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Dedicated to my master
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CALCULUS ON NON ORIENTABLE RIEMANN SURFACES
(CALCULUS ON N.R.S.)

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ABSTRACT

We show that calculus on nonorientable manifolds can be canonically reduced to calculus on orientable manifolds. We obtain all functions and differential forms on a nonorientable Riemann surface $\mathbf{X}$ in terms of the corresponding functions and forms on its unbranched, two-sheeted orientable covering $(\mathcal{O}_2, k)$. We prove Hodge's theorem concerning the harmonic differentials on a compact nonorientable Riemann surface,and show that a special type of de Rham cohomology appears on $(\mathcal{O}_2, k)$. We also compute the dimension of the de Rham's cohomology vector space of a compact nonorientable Riemann surface in function of the topological genus of the surface (*de Rham Theorem). We finally prove a Stokes theorem for chains of singular squares on nonorientable Riemann surfaces.*

Introduction

According to a result due to Klein (see Theorem 1), the existence of the nonorientable Riemann surface (N.R.S.) $\mathbf{X}$ *is equivalent with the existence of the symmetric Riemann surface* $(\mathcal{O}_2, k)$, *which is the unbranched two-sheeted orientable covering of* $\mathbf{X}$. *Thus,functions,differential forms,integration theory,and,in general, every concept defined on* $\mathbf{X}$ *is dependent on* $(\mathcal{O}_2, k)$. *An object or a concept occuring when* $(\mathcal{O}_2, k)$ *doesn't exist,or that persists even if* $(\mathcal{O}_2, k)$ *vanishes doesn't make much sense for* $\mathbf{X}$. *For example,even infinitesimal motions of the point representing* $(\mathcal{O}_2, k)$ *in its Teichmüller space,give rise to non-symmetric Riemann surfaces. These motions imply the disappearence of* $\mathbf{X}$ *and thus they must imply the disappearence of the objects and concepts on* $\mathbf{X}$.
The model of calculus proposed in this paper respects strictly the connection between the N.R.S. $\mathbf{X}$ *and its orientable covering* $(\mathcal{O}_2, k)$.

A century and a half of multivariable calculus,under the influence of physical and mechanical phenomena involving Green's theorem, Stokes's theorem and the divergence theorem of Gauss, has crystallyzed in E. Cartan's model of calculus on manifolds based on functions and differential forms subdue to the computation rules of the exterior differential calculus.
This paper extends naturally Cartan's model of calculus to the case of nonorientable manifolds.

In Section 1 we define the symmetric functions on $(\mathcal{O}_2, k)$ *and see how these functions represent the functions on* $\mathbf{X}$.

In Section 2 we introduce the symmetrization and the antisymmetrization operators which allow us to see the position of the algebra of symmetric functions inside the algebra of all complex functions on $(\mathcal{O}_2, k)$.

In Section 3 we give nontrivial examples of symmetric functions on $(\mathcal{O}_2, k)$, *namely the functions that appear by symmetrizing either the meromorphic or the antimeromorphic functions on* $(\mathcal{O}_2, k)$. *These functions sugest the definition of the N-meromorphic functions on* $\mathbf{X}$.

In Sections 4 and 5 we define the symmetric and the antisymmetric differential forms of degree 1 on $(\mathcal{O}_2, k)$.

In Section 6 we see how the space $\mathcal{D}^1_s(\mathcal{O}_2)$ *of symmetric differential forms of degree 1 on* $\mathcal{O}_2$ *can be identified with the space* $\mathcal{D}^1(\mathbf{X})$ *of all differential forms of degree 1 on* $\mathbf{X}$.

In Section 7 we define the symmetrization and the antisymmetrization operators on the space $\mathcal{D}^1(\mathcal{O}_2)$ *of all differential forms of degree 1 on* $(\mathcal{O}_2, k)$.

In Section 8 we introduce the line integral on $\mathbf{X}$ *by means of the line integral on* $(\mathcal{O}_2, k)$.

In Section 9 (as in Section 3) we give examples of symmetric differential forms, namely those which appear by symmetrising either the meromorphic or the antimeromorphic differentials (the Abelian differentials) on $(\mathcal{O}_2, k)$.

Section 10 deals with the definitions of the symmetric and the antisymmetric differential forms of degree 2 on $(\mathcal{O}_2, k)$. *The symmetrization and the antisimmetization operators are defined and their principal properties are presented.*

In Section 11 we see how the symmetric differential forms of degree 2 on $(\mathcal{O}_2, k)$ *represent the differential forms of degree 2 on* $\mathbf{X}$.

We further offer concrete examples of functions and differential forms on N.R.S. We give the complete lists of these objects in the case of the Möbius band.

As in Section 8,in Section 12 we define the double integral on $\mathbf{X}$.

In Section 13 we define the differential operators d_s *and* d_a *which give the canonical decomposition*

$$d = d_s + d_a$$

of d, *the usual operator of (exterior) differentiation on* $(\mathcal{O}_2, k)$.

In Section 14 we deal with the study of the symmetric and antisymmetric harmonic differentials on $(\mathcal{O}_2, k)$. *The main result is the Hodge Theorem on compact N.R.S. (Theorem 20).*

In Section 15 we study the de Rham cohomology appearing on $(\mathcal{O}_2, k)$ *and which is specific to symmetric manifolds.*

The de Rham symmetric cohomology vector space $\mathbf{R}^1_s(\mathcal{O}_2)$ *defined here is canonicaly isomorphic with the de Rham cohomology vector space of* $\mathbf{X}$. *In the case when* $\mathbf{X}$ *is compact, we give the de Rham Theorem on* $\mathbf{X}$ *(Corollary to Theorem 23).*

In Section 16 we present a Stokes Theorem for chains of singular squares on N.R.S.

We point out now that there exist three other models of calculus on nonorientable manifolds, proposed by M.Schiffer and D.Spencer [13], G. de Rham [12],and N.Alling and N. Greenleaf [1].

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Preliminaries

Definition 1 *A dianalytic atlas ([1] p.5) on the surface* $\mathbf{X}$ *is a family*

$$\mathcal{A} = \{(\mathbf{U}_i, \varphi_i, \mathbf{V}_i)\}_{i \in I}$$

where:
a) $\{\mathbf{U}_i \mid i \in I\}$ *is an open covering of* $\mathbf{X}$ *;*
b) $\mathbf{V}_i$ *is an open subset of the complex plane* $\mathbf{C}$*, for every* $i \in I$*;*
c) $\varphi_i : \mathbf{U}_i \longrightarrow \mathbf{V}_i$ *is a homeomorphism for every* $i \in I$*;*
d) If $i, j \in I$ *then either* $\mathbf{U}_i \cap \mathbf{U}_j = \Phi$ *or* $\mathbf{U}_i \cap \mathbf{U}_j \neq \Phi$ *and*

$$\varphi_i \circ {\varphi_j}^{-1} : \varphi_j(\mathbf{U}_i \cap \mathbf{U}_j) \longrightarrow \varphi_i(\mathbf{U}_i \cap \mathbf{U}_j)$$

is analytic or antianalytic on each connected component of $\varphi_j(\mathbf{U}_i \cap \mathbf{U}_j)$.

Remarks*:*
1. *In the case when* $\varphi_i \circ {\varphi_j}^{-1}$ *are analytic functions one gets the usual analytic atlases on (orientable) Riemann surfaces.*
2.*The surfaces we deal with in this paper are supposed to be connected.*

Definition 2 (Teichmüller) *A* **nonorientable Riemann surface (N.R.S.)** *is a pair* $(\mathbf{X}, \mathcal{A})$ *where* $\mathbf{X}$ *is surface and* $\mathcal{A}$ *is a maximal dianalytic atlas on* $\mathbf{X}$ *such that* $\mathcal{A}$ *does not contain any analytic subatlas.*

We shall denote throughout this work the N.R.S. $(\mathbf{X}, \mathcal{A})$ *by* $\mathbf{X}$ *because the atlas* $\mathcal{A}$ *will be fixed.*

Definition 3 (Felix Klein) *A* **symmetric Riemann surface** *is a pair* $(\mathcal{O}_2, k)$ *that consists of the (orientable) Riemann surface* $\mathcal{O}_2$ *and the antianalytic involution without fixed points*

$$k : \mathcal{O}_2 \longrightarrow \mathcal{O}_2.$$

Denoting by $\mathcal{H}$ *the two-elements group that consists of* k *and the identity of* $\mathcal{O}_2$*, we mention the following* **fundamental** *theorem due to Klein:*

Theorem 1 *If $(\mathcal{O}_2, k)$ is a symmetric Riemann surface then, on the orbit space $\mathcal{O}_2/\mathcal{H}$ (where the $\mathcal{H}$-equivalent points P and $k(P)$ are identified), there exists a dianalytic atlas $\mathcal{A}$ such that $(\mathcal{O}_2/\mathcal{H}, \mathcal{A})$ is a N.R.S.*

Conversely, if $(\mathbf{X}, \mathcal{A})$ is a N.R.S. then there exists a symmetric Riemann surface $(\mathcal{O}_2, k)$ such that $\mathbf{X}$ is dianalytically equivalent with $\mathcal{O}_2/\mathcal{H}$.
If $\mathbf{Y}$ is a N.R.S., dianalitically equivalent with $\mathbf{X}$ (i.e.isomorphic to $\mathbf{X}$), then there exists an antianalytic involution

$$k' : \mathcal{O}_2 \longrightarrow \mathcal{O}_2$$

conjugated with k in the group of all analytic or antianalytic automorphisms of $\mathcal{O}_2$, such that $\mathbf{Y}$ is isomorphic with $\mathcal{O}_2/\mathcal{H}'$, where $\mathcal{H}'$ consists of k' and the identity.

Remark:
In the following we use the notations from Klein's previous theorem and identify $\mathbf{X}$ with the orbit space $\mathcal{O}_2/\mathcal{H}$. We denote the canonical projection of $\mathcal{O}_2$ onto $\mathcal{O}_2/\mathcal{H}$ by p. Thus we get te commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_2 \arrow[rr, "k"] \arrow[dr, "p"'] & & \mathcal{O}_2 \arrow[dl, "p"] \\
& \mathbf{X} &
\end{tikzcd}$$

For $z \in \mathcal{O}_2$ denote by $\widetilde{z}$ its $\mathcal{H}$-orbit; this is the two-elements set $\{z; kz\}$. Thus,

$$\widetilde{z} = \widetilde{kz} = p(z) = p(kz) = \{z; kz\}.$$

1 Functions on N.R.S.

In this section we define the symmetric functions on $\mathcal{O}_2$ and see how these functions may be identified with the complex functions defined on $\mathbf{X}$.

