this theorem needs the uniqueness results for the exterior as well as the interior Oseen problem. The uniqueness result follow from the conditions at infinity (suggested by the asymptotic behavior of G):

(46) $$\Psi(x) = O(\ln|x|), \quad D^m\Psi = O\left(|x|^{-m/2}\right), m \geq 1, \quad \chi = O\left(|x|^{-1}\right),$$

as $|x| \to \infty$

where $\chi = \Delta\Psi - Re\Psi_{x_1}$, and supposing that the function χ satisfy:

$$\int_{C^i} \frac{\partial \chi}{\partial n} ds = 0, \quad i = \overline{1, N} \tag{47}$$

Because for the function Ψ we use the representation (43) the conditions (46) and (47) are immediately, as a consequence of Green formula.

Using the stream function, above determined, we obtain the velocity $\vec{U} = (\nabla\Psi)^{\perp}$, and the pressure P being harmonically conjugate of χ, but only locally because the domain Ω is not simply connected.

REFERENCES

1. R.C. Mac Camy, On a class of two - dimensional Stokes flows, Arch. Rat. Mech. Anal, 21(1966), $246-258$.

2. L. Dragos, A. Postelnicu, Metoda solutiilor fundamentale aplicata miscarii fluidelor vascoase incompresibile in prezenta profilului subtire, Stud. Cerc. Mat. 46(1994), $1, 27-34$.

3. L. Dragos, The principles of continous mechanics media, Edit. Tehnica, Bucuresti, 1983 (in Romanian).

4. G.C. Hsiao, R.C. Mac Camy, Solution of boundary value problems by integral equations of the first kind, SIAM,15(1973), $4, 687-705$.

5. G.C. Hsiao, Integral representation of solutions for two - dimensional viscous flow problems, Integral Equations and Operator Theory, 5(1982), 533--547.

6. G.C. Hsiao, R.C. Mac Camy, Singular perturbations for the two - dimensional viscous flow problem, Lecture Notes in Math. Springer - Verlag, Berlin,942(1982), $229-244$.

7. V. Mikhailov, Equations aux derivees partielles, Ed. Mir Moscow, 1980