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EXISTENCE AND UNIQUENESS OF MILD AND STRONG SOLUTIONS OF AN INTEGRODIFFERENTIAL EQUATION WITH NONLOCAL CONDITION

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Abstract: We prove the existence and uniqueness of mild and strong solutions of an integrodifferential equation with nonlocal initial conditions using the method of semigroups and the Banach fixed point theorem.

1. INTRODUCTION

Using the method of semigroups and the Banach fixed point theorem, Byszewski [2] studied about the existence and uniqueness of mild, strong and classical solutions of the following nonlocal Cauchy problem:

$$\frac{du(t)}{dt} + Au(t) = f(t,u(t)), \quad t\in(t_o,t_o+a]$$

$$u(t_o) + g(t_1,,\ldots,t_p,u(.)) = u_o,$$

where $-A$ is the infinitesimal generator of a C_o semigroup $T(t)$, $t\geq 0$, on a Banach space X, $0\leq t_o<t_1<\ldots<t_p\leq t_o+a$, $a>0$, $u_o\in X$ and $f:[t_o,t_o+a]xX\longrightarrow X$, $g:[t_o,t_o+a]^p xX\longrightarrow X$ are given functions. Balachandran and Ilamaran [1] proved the existence and uniqueness of mild and strong solutions of the problem

$$\frac{du(t)}{dt} + Au(t) = f(t,u(\sigma(t))), \quad t\in(t_o,t_o+a]$$

$$u(t_o) + g(t_1,,\ldots,t_p,u(.)) = u_o.$$

Moreover, Corduneanu [3] and Gripenberg [4] studied the existence of solutions of Volterra integral equations of various types using semigroups approach.

The aim of this paper is to prove two theorems about the existence and uniqueness of the mild and strong solutions of an integrodifferential equation with nonlocal initial condition.

2. PRELIMINARIES

Consider the initial value problem

$$\frac{du(t)}{dt} = Au(t) + f(t), \quad t\in(t_o,t_o+a], \qquad ---(1)$$

$$u(t_o) = u_o, \qquad ---(2)$$

where $f:[t_o,t_o+a] \longrightarrow X$, A is the infinitesimal generator of a C_o semigroup $T(t)$, $t\geq 0$, $u_o\in X$ and $t_o \geq 0$.

Throughout this paper we use the notation $I := [t_o,t_o+a]$.

Definition 1. A function u is said to be a strong solution of problem (1), (2) on I if u is differentiable almost everywhere on I, $\frac{du}{dt} \in L'((t_o,t_o+a],X)$, $u(t_o) = u_o$ and

$$\frac{du(t)}{dt} = Au(t) + f(t) \text{ a.e. on I.}$$

Theorem 1 [5]. If X is a reflexive Banach space, $u_o\in D(A)$ and f is Lipschitz continuous on I then problem (1), (2) has a unique strong solution u on I given by the formula

$$u(t) = T(t-t_o)u_o + \int_{t_o}^{t} T(t-s)f(s)ds, \qquad t\in I.$$

Consider the following integrodifferential equation

$$\frac{du(t)}{dt}+Au(t) = f(t,u(t))+\int_{t_o}^{t} K(t,s,u(s))ds, \quad t\in(t_o,t_o+a] \qquad ---(3)$$

with the nonlocal condition

$$u(t_o) + g(t_1,\dots,t_p,u(.)) = u_o, \qquad ---(4)$$

where $0\le t_o < t_1 < \dots < t_p \le t_o+a$, $-A$ is the infinitesimal generator of a C_o semigroup $T(t)$, $t\ge 0$, on a Banach space X, and $f:IxX\longrightarrow X$, $g(t_1,\dots,t_p,\ .):X\longrightarrow X$, $K:\triangle xX\longrightarrow X$ where $\triangle=\{(t,s): t_o\le s\le t\le t_o+a\}$. The symbol $g(t_1,\dots,t_p,u(.))$ is used in the sense that in the place of . we can substitute only elements of the set $\{t_1, \dots ,t_p\}$.

Definition 2. A continuous solution u of the integral equation

$$u(t) = T(t-t_o)u_o - T(t-t_o)g(t_1,\dots,t_p,u(.)) + \int_{t_o}^{t} T(t-s)f(s,u(s))ds+\int_{t_o}^{t} T(t-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds, \quad t\in I,$$

is said to be a mild solution of problem (3),(4) on I.