Definition 4 *A set $\Delta \subseteq \mathcal{O}_2$ is called symmetric (with respect to k) if*

$$k(\Delta) = \Delta.$$

Note: *For any $D \subseteq \mathcal{O}_2$ we shall denote by $\widetilde{D}$ the factor set*

$$\widetilde{D} = p(D) = \{\widetilde{z} \mid z \in D\}.$$

Clearly D is open if and only if $\widetilde{D}$ is open. It is also clear that

$$p^{-1}(\widetilde{D}) = D \cup k(D)$$

is a symmetric set.

Definition 5 *Let Δ be a symmetric subset of $\mathcal{O}_2$ and $f : \Delta \longrightarrow \hat{\mathbf{C}}$ be a function. Then f is called a symmetric (antisymmetric) function if*

$$f(z) = f(kz) \quad (resp. f(z) = -f(kz))$$

for every $z \in \Delta$.

Remark: *The function f is symmetric if and only if it is automorphic with respect to the two elements group that consists of the identity of Δ and the restriction of k to Δ.*

Let M be an open subset of X and $\Delta := p^{-1}(M)$. Clearly Δ is symmetric and $M = \widetilde{\Delta}$. Let $\widetilde{f} : \widetilde{\Delta} \longrightarrow \widehat{\mathbf{C}}$ be a function. For $\widetilde{z} = \{z; kz\} \in \widetilde{\Delta}$ we define

$$f(z) = f(kz) := \widetilde{f}(\widetilde{z}). \tag{1}$$

In this way the symmetric function $f : \Delta \longrightarrow \widehat{\mathbf{C}}$ has appeared, such that

$$f = \widetilde{f} \circ p \tag{2}$$

where $p = p\,|_{\Delta}$ is the restriction of p to Δ.

Conversely, let $\Delta = k(\Delta)$ be an open subset of $\mathcal{O}_2$ and let $f : \Delta \longrightarrow \widehat{\mathbf{C}}$ be a symmetric function. Then, for every $z \in \Delta$, $f(z) = f(kz)$, that is the value of f does not depend on z but only on the equivalence class $\widetilde{z}$ of z. This means that the function $\widetilde{f} : \widetilde{\Delta} \longrightarrow \widehat{\mathbf{C}}$ given by

$$\widetilde{f}(\widetilde{z}) := f(z) = f(kz) \tag{3}$$

is well defined.

It is clear that the sets $\mathcal{F}(\widetilde{\Delta})$ (resp. $\mathcal{F}_s(\Delta)$) that consist of all the functions $\widetilde{f} : \widetilde{\Delta} \longrightarrow \widehat{\mathbf{C}}$, for which $\widetilde{f}^{-1}(\infty)$ is a finite set (resp. $f : \Delta \longrightarrow \widehat{\mathbf{C}}$, $f = f \circ k$ and $f^{-1}(\infty)$ is a finite set) with respect to the usual algebraic operations, are algebras.

Thus we can formulate the following result in connection with Theorem 3e.

Theorem 2 *The sets of functions $\mathcal{F}(\mathbf{X})$ and $\mathcal{F}_s(\mathcal{O}_2)$ are algebras over the field of complex numbers. The function*

$$p_{\#} : \mathcal{F}_s(\mathcal{O}_2) \longrightarrow \mathcal{F}(\mathbf{X})$$

given by

$$p_{\#}(f) := \widetilde{f}, \tag{4}$$

where $\widetilde{f}$ is defined by (3), is an algebra isomorphism (called the natural isomorphism) whose inverse is the pull-back isomorphism $p^{\star}$ defined by

$$p^{\star}(\widetilde{f}) = f := \widetilde{f} \circ p. \tag{5}$$

Via the previous theorem one identifies the algebra $\mathcal{F}(\mathbf{X})$ with its isomorphic copy $\mathcal{F}_s(\mathcal{O}_2)$. For this reason we now concentrate on a deeper study of $\mathcal{F}_s(\mathcal{O}_2)$. [Statements similar to those that follow can be formulated in connection with algebras of the form $\mathcal{F}_s(\Delta)$ or $\mathcal{F}(\widetilde{\Delta})$, where $\Delta = k(\Delta) \subseteq \mathcal{O}_2$ and $\widetilde{\Delta} = p(\Delta)$, but we do not enter into details].

2 Symmetric functions on $\mathcal{O}_2$.

In this section we give the connection between the algebra $\mathcal{F}_s(\mathcal{O}_2)$ and the algebra $\mathcal{F}(\mathcal{O}_2)$ of all complex functions on $\mathcal{O}_2$.

In general, a function $f : \mathcal{O}_2 \longrightarrow \hat{\mathbf{C}}$ is far of being symmetric on $\mathcal{O}_2$. There exists, however, a **canonical** *way to symmetrize it, that is to identify a "part" of it that can be factorized by p.*
In this section we shall deal with this important question.

Denote by $\mathcal{F}^r(\mathcal{O}_2)$ the subalgebra of $\mathcal{F}(\mathcal{O}_2)$ which consists of all functions of class C^r for $0 \leq r \leq \infty$, and by $\mathcal{F}^\omega(\mathcal{O}_2)$ the subalgebra of the **R***-analytic ones. Let $H(\mathcal{O}_2)$ be the vector space of harmonic functions on $\mathcal{O}_2$. We have the following inclusions:*

$$H(\mathcal{O}_2) \subset \mathcal{F}^\omega(\mathcal{O}_2) \subset \mathcal{F}^r(\mathcal{O}_2) \subset \mathcal{F}(\mathcal{O}_2).$$

Now we can define the symmetrization operator $\mathcal{S}$ on $\mathcal{F}(\mathcal{O}_2)$, an operator which, as we shall see in all that follows, plays a crucial role in the whole function theory on **X**. *This operator has its analogue on the vector spaces of differential forms on $\mathcal{O}_2$ but they do not exist on non-symmetric Riemann surfaces.*

So, let $\mathcal{S} : \mathcal{F}(\mathcal{O}_2) \longrightarrow \mathcal{F}(\mathcal{O}_2)$ be defined by

$$\mathcal{S}f := f_s := \frac{1}{2}(f + f \circ k). \tag{6}$$

Since k is an involution, f_s is a symmetric function and it is easy to see that $f \in \mathcal{F}_s(\mathcal{O}_2)$ if and only if $\mathcal{S}f = f$; with other words f is symmetric if and only if it is a fixed point of $\mathcal{S}$.

Thus the difference $f - f_s$ gives a measure of the deviation of f from being symmetric. In this way we are led to the second important operator in connection with the function theory on N.R.S.,

$$\mathcal{A} : \mathcal{F}(\mathcal{O}_2) \longrightarrow \mathcal{F}(\mathcal{O}_2),$$

defined as:

$$\mathcal{A}f := f_a := f - f_s = \frac{1}{2}(f - f \circ k). \tag{7}$$

Observe that f_a is antisymmetric. For this reason we call $\mathcal{A}$ the antisymmetrization operator.

The following theorem describes important properties of the operators $\mathcal{S}$ and $\mathcal{A}$.

Theorem 3 *The following properties hold:*

a) $\mathcal{S}$ is a linear operator and its image is $Im(\mathcal{S}) = \mathcal{F}_s(\mathcal{O}_2)$;
b) $\mathcal{F}_s(\mathcal{O}_2)$ is a subalgebra of $\mathcal{F}(\mathcal{O}_2)$;
c) $\mathcal{F}_a(\mathcal{O}_2)$, the subset of antisymmetric elements of $\mathcal{F}(\mathcal{O}_2)$, is a vector subspace of $\mathcal{F}(\mathcal{O}_2)$;
d) The operator of antisymmetrization

$$\mathcal{A} : \mathcal{F}(\mathcal{O}_2) \longrightarrow \mathcal{F}(\mathcal{O}_2)$$

is linear and its image is $Im(\mathcal{A}) = \mathcal{F}_a(\mathcal{O}_2)$; *also* $f \in \mathcal{F}_a(\mathcal{O}_2)$ *if and only if* $\mathcal{A}f = f$;
e) $\mathcal{F}(\mathcal{O}_2) = \mathcal{F}_s(\mathcal{O}_2) \bigoplus \mathcal{F}_a(\mathcal{O}_2)$;
f) $\mathcal{S}$ *and* $\mathcal{A}$ *are orthogonal projectors i.e.*
$\mathcal{S} \circ \mathcal{S} = \mathcal{S}$, $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$ *and* $\mathcal{S} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{S} = \mathcal{O}$;
g) $\mathcal{S}$ *and* $\mathcal{A}$ *provide an orthogonal decomposition of the identity* $\mathcal{I}$ *of* $\mathcal{F}(\mathcal{O}_2)$:

$$\mathcal{I} = \mathcal{S} + \mathcal{A}.$$

h) For any $f, g \in \mathcal{F}(\mathcal{O}_2)$,

$$\mathcal{S}(fg) = \mathcal{S}(f)\mathcal{S}(g) + \mathcal{A}(f)\mathcal{A}(g),$$

$$\mathcal{A}(fg) = \mathcal{A}(f)\mathcal{S}(g) + \mathcal{A}(g)\mathcal{S}(f).$$

Proof: The points a)-d) are obvious.
e) If $g \in \mathcal{F}_s(\mathcal{O}_2) \cap \mathcal{F}_a(\mathcal{O}_2)$ *then* $g = 0$ *because* $g = g \circ k = -g \circ k$; *thus*

$$\mathcal{F}_s(\mathcal{O}_2) \cap \mathcal{F}_a(\mathcal{O}_2) = \{0\}.$$

For any $f \in \mathcal{F}(\mathcal{O}_2)$, f_s *and* f_a *are elements of* $\mathcal{F}_s(\mathcal{O}_2)$ *and* $\mathcal{F}_a(\mathcal{O}_2)$ *respectively and* $f = f_s + f_a$. *Thus e) holds.*
f) If $f \in \mathcal{F}(\mathcal{O}_2)$,

$(f_s)_s = \frac{1}{2}(f_s + f_s \circ k) = \frac{1}{2}(f_s + f_s) = f_s$ *because* f_s *is symmetric.*

$(f_a)_a = \frac{1}{2}(f_a - f_a \circ k) = \frac{1}{2}(f_a + f_a) = f_a$ *because* $f_a \circ k = -f_a$.

Now, $(f_s)_a = \frac{1}{2}(f_s - f_s \circ k) = 0$ *and* $(f_a)_s = \frac{1}{2}(f_a + f_a \circ k) = 0$.
g) is clear from e) and f).
(Remark: The constant functions lie in $\mathcal{F}_s(\mathcal{O}_2)$*).*
h) Let $f, g \in \mathcal{F}(\mathcal{O}_2)$ *and let*

$$f = f_s + f_a, \quad g = g_s + g_a$$

be their decompositions.

$$fg = f_s g_s + f_a g_a + f_a g_s + f_s g_a.$$

It is clear that $f_s g_s$, $f_a g_a \in \mathcal{F}_s(\mathcal{O}_2)$ *and* $f_a g_s$, $f_s g_a \in \mathcal{F}_a(\mathcal{O}_2)$. *Now, from b), c) and e) the result follows.* □

Remark: *Fig.1 gives a geometric image of the space* $\mathcal{F}(\mathcal{O}_2)$.
Notations:
$\mathcal{F}_{s,r}(\mathcal{O}_2) := \mathcal{F}_s(\mathcal{O}_2) \cap \mathcal{F}^r(\mathcal{O}_2)$ *and* $\mathcal{F}_{a,r}(\mathcal{O}_2) := \mathcal{F}_a(\mathcal{O}_2) \cap \mathcal{F}^r(\mathcal{O}_2)$ *are the sets of functions of class* C^r, $0 \leq r \leq \omega$, *which lie in* $\mathcal{F}_s(\mathcal{O}_2)$ *and* $\mathcal{F}_a(\mathcal{O}_2)$ *respectively.*
$\mathcal{F}^r(\mathbf{X})$ *is the set of functions of class* C^r *on* $\mathbf{X}$.

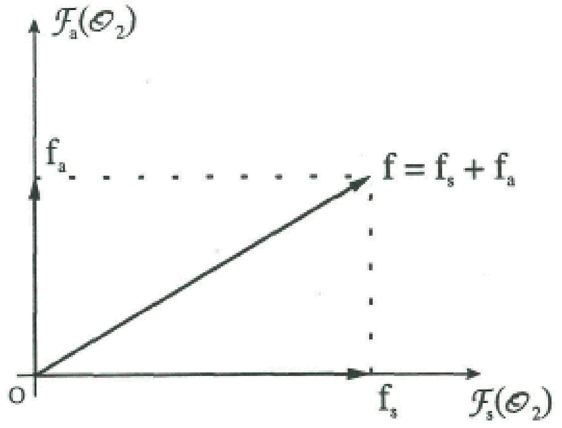

Fig. 1:

For $\Delta = k(\Delta) \subseteq \mathcal{O}_2$, *we denote by:*
$H(\Delta)$ *the vector space of harmonic functions on* Δ,
$H_s(\Delta)$ ($H_a(\Delta)$) *the vector space of symmetric (antisymmetric) harmonic functions on* Δ *and* $H(\widetilde{\Delta})$ *the vector space of harmonic functions on* $\widetilde{\Delta} \subseteq \mathbf{X}$.
Theorems 2 and 3 have the following two corollaries:

Corollary 1 *a)* $p^\star : \mathcal{F}^r(\mathbf{X}) \longrightarrow \mathcal{F}_{s,r}(\mathcal{O}_2)$, *defined by* $p^\star(\widetilde{f}) := \widetilde{f} \circ p$, *is an isomorphism of algebras;*
b) $p^\star : H(\widetilde{\Delta}) \longrightarrow H_s(\Delta)$ *is an isomorphism of vector spaces.*

Corollary 2 *The following direct sum decompositions hold:*
a)$\mathcal{F}^r(\mathcal{O}_2) = \mathcal{F}_{s,r}(\mathcal{O}_2) \bigoplus \mathcal{F}_{a,r}(\mathcal{O}_2)$
b) $H(\Delta) = H_s(\Delta) \bigoplus H_a(\Delta)$.

Proof: Since p is a dianalytic covering projection, $\widetilde{f}$ *is of class* C^r *(harmonic) if and only if* $p^\star(\widetilde{f})$ *is of class* C^r *(harmonic).*

If $f = f_s + f_a$, *clearly* f *lies in* $\mathcal{F}^r(\mathcal{O}_2)$ $(H(\Delta))$ *if and only if* f_s *and* f_a *lie in* $\mathcal{F}^r(\mathcal{O}_2)$ $(H(\Delta))$. $\square$

We point out now some conclusions.
The study of functions on $\mathbf{X}$ *is canonically reduced to the study of functions on* $\mathcal{O}_2$. *More precisely, the elements of* $\mathcal{F}(\mathbf{X})$ $(H(\widetilde{\Delta}))$ *are identified with the symmetric functions on* $\mathcal{O}_2$ *and of the same class (or which are harmonic on* Δ*). Each function on* $\mathcal{O}_2$ *has a canonical decomposition*

$$f = f_s + f_a,$$

and only the component f_s *has a meaning on* $\mathbf{X}$.

3 N-meromorphic functions.

We further take a look at functions $\tilde{f} \in \mathcal{F}(\mathbf{X})$ having the property that $p^*(\tilde{f})$ is obtained by symmetrization from a meremorphic or an antimeromorphic function $f \in \mathcal{F}(\mathcal{O}_2)$.

Let $\mathcal{M}(\mathcal{O}_2)$ be the field of meromorphic functions on $\mathcal{O}_2$. We denote by $\overline{\mathcal{M}(\mathcal{O}_2)}$ the field of antimeromorphic functions on $\mathcal{O}_2$ or, equivalently, the field of meromorphic functions on $\mathcal{O}_2$ endowed with its second analytic structure. The covering projection p mixes the two orientations of $\mathcal{O}_2$ leading, this way, to the dianalytic structure of $\mathbf{X}$. This phenomenon is reflected in the structure of some elements of $\mathcal{F}(\mathbf{X})$, elements which will be called **N-meromorphic functions**.

The symmetry k induces the isomorphism

$$k^* : \mathcal{M}(\mathcal{O}_2) \longrightarrow \overline{\mathcal{M}(\mathcal{O}_2)},$$

which is defined by

$$k^*(f) := f \circ k.$$

The inverse of k^* is

$$k^{*-1} : \overline{\mathcal{M}(\mathcal{O}_2)} \longrightarrow \mathcal{M}(\mathcal{O}_2)$$

and it has the same form as k^*:

$$k^{*-1}(f) = f \circ k.$$

It is clear that

$$\mathcal{S} \circ k^* = \mathcal{S}|_{\mathcal{M}(\mathcal{O}_2)} \quad \text{and} \quad \mathcal{S} \circ k^{*-1} = \mathcal{S}|_{\overline{\mathcal{M}(\mathcal{O}_2)}}.$$

Theorem 4 $\mathcal{S}(\mathcal{M}(\mathcal{O}_2)) = \mathcal{S}(\overline{\mathcal{M}(\mathcal{O}_2)})$.

Proof: $\frac{1}{2}[f + f \circ k] = \frac{1}{2}[f \circ k + (f \circ k) \circ k]$
and $f \in \mathcal{M}(\mathcal{O}_2) \Leftrightarrow f \circ k \in \overline{\mathcal{M}(\mathcal{O}_2)}$. □

Definition 6 *The function $\tilde{f} : \mathbf{X} \longrightarrow \hat{\mathbf{C}}$ is called an* **N-meromorphic function**[*] *if*

$$p^*(\tilde{f}) \in \mathcal{S}(\mathcal{M}(\mathcal{O}_2)).$$

We close this section with a remark concerning the harmonic functions on $\mathbf{X}$.

Proposition 1 *Let M be an open subset of $\mathbf{X}$ and $h : M \longrightarrow \mathbf{R}$ an harmonic function. Then there exists an N-meromorphic function $\tilde{f} : M \longrightarrow \mathbf{C}$ (possibly multivalued) such that*

$$h = \mathcal{R}e(\tilde{f}).$$

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[*] The N-meromorphic functions were defined in [2].

4 The Cauchy-Pompeiu derivatives.

Now we shall present some properties of the derivation operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$, *which will be of use in our endeavours.*
Define[†]*:*

$$(8) \qquad \begin{cases} \dfrac{\partial}{\partial z} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right) & \textit{(Cauchy)} \\[2ex] \dfrac{\partial}{\partial \bar{z}} := \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right) & \textit{(Pompeiu)} \end{cases}$$

Thus, if $h = h_1 + ih_2$ *is a function of class* C^1 *on an open set lying in the complex plane of the variable* $z = x + iy$, *then*

$$(9) \qquad \begin{cases} \dfrac{\partial h}{\partial z} := \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} - i\dfrac{\partial h}{\partial y}\right) = \dfrac{1}{2}\left(\dfrac{\partial h_1}{\partial x} + \dfrac{\partial h_2}{\partial y}\right) - \dfrac{i}{2}\left(\dfrac{\partial h_1}{\partial y} - \dfrac{\partial h_2}{\partial x}\right) \\[2ex] \dfrac{\partial h}{\partial \bar{z}} := \dfrac{1}{2}\left(\dfrac{\partial h}{\partial x} + i\dfrac{\partial h}{\partial y}\right) = \dfrac{1}{2}\left(\dfrac{\partial h_1}{\partial x} - \dfrac{\partial h_2}{\partial y}\right) + +\dfrac{i}{2}\left(\dfrac{\partial h_1}{\partial y} + \dfrac{\partial h_2}{\partial x}\right) \end{cases}$$

From (9) we obtain:

$$(10) \qquad \frac{\partial \bar{h}}{\partial \bar{z}} = \overline{\left(\frac{\partial h}{\partial z}\right)} \quad \text{and} \quad \frac{\partial \bar{h}}{\partial z} = \overline{\left(\frac{\partial h}{\partial \bar{z}}\right)};$$

$$(11) \qquad \frac{\partial h}{\partial z} = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{\partial h_1}{\partial x} = -\dfrac{\partial h_2}{\partial y} \\[2ex] \dfrac{\partial h_1}{\partial y} = \dfrac{\partial h_2}{\partial x} \end{cases}$$

(Cauchy-Riemann equations for $\bar{h}$*)* $\Rightarrow$

$$(12) \qquad \frac{\partial \bar{h}}{\partial z} = \frac{\partial h_1}{\partial x} - i\frac{\partial h_1}{\partial y} = \frac{d\bar{h}}{dz}.$$