3. Existence of a mild solution

Theorem 2. Assume that

(i) X is a Banach space with norm $\|\cdot\|$ and $u_o \in X$.

(ii) $0 \le t_o < t_1 < \dots < t_p \le t_o+a$ and $B_r := \{v: \|v\|\le r\} \subset X$.

(iii) $f:IxX \longrightarrow X$ is continuous in t on I and there exists a constant $L > 0$ such that

$$\|f(s,v_1) - f(s,v_2)\| \le L\|v_1 - v_2\| \text{ for } s \in I, \quad v_1,v_2 \in B_r.$$

(iv) $K: \triangle xX\longrightarrow X$ is continuous and there exists a constant $K_o>0$ such that

$$\|K(t,s,x(s)) - K(t,s,y(s))\| \le K_o\|x(s) - y(s)\|$$

(v) $g: I^p x X \longrightarrow X$ and there exists a constant $G_o > 0$ such that

$$\|g(t_1,\dots,t_p,u_1(\cdot)) - g(t_1,\dots,t_p,u_2(\cdot))\| \le G_o \sup_{t\in I} \|u_1(t) - u_2(t)\| \quad \text{for } u_1, u_2 \in C(I,B_r).$$

(vi) $-A$ is the infinitesimal generator of a C_o semigroup $T(t)$, $t \ge 0$, on X

(vii) $M = \max_{t\in[0,a]}\|T(t)\|$, $\qquad N = \max_{s\in I} \|f(s,0)\|$,

$K_1 = \max_{t_o\le s\le t\le t_o+a}\|K(t,s,0)\|$, $\qquad G_1 = \max_{u\in C(I,B_r)}\|g(t_1,\dots,t_p,u(.))\|$.

(viii) $M(\|u_o\| + G_1 + raL + aN + K_o ra^2 + K_1 a^2) \le r$, and

$$MK + MLa + MK_o a^2 < 1.$$

Then problem (3),(4) has a unique mild solution on I.

Proof:

Take $E := C(I,B_r)$ and define an operator F on E by

$$(Fv)(t) = T(t-t_o)u_o - T(t-t_o)g(t_1,\dots,t_p,v(.)) + \int_{t_o}^{t} T(t-s)f(s,v(s))ds + \int_{t_o}^{t} T(t-s)\int_{t_o}^{s} K(s,\tau,v(\tau))d\tau ds, \quad t\in I,$$

From our assumption, we have

$$\|(Fv)(t)\| = \|T(t-t_o)u_o\| + \|T(t-t_o)g(t_1,\dots,t_p,v(.))\| + \left\|\int_{t_o}^{t} T(t-s)f(s,v(s))ds\right\| + \left\|\int_{t_o}^{t} T(t-s)\int_{t_o}^{s} K(s,\tau,v(\tau))d\tau ds\right\|,$$

$$\le M\|u_o\| + MG_1 + M\int_{t_o}^{t} (\|f(s,v(s)) - f(s,0)\| + \|f(s,0)\|)ds$$

$$+M\int_{t_o}^{t}\int_{t_o}^{s}[\|K(s,\tau,v(\tau)) - K(s,\tau,0)\| + \|K(s,\tau,0)\|]d\tau ds$$

$$\le M\|u_o\| + MG_1 + M\int_{t_o}^{t} (L\|v(s)\| + N)ds$$

$$+M\int_{t_o}^{t}\int_{t_o}^{s}[K_o\|v(\tau)\| + K_1]d\tau ds$$

$$\leq M(\|u_o\| + G_1 + raL + aN + K_o ra^2 + K_1 a^2) \leq r \quad \text{for } v \in E.$$

Therefore, $FE \subset E$.

Now, for every $v_1, v_2 \in E$ and $t \in I$, we have

$$\|(Fv_1)(t)-(Fv_2)(t)\| \leq \int_{t_o}^{t} \|T(t-s)\| \|f(s,v_1(s)) - f(s,v_2(s))\| ds$$

$$+ \|T(t-t_o)\| \|g(t_1,\ldots,t_p,v_1(.)) - g(t_1,\ldots,t_p,v_2(.))\|$$

$$+\int_{t_o}^{t} \|T(t-s)\| \int_{t_o}^{s} \|[K(s,\tau,v_1(\tau)) - K(s,\tau,v_2(\tau))]\| d\tau ds$$

$$\leq ML\int_{t_o}^{t} \|v_1(s) - v_2(s)\| ds + MG_o \sup_{t \in I} \|v_1(t) - v_2(t)\|$$

$$+ MK_o a^2 \sup_{t \in I} \|v_1(t) - v_2(t)\|$$

$$\leq (MLa + MG_o + MK_o a^2) \sup_{t \in I} \|v_1(t) - v_2(t)\|$$

If we take $q := MLa + MG_o + MK_o a^2$ then

$$\sup_{t \in I} \|(Fv_1)(t) - (Fv_2)(t)\| \leq q \sup_{t \in I} \|v_1(t) - v_2(t)\|$$

with $0 < q < 1$.