If k *is an antianalytic involution of an open subset of the complex plane and if*

$w = u + iv = kz = k_1(x, y) + ik_2(x, y)$,

with k*'s Jacobian* J *at the point* z *given by*

$$J = \frac{\partial k_1}{\partial x}(z)\frac{\partial k_2}{\partial y}(z) - \frac{\partial k_1}{\partial y}(z)\frac{\partial k_2}{\partial x}(z) = \frac{D(k_1, k_2)}{D(x, y)}(z) = \frac{D(u, v)}{D(x, y)}(z),$$

[†] The operator $\frac{\partial}{\partial z}$ was introduced by Cauchy. The operator $\frac{\partial}{\partial \bar{z}}$ first appeared in Pompeiu's work[11] under the name "the areolar derivative".

the following equalities hold:

$$\left\{\begin{array}{l} \dfrac{\partial k_1}{\partial u}(w) = \dfrac{1}{J}\ \dfrac{\partial k_2}{\partial y}(z) \\ \\ \dfrac{\partial k_1}{\partial v}(w) = -\dfrac{1}{J}\ \dfrac{\partial k_1}{\partial x}(z), \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l} \dfrac{\partial k_2}{\partial u}(w) = -\dfrac{1}{J}\ \dfrac{\partial k_2}{\partial x}(z) \\ \\ \dfrac{\partial k_2}{\partial v}(w) = \dfrac{1}{J}\ \dfrac{\partial k_1}{\partial x}(z), \end{array}\right. \tag{14}$$

$$\left\{\begin{array}{l} \dfrac{\partial k}{\partial \bar{w}}(w) = -\dfrac{1}{J}\ \dfrac{\partial k}{\partial \bar{z}}(z) \\ \\ \dfrac{\partial \bar{k}}{\partial w}(w) = -\dfrac{1}{J}\ \dfrac{\partial \bar{k}}{\partial z}(z), \end{array}\right. \tag{15}$$

$$\left\{\begin{array}{l} \dfrac{\partial k}{\partial \bar{z}}(z)\dfrac{\partial \bar{k}}{\partial z}(z) \ = \ \left|\dfrac{\partial k}{\partial \bar{z}}(z)\right|^2 = -\dfrac{D(u,v)}{D(x,y)}(z) \\ \\ \dfrac{\partial k}{\partial \bar{w}}(w)\dfrac{\partial \bar{k}}{\partial w}(w) = \left|\dfrac{\partial k}{\partial \bar{w}}(w)\right|^2 = -\dfrac{D(x,y)}{D(u,v)}(w) \end{array}\right. \tag{16}$$

$$\frac{D(x,y)}{D(u,v)}(w)\frac{D(u,v)}{D(x,y)}(z) = 1,$$

(The chain rule for Cauchy-Pompeiu derivatives)

$$\left\{\begin{array}{l} \dfrac{\partial}{\partial z}(f \circ k)(z) = \dfrac{\partial f}{\partial w}(kz)\dfrac{\partial k}{\partial z}(z) + \dfrac{\partial f}{\partial \bar{w}}(kz)\dfrac{\partial \bar{k}}{\partial z}(z) = \\ = \dfrac{\partial f}{\partial u}(kz)\dfrac{\partial u}{\partial z}(z) + \dfrac{\partial f}{\partial v}(kz)\dfrac{\partial v}{\partial z}(z) \\ \dfrac{\partial}{\partial \bar{z}}(f \circ k)(z) = \dfrac{\partial f}{\partial w}(kz)\dfrac{\partial k}{\partial \bar{z}}(z) + \dfrac{\partial f}{\partial \bar{w}}(kz)\dfrac{\partial \bar{k}}{\partial \bar{z}}(z) = \\ = \dfrac{\partial f}{\partial u}(kz)\dfrac{\partial u}{\partial \bar{z}}(z) + \dfrac{\partial f}{\partial v}(kz)\dfrac{\partial v}{\partial \bar{z}}(z). \end{array}\right. \tag{17}$$

The mapping k is antianalytic,therefore

$$\frac{\partial k}{\partial z} = \frac{\partial \bar{k}}{\partial \bar{z}} = 0.$$

Thus we obtain:

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial z}(f \circ k)(z) = \dfrac{\partial f}{\partial \bar{w}}(kz)\dfrac{\partial \bar{k}}{\partial z}(z) \\ \dfrac{\partial}{\partial \bar{z}}(f \circ k)(z) = \dfrac{\partial f}{\partial w}(kz)\dfrac{\partial k}{\partial \bar{z}}(z). \end{array} \right. \tag{18}$$

5 Symmetric and antisymmetric differential forms of degree 1.

In this section we define the symmetric and antisymmetric differential forms of degree 1 on $\mathcal{O}_2$. The symmetric ones play a crucial role in all calculus on $\mathbf{X}$ by means of their connection with the differential forms of degree 1 on $\mathbf{X}$.

In order to obtain the differential forms on the N.R.S. $\mathbf{X}$, we have to establish, for the differential forms on $\mathcal{O}_2$, results analogous to the ones given in Section 2 for functions (= differential forms of degree 0). For this it is enough to re-write the definition (6) of the symmetrization operator $\mathcal{S}$ as :

$$\mathcal{S}f = f_s = \frac{1}{2}(f + k^* f),$$

where $k^* f$ is the pull-back by k of f :

$$(k^* f)(z) := (f \circ k)(z), \quad \text{for every } z \in \mathcal{O}_2.$$

We consider that $\Delta = k(\Delta) \subseteq \mathcal{O}_2$ is a symmetric open subset of $\mathcal{O}_2$,and that

$$\omega : \Delta \longrightarrow \mathbf{T}^*(\mathcal{O}_2)$$

is a differential form of degree 1 on Δ.
($\mathbf{T}^*(M)$ is the cotangent fiber bundle of the manifold M).
We can now formulate

Definition 7 *The differential form ω is called symmetric (antisymmetric) if*

$$k^*\omega = \omega \quad (k^*\omega = -\omega).$$

To avoid sofisticated notations while dealing with the local study of differential forms, we identify different parametric disks D on $\mathcal{O}_2$ with their images in the Euclidian planes.

Since the preimage by p of each parametric disk $\tilde{D}$ on $\mathbf{X}$ consists of the pair D and kD of symmetric disks on $\mathcal{O}_2$, it is more natural to consider the restrictions of the type $\omega\,|_{D\cup kD}$ rather then the restrictions $\omega\,|_D$ of ω, in the local study of differential forms on $\mathcal{O}_2$. The restrictions $k\,|_{D\cup kD}$ have also the property that they are involutions of $D\cup kD$ whereas the restrictions $k\,|_D$ are not involutions of D. Fig.2 gives a description of $k\,|_{D\cup kD}$.

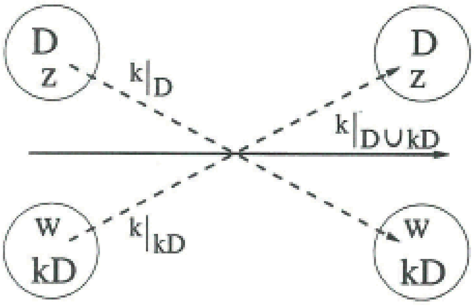

Fig. 2:

Since k is an involution without fixed points, we may consider that

$$D\cap kD=\emptyset.$$

We denote by z a local uniformiser in D, by w an uniformiser in kD and by $\omega\,|_{D\cup kD}$ the restriction of ω to $D\cup kD$,

$$\omega\,|_{D\cup kD}=\begin{cases} A(z)\,dz+B(z)\,d\bar{z}, & z\in D\\[2ex] A'(w)\,dw+B'(w)\,d\bar{w}, & w\in kD.\end{cases}\tag{19}$$

With the previous convention, D (or kD) will be a disk in the plane of the complex variable $z=x+iy$ (or a disk in the plane of the complex variable $w=u+iv$). Notice that k fulfils the equations (13)-(18).

It is obvious that ω is symmetric if and only if $\omega\,|_{D\cup kD}$ is symmetric for every disk

$D \subsetneq \Delta$. *(In this case $kD \subsetneq \Delta$ as well.) For this reason it is enough to give rules by means of which one can recognize the symmetric differential forms defined on sets of the type $D \cup kD$.*

The main result concerning this question is the following:

Theorem 5 a. *The differential form ω is symmetric if and only if the coefficients A, B, A', B' satisfy*

$$(20) \qquad \begin{cases} A(z) = \epsilon B'(kz)\dfrac{\partial \bar{k}}{\partial z}(z) \\[2ex] B(z) = \epsilon A'(kz)\dfrac{\partial k}{\partial \bar{z}}(z) \end{cases} \text{and} \quad \begin{cases} A'(w) = \epsilon B(kw)\dfrac{\partial \bar{k}}{\partial w}(w) \\[2ex] B'(w) = \epsilon A(kw)\dfrac{\partial k}{\partial \bar{w}}(w), \end{cases}$$

with $\epsilon = 1$.

b. *ω is antisymmetric if and only if the coefficients A, B, A', B' satisfy the previous relations with $\epsilon = -1$.*

Proof: The restriction to $D \cup kD$ of $k^*\omega$ is given by:

$$(21) \qquad (k^*\omega)\mid_{D\cup kD} = \begin{cases} B'(kz)\dfrac{\partial \bar{k}}{\partial z}(z)\, dz + A'(kz)\dfrac{\partial k}{\partial \bar{z}}(z)\, d\bar{z} \\[2ex] B(kw)\dfrac{\partial \bar{k}}{\partial w}(w)\, dw + A(kw)\dfrac{\partial k}{\partial \bar{w}}(w)\, d\bar{w}. \end{cases}$$

The rest of the proof follows now from the definitions. □

Remark. *The second pair of equalities in the symmetry (antisymmetry) conditions for ω is a consequence of the first one. Indeed, the multiplication of*

$$A(z) = \epsilon B'(kz)\frac{\partial \bar{k}}{\partial z}(z)$$

by $\dfrac{\partial k}{\partial \bar{z}}(z)$ and the use of relation (16) yields

$$A(z)\frac{\partial k}{\partial \bar{z}}(z) = \epsilon B'(w)\frac{\partial \bar{k}}{\partial z}(z)\frac{\partial k}{\partial \bar{z}}(z) = \epsilon B'(w)(-J)$$

Thus,

$$B'(w) = \epsilon A(z)\left(-\frac{1}{J}\frac{\partial k}{\partial \bar{z}}(z)\right) = \epsilon A(kw)\frac{\partial k}{\partial \bar{w}}(w).$$

6 The pull-back by p of differential forms of degree 1 on X.

In this section we see how the differential forms on $\mathbf{X}$ *are identified with the symmetric differential forms on* $\mathcal{O}_2$.

Let D *be an open disk on* $\mathcal{O}_2$ *such that* $D \cap kD = \emptyset$. *We denote*

$$\widetilde{D} = \widetilde{kD} = \{\widetilde{z} \mid z \in D\} \quad \text{where } \widetilde{z} = \widetilde{w} = \{z; w = kz\}.$$

By means of p, *the points* z *and* w *are pasted together leading to the point* $\widetilde{z} \in \mathbf{X}$.

Let

$$\widetilde{\omega} : \mathbf{X} \longrightarrow \mathbf{T}^*(\mathbf{X})$$

be a differential form on $\mathbf{X}$.

On $\widetilde{D}$ *we have two local parameters:* z *and* w; *any other local parameter is analytically coherent to either* z *or* w.

For $\widetilde{z} = \widetilde{w} \in \widetilde{D}$, $\{dz; d\bar{z}\}$ *and* $\{dw; d\bar{w}\}$ *are two bases of the same vector space*

$$\mathbf{T}^*_{\widetilde{z}}(\mathbf{X}) = \mathbf{T}^*_{\widetilde{w}}(\mathbf{X}).$$

$$\widetilde{\omega}(\widetilde{z}) = \widetilde{\omega}(\widetilde{w}) \in \mathbf{T}^*_{\widetilde{z}}(\mathbf{X}) \quad \text{for every} \quad \widetilde{z} \in \widetilde{D}.$$

The vector $\widetilde{\omega}(\widetilde{z})$ *is a linear combination of the vectors in each base:*

$\widetilde{\omega}(\widetilde{z}) = \alpha dz + \beta d\bar{z}$ *in the base* $\{dz; d\bar{z}\}$;

$\widetilde{\omega}(\widetilde{w}) = \alpha' dw + \beta' d\bar{w}$ *in the base* $\{dw; d\bar{w}\}$.

When $\widetilde{z}$ *varies in* $\widetilde{D}$ *the coefficients* $\alpha, \beta, \alpha', \beta'$ *are also variable.*
With other words, $\alpha, \beta, \alpha', \beta'$ *are complex functions defined on* $\widetilde{D}$:

$$\widetilde{\omega}(\widetilde{z}) = \alpha(\widetilde{z})dz + \beta(\widetilde{z})d\bar{z}$$

and

$$\widetilde{\omega}(\widetilde{w}) = \alpha'(\widetilde{w})dw + \beta'(\widetilde{w})d\bar{w} = \alpha'(\widetilde{z})dw + \beta'(\widetilde{z})d\bar{w}.$$

Now, we can state:

Theorem 6 *The coefficients* $\alpha, \beta, \alpha', \beta'$ *satisfy:*

$$(22) \qquad \begin{cases} \alpha(\widetilde{z}) = \beta'(\widetilde{z}) \, \dfrac{\partial \bar{k}}{\partial z}(z) \\[2ex] \beta(\widetilde{z}) = \alpha'(\widetilde{z}) \, \dfrac{\partial k}{\partial \bar{z}}(z) \end{cases}$$

and

$$\begin{cases} \alpha'(\widetilde{z}) = \beta(\widetilde{z})\, \dfrac{\partial \bar{k}}{\partial w}(w) \\ \\ \beta'(\widetilde{z}) = \alpha(\widetilde{z})\, \dfrac{\partial k}{\partial \bar{w}}(w). \end{cases}$$

Proof: $\widetilde{\omega}(\widetilde{z}) = \alpha'(\widetilde{z})d(kz) + \beta'(\widetilde{z})d(\overline{kz}) = \alpha'(\widetilde{z})\, \frac{\partial k}{\partial \bar{z}}(z)d\bar{z} + \beta'(\widetilde{z})\frac{\partial \bar{k}}{\partial z}(z)dz = \alpha(\widetilde{z})dz + \beta(\widetilde{z})d\bar{z}$.

On the other hand,
$\widetilde{\omega}(\widetilde{z}) = \widetilde{\omega}(\widetilde{w}) = \alpha(\widetilde{z})dz + \beta(\widetilde{z})d\bar{z} = \alpha(\widetilde{z})\, \frac{\partial k}{\partial \bar{w}}(w)d\bar{w} + \beta(\widetilde{z})\, \frac{\partial \bar{k}}{\partial w}(w)dw = \alpha'(\widetilde{z})dw + \beta'(\widetilde{z})d\,\bar{w}$, *etc.*□

Let $\omega : \mathcal{O}_2 \longrightarrow \mathbf{T}^*(\mathcal{O}_2)$ *be the pull-back of* $\widetilde{\omega}$ *by* p^*.
On $D \cup kD$, ω *is defined,as it is known, by:*

$$\omega\mid_{D\cup kD} = \begin{cases} A(z)\, dz + B(z)\, d\bar{z}, \quad z \in D \\ \\ A'(w)\, dw + B'(w)\, d\bar{w}, \quad w \in kD \quad \textit{where} \end{cases}$$

$$\begin{cases} A(z) := \alpha(p(z)) = \alpha(\widetilde{z}) \\ B(z) := \beta(p(z)) = \beta(\widetilde{z}) \end{cases} \text{and} \quad \begin{cases} A'(w) := \alpha'(p(w)) = \alpha'(\widetilde{w}) \\ B'(w) := \beta'(p(w)) = \beta'(\widetilde{w}). \end{cases}$$

From these relations and (22) we obtain

$$A(z) = \alpha(\widetilde{z}) = \beta'(\widetilde{z})\, \frac{\partial \bar{k}}{\partial z}(z) = \beta'(\widetilde{w})\, \frac{\partial \bar{k}}{\partial z}(z) = B'(w)\, \frac{\partial \bar{k}}{\partial z}(z) = B'(kz)\, \frac{\partial \bar{k}}{\partial z}(z).$$

In a similar way,

$$B(z) = A'(kz)\, \frac{\partial k}{\partial \bar{z}}(z) \quad \text{for each } z \in D.$$

According to Theorem 5, ω *is a symmetric differential form.*
Now let us start out with a symmetric differential form ω. *For each point* $z \in \mathcal{O}_2$, $\omega(z) = \omega(kz)$ *(because* $\omega = k^*\omega$*), that is* $\omega(z)$ *does not depend on* z *but on its equivalence class* $\widetilde{z}$ *only.*
With other words one can define the differential form $p_{\#}\omega := \widetilde{\omega}$ *by*

$$\widetilde{\omega}(\widetilde{z}) := \omega(z). \tag{23}$$

Denote by $\mathcal{D}^1(\mathcal{O}_2)$ *the vector space of differential forms of degree 1 on* $\mathcal{O}_2$ *and by* $\mathcal{D}^1_s(\mathcal{O}_2)$ *the subspace of all symmetric differential forms. We can now formulate the following theorem:*

Theorem 7 *The operator*

$$p_{\#} : \mathcal{D}^1_s(\mathcal{O}_2) \longrightarrow \mathcal{D}^1(\mathbf{X})$$

is linear, bijectiv, and $p_{\#}{}^{-1} = p^*$.

Proof: It remains only to verify $p_{\#}$*'s linearity, which is obvious.* □
In all that follows we identify $\widetilde{\omega}$ *with* ω *by means of (23).*

7 The operators $\mathcal{S}$ and $\mathcal{A}$ on $\mathcal{D}^1(\mathcal{O}_2)$.

Since $\mathcal{D}^1_s(\mathcal{O}_2)$ *represents* $\mathcal{D}^1(\mathbf{X})$, *it is important to localize it inside the space* $\mathcal{D}^1(\mathcal{O}_2)$. *This will be done now.*

Denote

$$\mathcal{D}^1_s(\mathcal{O}_2) := \{\omega \in \mathcal{D}^1(\mathcal{O}_2) \mid k^*\omega = \omega\}$$

and

$$\mathcal{D}^1_a(\mathcal{O}_2) := \{\omega \in \mathcal{D}^1(\mathcal{O}_2) \mid k^*\omega = -\omega\}.$$

Let $\mathcal{S}, \mathcal{A} : \mathcal{D}^1(\mathcal{O}_2) \longrightarrow \mathcal{D}^1(\mathcal{O}_2)$ *be defined by:*

$$\left\{ \begin{array}{l} \mathcal{S}\omega := \omega_s := \dfrac{1}{2}\,(\omega + k^*\omega) \\[2ex] \mathcal{A}\omega := \omega_a := \dfrac{1}{2}\,(\omega - k^*\omega). \end{array} \right. \tag{24}$$

The next result is analogous to Theorem 3.

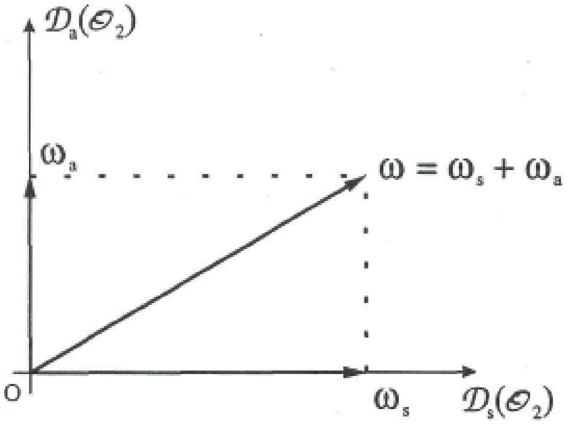

Fig. 3:

Theorem 8 *The following properties of* $\mathcal{S}$ *and* $\mathcal{A}$ *hold:*
a) $\mathcal{S}$ *and* $\mathcal{A}$ *are linear operators and their images are* $\mathcal{I}m(\mathcal{S}) = \mathcal{D}^1_s(\mathcal{O}_2)$ *and* $\mathcal{I}m(\mathcal{A}) = \mathcal{D}^1_a(\mathcal{O}_2)$*;*
b) For $\omega \in \mathcal{D}^1(\mathcal{O}_2)$,

$$\left\{ \begin{array}{l} \mathcal{S}\omega = \omega \Leftrightarrow \omega \in \mathcal{D}^1_s(\mathcal{O}_2) \\[2ex] \mathcal{A}\omega = \omega \Leftrightarrow \omega \in \mathcal{D}^1_a(\mathcal{O}_2); \end{array} \right.$$

c) $\mathcal{S}$ *and* $\mathcal{A}$ *are ortogonal projectors:*
$\mathcal{S} \circ \mathcal{S} = \mathcal{S}$, $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$ *and* $\mathcal{S} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{S} = O$*;*

d) $\mathcal{D}^1(\mathcal{O}_2) = \mathcal{D}^1_s(\mathcal{O}_2) \bigoplus \mathcal{D}^1_a(\mathcal{O}_2)$*;*

e) If $\mathcal{I}$ *is the identity of* $\mathcal{D}^1(\mathcal{O}_2)$ *,*

$$\mathcal{I} = \mathcal{S} + \mathcal{A},$$

that is $\mathcal{S}$ *and* $\mathcal{A}$ *provide an orthogonal decomposition of the identity of* $\mathcal{D}^1(\mathcal{O}_2)$*.*

The proof is similar with that of Theorem 3, and will be omited.
Fig. 3 gives a geometric image of the space $\mathcal{D}^1(\mathcal{O}_2)$.