This shows that operator F is a contraction on the complete metric space E. Applying Banach fixed point theorem we get a unique fixed point for F in space E and this point is the mild solution of problem (3), (4) on I.

4. Existence of a strong solution

Definition 3. A function u is said to be a strong solution of problem (3), (4) on I if u is differentiable a.e. on I,

$$\frac{du}{dt} \in L'((t_o,t_o+a],\ X)$$

$$u(t_o) + g(t_1,\ldots,t_p,u(.)) = u_o$$

and $$\frac{du(t)}{dt} + Au(t) = f(t,u(t)) + \int_{t_o}^{t} K(t,s,u(s))ds, \quad t\in(t_o,t_o+a]$$

Theorem 3. Assume that

(i) X is a reflexive Banach space with norm $\|.\|$ and $u_o \in X$.

(ii) $0 \le t_o < t_1 < \ldots < t_p \le t_o+a$ and $B_r := \{v: \|v\| \le r\} \subset X$.

(iii) $f: I\times X \longrightarrow X$ is continuous in t on I and there exists a constant $L > 0$ such that

$$\|f(s_1,v_1) - f(s_2,v_2)\| \le L(\|s_1 - s_2\| + \|v_1 - v_2\|)$$

for $s_1,s_2 \in I$, $v_1,v_2 \in B_r$.

(iv) $K: \triangle \times X \longrightarrow X$ is continuous and there exists a constant $K_o>0$ such that

$$\|K(t_1,s,x(s)) - K(t_2,s,y(s))\| \le K_o(|t_1 - t_2| + \|x(s) - y(s)\|)$$

(v) $g: I^p \times X \longrightarrow X$ and there exists a constant $G_o > 0$ such that

$$\|g(t_1,\ldots,t_p,u_1(\cdot)) - g(t_1,\ldots,t_p,u_2(\cdot))\| \le G_o \sup_{t\in I} \|u_1(t) - u_2(t)\| \quad \text{for } u_1,\ u_2 \in C(I,B_r),$$

and $g(t_1,\ldots,t_p,\ .\ ) \in D(A)$.

(vi) $-A$ is the infinitesimal generator of a C_o semigroup $T(t)$, $t \ge 0$, on X.

(vii) $u_o \in D(A)$.

(viii) $M = \max_{t\in[0,a]} \|T(t)\|$, $\quad N = \max_{s\in I} \|f(s,0)\|$,

$K_1 = \max_{t_o \le s \le t \le t_o+a} \|K(t,s,0)\|$, $\quad G_1 = \max_{u\in C(I,B_r)} \|g(t_1,\ldots,t_p,u(.))\|$.

(ix) $M(\|u_o\| + G_1 + raL + aN + K_o ra^2 + K_1 a^2) \le r$, and

$$MG_o + MLa + MK_oa^2 < 1.$$

Then the problem (3), (4) has a strong solution on I.

Proof:

Since all the assumptions of Theorem 2 are satisfied then problem (3), (4) possesses a unique mild solution belonging to C(I;X) which we denote it by u.

Now, we shall show that this mild solution is a strong solution of problem (3),(4) on I.