8 The line integral.

In this section we define the line integral on $\mathbf{X}$. The definition of the line integral is reduced to the well established line integral on the orientable surface $\mathcal{O}_2$.
We start out with the following important theorem:

Theorem 9 *Let* Δ *be a symmetric open subset of* $\mathcal{O}_2$ *and* $\gamma : [0,1] \longrightarrow \Delta$ *a picewise continuously differentiable curve in* Δ*. Let* $\omega \in \mathcal{D}^1_s(\Delta)$ *be continuous. Then*

$$\int_\gamma \omega = \int_{k\circ\gamma} \omega.$$

Proof: (Intuitively everything is obvious). There exists a division

$$0 = t_0 < t_1 < ... < t_i < t_{i+1} < ... < t_n = 1$$

such that the restriction γ_i of γ to $I_i := [t_i, t_{i+1}]$ is differentiable (with the derivative γ' continuous) and such that $\gamma(I_i) = \gamma_i(I_i)$ lies in a single parametric disk. Then the curve $k \circ \gamma$ has similar properties. To prove the theorem it is enough to consider the case when γ is of class C^1 and $\gamma([0,1])$ lies in the disk D. Then $k \circ \gamma$ lies on the disk kD. Since everything happens in D and kD (and we suppose that $D \cap kD = \emptyset$),we consider that the disks lie in complex planes. In this way we have reduced the problem to the following one:
Let D be a disk in the plane of the complex variable $z = x + iy$,
let $k :\longrightarrow kD$ be an anticonformal representation, $w = kz = u + iv$, and let ω be a continuous symmetric differential form defined on $D \cup kD$. Let $\gamma : [0,1] \longrightarrow D$ be differentiable with the derivative γ' continuous and $\delta(t) = k(\gamma(t))$ for $t \in [0,1]$.
Under these circumstances,

$$\int_\gamma \omega = \int_\delta \omega.$$

The form ω is given as in (19) and the coefficients A, B, A', B' satisfy (20) and are continuous.
Then, according to the usual definition of the line integral in the plane, one has:

$$\int_\delta \omega := \int_0^1 h(t)dt \quad \text{and} \int_\gamma \omega := \int_0^1 g(t)dt,$$

where

$$(i)\ \ h(t)dt = \delta^*(\omega)(t) = \left[A'(\delta(t))\ \frac{d\delta}{dt}(t) + B'(\delta(t))\ \frac{d\bar{\delta}}{dt}(t)\right] dt \quad \text{and}$$

$$(ii)\ \ g(t)dt = \gamma^*(\omega)(t) = \left[A(\gamma(t))\ \frac{d\gamma}{dt}(t) + B(\gamma(t))\ \frac{d\bar{\gamma}}{dt}(t)\right] dt.$$

We obtain, successively:

$$h(t) = A'(\delta(t))\ \frac{\partial k}{\partial \bar{z}}(\gamma(t))\ \frac{d\bar{\gamma}}{dt}(t) + B'(\delta(t))\ \frac{\partial \bar{k}}{\partial z}(\gamma(t))\ \frac{d\gamma}{dt}(t) =$$

$$= B(\gamma(t))\ \frac{d\bar{\gamma}}{dt}(t) + A(\gamma(t))\ \frac{d\gamma}{dt}(t) = g(t).$$

Thus, $\int_\gamma \omega = \int_\delta \omega$, *and the proof is complete.* □

Corollary 3 *In the context of the previous theorem, if* $\omega \in \mathcal{D}^1_a(\mathcal{O}_2)$ *then*

$$\int_{k\circ\gamma} \omega = -\int_\gamma \omega.$$

Proof: With the same notations, one gets $h(t) = -g(t)$ *as a consequence of (20) with* $\epsilon = -1$ □

Theorem 9 is to be seen as a partial result of the following characterization theorem:

Theorem 10 *The following statements, in the hypotheses of Theorem 9, are equivalent:*
a) $\omega \in \mathcal{D}^1_s(\mathcal{O}_2)$;
b) $\int_\gamma \omega = \int_{k\circ\gamma} \omega$ *for any* γ;
c) $\int_\gamma \omega_a = 0$;
d) $\omega_a = 0$.

Proof: "a) ⇒ b)". This is the content of Theorem 9.
"b) ⇒ c)". By using b) and Corollary 3, one gets, succesively :

$$\int_\gamma \omega_s + \int_\gamma \omega_a = \int_\gamma (\omega_s + \omega_a) = \int_\gamma \omega = \int_{k\circ\gamma} \omega = \int_{k\circ\gamma} (\omega_s + \omega_a) = \int_\gamma \omega_s - \int_\gamma \omega_a.$$

Thus, $\int_\gamma \omega_a = -\int_\gamma \omega_a$, *which means nothing but c).*
"c) ⇒ d)". The form ω *is supposed continuous. Thus* ω_a *is continuous. If* $\omega_a \neq 0$ *there exists a continuously differentiable path* $\gamma : [0,1] \longrightarrow \Delta$ *with the carrier* $\gamma([0,1])$ *lying in a single parametric disk D, such that* $\int_\gamma \omega_a \neq 0$, *a contradiction with c).*
"d) ⇒ a)". $\omega = \omega_s + \omega_a = \omega_s \in \mathcal{D}^1_s(\mathcal{O}_2)$ □

We are now able to define the line integral on $\mathbf{X}$.

Let $\widetilde{\Delta}$ *be an open subset of* $\mathbf{X}$ *and* $\Delta = p^{-1}(\widetilde{\Delta})$. *Let* $\widetilde{\omega}$ *be a continuous differential form on* $\widetilde{\Delta}$ *and* $\omega = p^*\widetilde{\omega}$ *be the pull-back of* ω *by* p.
Then $\omega \in \mathcal{D}^1_s(\Delta)$ *and is continuous. Let* $\delta : [0,1] \longrightarrow \widetilde{\Delta}$ *be a piecewise continuously differentiable curve in* $\widetilde{\Delta}$. *The curve* δ *has exactly two liftings on* Δ. *If* $\delta(0) = \widetilde{z}_0 = \{z_0; w_0 = kz_0\}$ *and if* $\gamma : [0,1] \longrightarrow \Delta$ *is the lifting of* δ *at* z_0, *then* $k \circ \gamma$ *is the lifting of* δ *at* kz_0.

Definition 8 *The integral of* $\widetilde{\omega}$ *on the curve* δ *is the common value of the integrals* $\int_\gamma \omega$ *and* $\int_{k\circ\gamma}\omega$:

$$\int_\delta \widetilde{\omega} := \int_\gamma \omega = \int_{k\circ\gamma} \omega, \quad \delta = p \circ \gamma. \tag{25}$$

Thus, with the earlier notations,

$$\int_{p\circ\gamma} p_{\#}(\omega) := \int_\gamma \omega = \int_{k\circ\gamma} \omega,$$

the second equality being guaranteed by Theorem 9.

The properties of the line integral on N.R.S. are, according to its definition, the same as the ones of the line integral on orientable Riemann surfaces. For example:

$$\int_{\widetilde{\gamma}_1+\widetilde{\gamma}_2} \widetilde{\omega} = \int_{\widetilde{\gamma}_1} \widetilde{\omega} + \int_{\widetilde{\gamma}_2} \widetilde{\omega}; \ \int_{\widetilde{\gamma}^{-1}} \widetilde{\omega} = -\int_{\widetilde{\gamma}} \widetilde{\omega}.$$

9 N-meromorphic differential forms.

We consider now the elements of $\mathcal{D}^1(\mathcal{O}_2)$ *which occur by symmetrizing the meromorphic or antimeromorphic differentials on* $\mathcal{O}_2$ *(the abelian differentials).*

Let $\mathcal{M}^1(\mathcal{O}_2)$ ($\overline{\mathcal{M}^1(\mathcal{O}_2)}$) *be the vector space of meromorphic (antimeromorphic) differential forms on* $\mathcal{O}_2$. *We shall consider the pair* D, kD *of disjoint symmetric parametric disks and* $k\,|_{D\cup kD}$ *looking like in Fig. 2.*

Let $\eta \in \mathcal{M}^1(\mathcal{O}_2)$ *and* $\theta \in \overline{\mathcal{M}^1(\mathcal{O}_2)}$) *be two differentials.*
Suppose that

$$\eta = \begin{cases} f(z)dz, & z \in D \\ g(w)dw, & w \in kD \quad \text{and} \end{cases} \tag{26}$$

$$\theta = \begin{cases} f_1(z)d\bar{z}, & z \in D \\ g_1(w)d\bar{w}, & w \in kD. \end{cases} \tag{27}$$

Note 1. In the case when f is not holomorphic but meromorphic (i.e. it has at least one pole in D), the notation "$z \in D$" means "$z \in D$ and z is not a pole of f".
Note 2. We eliminate the specifications "$z \in D$" and "$w \in kD$" in the writing of the restrictions to $D \cup kD$ of differential forms.