For any $t \in I$, we have

$$\begin{aligned}
u(t+h) - u(t) &= [T(t+h-t_o)u_o - T(t-t_o)u_o] \\
&\quad - [T(t+h-t_o)g(t_1,\ldots,t_p,u(.)) - T(t-t_o)g(t_1,\ldots,t_p,u(.))] \\
&\quad + \int_{t_o}^{t_o+h} T(t+h-s)f(s,u(s))ds + \int_{t_o+h}^{t+h} T(t+h-s)f(s,u(s))ds \\
&\quad - \int_{t_o}^{t} T(t-s)f(s,u(s))ds + \int_{t_o}^{t_o+h} T(t+h-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds \\
&\quad + \int_{t_o+h}^{t+h} T(t+h-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds - \int_{t_o}^{t} T(t-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds \\
&= [T(t+h-t_o)u_o - T(t-t_o)u_o] \\
&\quad - [T(t+h-t_o)g(t_1,\ldots,t_p,u(.)) - T(t-t_o)g(t_1,\ldots,t_p,u(.))] \\
&\quad + \int_{t_o}^{t_o+h} T(t+h-s)f(s,u(s))ds \\
&\quad\quad + \int_{t_o}^{t} T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds \\
&\quad + \int_{t_o}^{t_o+h} T(t+h-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds \\
&\quad + \int_{t_o}^{t} T(t-s)\int_{t_o}^{s+h} K(s+h,\tau,u(\tau))d\tau ds - \int_{t_o}^{t} T(t-s)\int_{t_o}^{s} K(s,\tau,u(\tau))d\tau ds
\end{aligned}$$

$$= T(t-t_o)[T(h) - I]u_o - T(t-t_o)[T(h) - I]g(t_1,\ldots,t_p,u(.))]$$

$$+ \int_{t_o}^{t_o+h} T(t+h-s)[f(s,u(s)) - f(s,0)]ds + \int_{t_o}^{t_o+h} T(t+h-s)f(s,0)ds$$

$$+ \int_{t_o}^{t} T(t-s)[f(s+h,u(s+h)) - f(s,u(s))]ds$$

$$+ \int_{t_o}^{t_o+h} T(t+h-s)\int_{t_o}^{s} [K(s,\tau,u(\tau)) - K(s,\tau,0)]d\tau ds$$

$$+ \int_{t_o}^{t_o+h} T(t+h-s)\int_{t_o}^{s} K(s,\tau,0)d\tau ds$$

$$+ \int_{t_o}^{t} T(t-s)\int_{t_o}^{s} [K(s+h,\tau,u(\tau)) - K(s,\tau,u(\tau))]d\tau ds$$

$$+ \int_{t_o}^{t} T(t-s)\int_{s}^{s+h} [K(s+h,\tau,u(\tau)) - K(s+h,\tau,0)]d\tau ds$$

$$+ \int_{t_o}^{t} T(t-s)\int_{s}^{s+h} K(s+h,\tau,0)d\tau ds$$

Using our assumptions we observe that

$$\begin{aligned}\|u(t+h) - u(t)\| &\le hM\|Au_o\| + hM\|Ag(t_1,\ldots,t_p,u(.))\| + hMLr + MNh \\ &\quad + MLah + ML\int_{t_o}^{t}\|u(s+h) - u(s)\|ds \\ &\quad + MK_orah + MK_1ah + MK_oha^2 + MK_orah + MK_1ah \\ &\le Ph + ML\int_{t_o}^{t}\|u(s+h) - u(s)\|ds,\end{aligned}$$

where $P := M\|Au_o\| + M\|Ag(t_1,\ldots,t_p,u(.))\| + MLr + MN + MLa$

$$+ MK_ora + MK_1a + MK_oa^2 + MK_ora + MK_1a.$$

Using Gronwall's inequality, we get

$$\|u(t+h) - u(t)\| \le Phe^{MLa} \quad \text{for} \quad t\in I.$$

Therefore, u is Lipschitz continuous on I.

The Lipschitz continuity of u on I combined with (iii) give that $t \longrightarrow f(t,u(t))$ is Lipschitz continuous on I. Also by assumption (iv), $t \longrightarrow \int_{t_0}^{t} K(t,s,u(s))ds$ is Lipschitz continuous on I. Using Theorem 1 we observe that the equation

$$\frac{dv(t)}{dt} + Av(t) = f(t,u(t)) + \int_{t_0}^{t} K(t,s,u(s))ds, \quad t \in (t_0, t_0+a]$$

$$v(t_0) = u_0 - g(t_1,\dots,t_p,u(.))$$

has a unique strong solution v on I satisfying the equation

$$v(t) = T(t-t_0)u_0 - T(t-t_0)g(t_1,\dots,t_p,u(.)) + \int_{t_0}^{t} T(t-s)f(s,u(s))ds + \int_{t_0}^{t} T(t-s)\int_{t_0}^{s} K(s,\tau,u(\tau))d\tau ds,$$

$$= u(t), \quad t \in I.$$

Consequently, u is a strong solution of problem (3), (4) on I.

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