The pull-backs of η and θ by k are

$$k^*\eta = \begin{cases} g(kz)d(kz) \\ f(kw)d(kw) \end{cases} = \begin{cases} g(kz)\, \dfrac{\partial k}{\partial \bar{z}}(z)d\bar{z} \\ f(kw)\, \dfrac{\partial k}{\partial \bar{w}}(w)d\bar{w} \end{cases}, \tag{28}$$

$$k^*\theta = \begin{cases} g_1(kz)d(\bar{k}z) \\ f_1(kw)d(\bar{k}w) \end{cases} = \begin{cases} g_1(kz)\, \dfrac{\partial \bar{k}}{\partial z}(z)dz \\ f_1(kw)\, \dfrac{\partial \bar{k}}{\partial w}(w)dw \end{cases}. \tag{29}$$

We see that $k^*\eta$ is antimeromorphic and $k^*\theta$ is meromorphic.
The symmetric components η_s and θ_s are given by:

$$2\eta_s = \begin{cases} f(z)dz + g(kz)\, \dfrac{\partial k}{\partial \bar{z}}(z)d\bar{z} \\ g(w)dw + f(kw)\, \dfrac{\partial k}{\partial \bar{w}}(w)d\bar{w} \end{cases} \tag{30}$$

and

$$2\theta_s = \begin{cases} g_1(kz)\, \dfrac{\partial \bar{k}}{\partial z}(z)dz + f_1(z)d\bar{z} \\ f_1(kw)\, \dfrac{\partial \bar{k}}{\partial w}(w)dw + g_1(w)d\bar{w} \end{cases}. \tag{31}$$

Thus, the mixing of the two analytic structures of $\mathcal{O}_2$ mentioned in Section 3 is reflected also in the structure of the differential forms lying in $\mathcal{S}(\mathcal{M}^1(\mathcal{O}_2))$ and $\mathcal{S}(\overline{\mathcal{M}^1(\mathcal{O}_2)})$. By means of (29) and (30) we are led to the following analogous of Theorem 4:

Theorem 11 $\mathcal{S}(\mathcal{M}^1(\mathcal{O}_2)) = \mathcal{S}(\overline{\mathcal{M}^1(\mathcal{O}_2)})$.

Proof: $\dfrac{1}{2}[\eta + k^*\eta] = \dfrac{1}{2}[k^*\eta + k^*(k^*\eta)]$

and $\eta \in \mathcal{M}^1(\mathcal{O}_2) \Longleftrightarrow k^*\eta \in \overline{\mathcal{M}^1(\mathcal{O}_2)}$ □

We are now led to the definition of the "abelian" differentials on $\mathbf{X}$.

Definition 9 *The map* $\widetilde{\omega} : \mathbf{X} \longrightarrow \mathbf{T}^*(\mathbf{X})$ *is called an N-meromorphic differential form if* $p^*\widetilde{\omega} \in \mathcal{S}(\mathcal{M}^1(\mathcal{O}_2))$.

Some information about these differential forms can be found in [3].

10 Differential forms of degree 2.

We establish now, for differential forms of degree 2 on $\mathcal{O}_2$, *the results which are similar to those given in Sections 5 and 7 for the differential forms of degree 1.*
Let Δ *be a symmetric open subset of* $\mathcal{O}_2$ and

$$\Omega : \Delta \longrightarrow \Lambda^2 \left(\mathbf{T}^* \left(\mathcal{O}_2\right)\right)$$

a differential form of degree 2.

$\Lambda^2 \left(\mathbf{T}^* \left(\mathcal{O}_2\right)\right) := \bigcup_{z \in \mathcal{O}_2} \Lambda^2 \left(\mathbf{T}_z^* \left(\mathcal{O}_2\right)\right)$ and

$\Lambda^2 \left(\mathbf{T}_z^* \left(\mathcal{O}_2\right)\right)$ *is the 2-nd exterior power of the cotangent space of* $\mathcal{O}_2$ at z.

If $P_0 \in \Delta$ *and if* z *is a local uniformiser in a neighborhood* V *of* P_0 *then, in* V, Ω *can be written as*

$$\Omega(z) = A(z)\, dz \wedge d\bar{z},$$

where $dz \wedge d\bar{z} = 2i dx \wedge dy$ *is the usual base of the 1-dimentional vector space of alternating tensors of degree 2 on* $\mathbf{C} \times \mathbf{C}$.

Suppose that the restriction of Ω *to* $D \cup kD$ *is given by*

$$\Omega|_{D \cup kD} = \begin{cases} A(z)\, dz \wedge d\bar{z} & , z \in D \\ A'(w)\, dw \wedge d\bar{w} & , w \in kD. \end{cases} \tag{32}$$

[Even in the sequel we eliminate the specifications $z \in D$ and $w \in kD$*].*

Proposition 2 *The restriction of the pull-back by* k *of* Ω *to* $D \cup kD$ *is given by*

$$(k^*\Omega)|_{D \cup kD} = \begin{cases} A'(kz)\, \dfrac{D(u,v)}{D(x,y)}(z)\, dz \wedge d\bar{z} \\ A(kw)\, \dfrac{D(x,y)}{D(u,v)}(w)\, dw \wedge d\bar{w}. \end{cases} \tag{33}$$

Proof: According to the definition of the pull-back mapping,

$$(k^*\Omega)|_D(z) = k|_D^*\,(\Omega|_{kD})\,(z) :=$$
$$= A'(kz)d(kz)\wedge d(\bar{k}z) = A'(kz)\left(\frac{\partial k}{\partial\bar{z}}(z)d\bar{z}\right)\wedge\left(\frac{\partial\bar{k}}{\partial z}(z)dz\right) =$$
$$= A'(kz)\left(-\frac{\partial k}{\partial\bar{z}}(z)\frac{\partial\bar{k}}{\partial z}(z)\right)dz\wedge d\bar{z} = A'(kz)\,\frac{D(u,v)}{D(x,y)}(z)\,dz\wedge d\bar{z}.$$

In the same way one computes the restriction of $k^*\Omega$ to kD. □

We state now the analogous of Definition 6.

Definition 10 *The differential Ω is called symmetric (antisymmetric) if $k^*\Omega = \Omega$ ($k^*\Omega = -\Omega$, respectively.)*

The following theorem characterizes the two types of differentials mentioned in the previous definition.

Theorem 12 *The following statements are equivalent:*
a) Ω is symmetric (antisymmetric, respectively);
b) $\Omega|_{D\cup kD}$ is symmetric (antisymmetric, respectively) for every pair of symmetric disks D and kD;
c) The coefficients A and A' satisfy :

$$\text{(34)}\qquad \begin{cases} A(z) = \epsilon\,A'(kz)\,\dfrac{D(u,v)}{D(x,y)}(z) \\[2ex] A'(w) = \epsilon\,A(kw)\,\dfrac{D(x,y)}{D(u,v)}(w). \end{cases}$$

with $\epsilon = 1$ ($\epsilon = -1$, respectively).

Proof: The equivalence "a) $\Longleftrightarrow$ b)" is obvious and the equivalence "b) $\Longleftrightarrow$ c)" follows from Proposition 2 and the definitions. □

Notations:
$\mathcal{D}^2(\Delta)$ =*the vector space of the differential forms of degree 2 on* Δ;

$$\mathcal{D}_s^2(\Delta) := \{\Omega\in\mathcal{D}^2(\Delta)\mid k^*\Omega = \Omega\};$$
$$\mathcal{D}_a^2(\Delta) := \{\Omega\in\mathcal{D}^2(\Delta)\mid k^*\Omega = -\Omega\}.$$

The symmetrization operator $\mathcal{S}$ and the antisymmetrization operator $\mathcal{A}$ are defined, as expected, by:

$$\text{(35)}\qquad \begin{cases} \mathcal{S}\Omega := \Omega_s := \dfrac{1}{2}(\Omega + k^*\Omega) \\[2ex] \mathcal{A}\Omega := \Omega_a := \dfrac{1}{2}(\Omega - k^*\Omega) \end{cases}$$

for every $\Omega \in \mathcal{D}^2(\Delta)$.

We mention now the analogous of the Theorem 8 :

Theorem 13 *The following properties of* $\mathcal{S}$ *and* $\mathcal{A}$ *hold:*
a) $\mathcal{S}$ *and* $\mathcal{A}$ *are linear operators and their images are:*
$\mathcal{I}m(\mathcal{S}) = \mathcal{D}^2_s(\Delta)$ *and* $\mathcal{I}m(\mathcal{A}) = \mathcal{D}^2_a(\Delta)$;

b) For $\Omega \in \mathcal{D}^2(\Delta)$, $\Longrightarrow \begin{cases} \Omega \in \mathcal{D}^2_s(\Delta) \Longleftrightarrow \mathcal{S}\Omega = \Omega \\ \\ \Omega \in \mathcal{D}^2_a(\Delta) \Longleftrightarrow \mathcal{A}\Omega = \Omega; \end{cases}$

c) $\mathcal{S}$ *and* $\mathcal{A}$ *are orthogonal projectors:*
$\mathcal{S} \circ \mathcal{S} = \mathcal{S}$, $\mathcal{A} \circ \mathcal{A} = \mathcal{A}$ *and* $\mathcal{S} \circ \mathcal{A} = \mathcal{A} \circ \mathcal{S} = O$;

d) $\mathcal{D}^2(\Delta) = \mathcal{D}^2_s(\Delta) \bigoplus \mathcal{D}^2_a(\Delta)$;

e) $\mathcal{S}$ *and* $\mathcal{A}$ *provide an orthogonal decomposition of the identity* $\mathcal{I} = \mathcal{S} + \mathcal{A}$ *of* $\mathcal{D}^2(\Delta)$.

11 The pull-back by p of the differential forms of degree 2.

In this section we shall see how the space $\mathcal{D}^2_{\overline{s}}(\mathcal{O}_2)$ *represents the space* $\mathcal{D}^2(\mathbf{X})$. *Let* $\widetilde{\Delta}$ *be an open subset of* $\mathbf{X}$ *and* $\Delta = p^{-1}(\widetilde{\Delta})$. *Let* $\widetilde{\Omega}$ *be a differential form of degree 2 on* $\widetilde{\Delta}$:

$$\widetilde{\Omega} : \widetilde{\Delta} \longrightarrow \Lambda^2\left(\mathbf{T}^*(\mathbf{X})\right) := \bigcup_{\widetilde{z} \in \mathbf{X}} \Lambda^2\left(\mathbf{T}^*_{\widetilde{z}}(\mathbf{X})\right).$$

To avoid confusions we present the subject in detail. It is enough to do a local study.

Let $\widetilde{D} \subseteq \widetilde{\Delta}$ *be a parametric disk such that*

$$p^{-1}(\widetilde{D}) = D \cup kD \text{ and } D \cap kD = \emptyset.$$

Assume that z *is a local parameter in* D *and* w *is a local parameter in* kD. *Both* z *and* w *are local parameters in* $\widetilde{D}$ *and* $\widetilde{z} = \widetilde{w} = \{z; w = kz\}$ *for each* $\widetilde{z} \in \widetilde{D}$.

Each space $\Lambda^2\left(\mathbf{T}^*_{\widetilde{z}}(\mathbf{X})\right)$ *has the dimention 1 and the singletons* $\{dz \wedge d\bar{z}\}$ *and* $\{dw \wedge d\bar{w}\}$ *are two bases of this space.*

$$\widetilde{\Omega}(\widetilde{z}) \in \Lambda^2\left(\mathbf{T}^*_{\widetilde{z}}(\mathbf{X})\right) \quad \text{for each } \widetilde{z} \in \widetilde{\Delta}.$$

Thus $\widetilde{\Omega}(\widetilde{z})$ *is a multiple of the vector of each base:*

$$\widetilde{\Omega}(\widetilde{z}) = \widetilde{A}(\widetilde{z})\, dz \wedge d\bar{z} = \widetilde{A}'(\widetilde{w})\, dw \wedge d\bar{w},$$

where $\widetilde{A}, \widetilde{A}' : \tilde{D} \longrightarrow \mathbf{C}$ *are functions.*

With the old notations, (i.e. $z = x + iy; w = kz = u + iv$*),*

$$dw \wedge d\bar{w} = \frac{D(u,v)}{D(x,y)}(z)\; dz \wedge d\bar{z}$$

and thus:

$$\widetilde{A}(\widetilde{z}) = \widetilde{A}'(\widetilde{kz})\; \frac{D(u,v)}{D(x,y)}(z). \tag{36}$$

The pull-back by p *of* $\widetilde{\Omega}$ *is the differential form*

$$p^*\widetilde{\Omega} := \Omega : \Delta \longrightarrow \Lambda^2\left(\mathbf{T}^*\left(\mathcal{O}_2\right)\right)$$

that is defined, on $D \cup kD$ *as in (32) where:*

$A(z) := \widetilde{A}(\widetilde{z})$ *for every* $z \in D$ *and*

$A'(w) := \widetilde{A}'(\widetilde{w})$ *for every* $w \in kD$.

From (36) one gets

$$A(z) = A'(kz)\; \frac{D(u,v)}{D(x,y)}(z), \tag{37}$$

and, according to Theorem 12, Ω *is symmetric.*

Conversely, if Δ *is a symmetric open subset of* $\mathcal{O}_2$ *and if* Ω *is a symme- tric differential form of degree 2 on* Δ*, then*

$$p_{\#}\Omega := \widetilde{\Omega},$$

where

$$\widetilde{\Omega}(\widetilde{z}) := \Omega(z), \quad \text{for every } z \in \Delta, \tag{38}$$

is a well defined differential form of degree 2 on $\widetilde{\Delta} = p(\Delta)$*, and* $p^*\widetilde{\Omega} = \Omega$.
We summarize the previous results in the following theorem:

Theorem 14 *The sets* $\mathcal{D}^2(\widetilde{\Delta})$ *and* $\mathcal{D}^2_s(\Delta)$, $\left(\Delta = p^{-1}(\widetilde{\Delta})\right)$, *are vector spaces and the pull-back map*

$$p^* : \mathcal{D}^2(\widetilde{\Delta}) \longrightarrow \mathcal{D}^2_s(\Delta)$$

is an isomorphism whose inverse is $p_{\#}$.

Thus, one can identify the space of differential forms of degree 2 on $\widetilde{\Delta}$ *with the space of* **symmetric** *differential forms of degree 2 on* Δ*,by means of (38):*

$$\tilde{\Omega} \equiv \Omega. \;\square$$

Examples.

Let us consider the annulus $\mathbf{A}$ *of module* $R > 1$ *represented as*

$$\mathbf{A} = \{z \in \mathbf{C} \mid \frac{1}{r} < |z| < r\},$$

where $r = \sqrt{R}$ *and let*

$$k : \mathbf{A} \longrightarrow \mathbf{A}, \quad kz = -\frac{1}{\bar{z}}.$$

k *is an antianalytic involution without fixed points of* $\mathbf{A}$ *and with* $\mathcal{O}_2 = \mathbf{A}$*, the pair* $(\mathcal{O}_2, k)$ *is a symmertic Riemann surface.*
The orbit space $\mathcal{O}_2/\mathcal{H}$, $\mathcal{H} = \{Identity; k\}$*, is the Möbius band*

$$\mathbf{M} = \{\tilde{z} \mid \tilde{z} = \{z; kz\}; z \in \mathbf{A}\}.$$

We formulate:

Proposition 3 *The following lists give the sets of* **all** *functions, differential forms of degree 1 and differential forms of degree 2 on* $\mathbf{M}$*:*
a) $\tilde{f}(\tilde{z}) = f(z) + f(-\frac{1}{\bar{z}})$*, where* $f : \mathbf{A} \longrightarrow \mathbf{C}$ *is arbitrary;*
b) $\tilde{\omega}(\tilde{z}) = \alpha(z)dz + \frac{1}{\bar{z}^2}\alpha(-\frac{1}{\bar{z}})d\bar{z}$*, where* $\alpha : \mathbf{A} \longrightarrow \mathbf{C}$ *is arbitrary;*
c) $\tilde{\Omega}(\tilde{z}) = \frac{1}{2}\left[\lambda(z) - \frac{1}{|z|^4}\lambda(-\frac{1}{\bar{z}})\right] dz \wedge d\bar{z}$*,with* $\lambda : \mathbf{A} \longrightarrow \mathbf{C}$ *arbitrary.*

Proof: The point a) follows from Theorem 2 and the relation (3) on p.5 applied to $\Delta = \mathcal{O}_2 = \mathbf{A}$.

To prove b) we use Theorem 7:
For $\tilde{\omega} : \mathbf{M} \longrightarrow \mathbf{T}^\star(\mathbf{M})$ *there exists* $\omega : \mathbf{A} \longrightarrow \mathbf{T}^\star(\mathbf{A})$ **symmetric** *such that*

$$\tilde{\omega}(\tilde{z}) = \omega(z) \quad \text{for every} \quad \tilde{z} \in \mathbf{M}.$$

The differential form ω *is given by:*

$$\omega(z) = \alpha(z)dz + \beta(z)d\bar{z}, \tag{39}$$

where $\alpha, \beta : \mathbf{A} \longrightarrow \mathbf{C}$ are functions. We obtain, successively:

$$(k^\star\omega)(z) = \alpha(-\frac{1}{\bar{z}})d(-\frac{1}{\bar{z}}) + \beta(-\frac{1}{\bar{z}})d(-\frac{1}{z}) = \alpha(-\frac{1}{\bar{z}})\frac{1}{\bar{z}^2}d\bar{z} + \beta(-\frac{1}{\bar{z}})\frac{1}{z^2}dz.$$

The condition $\omega = k^\star\omega$ gives:

$$\beta(z) = \frac{1}{\bar{z}^2}\alpha(-\frac{1}{\bar{z}}).$$

From (39) we obtain:

$$\tilde{\omega}(\tilde{z}) = \alpha(z)dz + \frac{1}{\bar{z}^2}\alpha(-\frac{1}{\bar{z}})d\bar{z},$$

which means that $\widetilde{\omega}$ *has the form announced in b).*
Conversely, if $\omega(z) = \alpha(z)dz + \frac{1}{\bar{z}^2}\alpha(-\frac{1}{\bar{z}})d\bar{z}$ *we have* $\omega(z) = \omega(-\frac{1}{\bar{z}})$,
i.e. $\widetilde{\omega}$, *given by*

$$\widetilde{\omega}(\tilde{z}) := \omega(z),$$

is a well-defined differential form on $\mathbf{M}$. *This means that b) gives the* **complete** *list of differential forms of degree 1 on* $\mathbf{M}$.
To prove c) we shall use Theorem 14 with $\widetilde{\Delta} = \mathbf{M}$:
$\widetilde{\Omega}$ *is a differential form of degree 2 on* $\mathbf{M}$ *if and only if there exists a* **symmetric** *differential form* Ω *on* $\mathbf{A}$, *having degree 2, such that*

$$\widetilde{\Omega}(\tilde{z}) = \Omega(z)$$

for every $z \in \mathbf{A}$.

According to Theorem 13, $\Omega = k^*\Omega$ *if and only if there exists* $\Lambda \in \mathcal{D}^2(\mathbf{A})$ *such that*

$$\Omega = \Lambda_s = \frac{1}{2}(\Lambda + k^*\Lambda).$$

Notice that $\Lambda(z) = \lambda(z)dz \wedge d\bar{z}$, *with* $\lambda : \mathbf{A} \longrightarrow \mathbf{C}$. *We obtain:*

$$(k^*\Lambda)(z) = \lambda(-\frac{1}{\bar{z}})d(-\frac{1}{\bar{z}}) \wedge d(-\frac{1}{z}) = -\frac{1}{|z|^4}\lambda(-\frac{1}{\bar{z}})dz \wedge d\bar{z},$$

and finally

$$\Omega(z) = \Lambda_s(z) = \frac{1}{2}\left[\lambda(z) - \frac{1}{|z|^4}\lambda(-\frac{1}{\bar{z}})\right] dz \wedge d\bar{z},$$

which is the expression announced in c).
Conversely, if $\lambda : \mathbf{A} \longrightarrow \mathbf{C}$ *is an arbitrary function and*

$$\Omega(z) = \frac{1}{2}\left[\lambda(z) - \frac{1}{|z|^4}\lambda(-\frac{1}{\bar{z}})\right] dz \wedge d\bar{z},$$

then clearly $\Omega(z) = \Omega(kz)$, *and thus*

$$\widetilde{\Omega}(\tilde{z}) := \Omega(z)$$

is a well-defined differential form on $\mathbf{M}$.

In conclusion, c) gives the complete list of differential forms of degree 2 on $\mathbf{M}$.

REMARK: *Prof.***C.Constantinescu** *from ETH-Zentrum,Zürich, has observed that the vector space of the odd de Rham differential forms ([12]p.19) of degree* $r \in \{0, 1, 2\}$ *on* $\mathbf{X}$, *is isomorphic with the vector space of* **antisymmetric** *differential forms on* $\mathcal{O}_2$, *of the same degree* r. *This way we have a very simple description of the odd de Rham differential forms on the N.R.S* $\mathbf{X}$. *For example, the vector space of odd de Rham functions on the Möbius band* $\mathbf{M}$ *is isomorphic with the vector space of antisymmetric functions on* $\mathbf{A}$:

$$h(z) - h(-\frac{1}{\bar{z}})$$

with $h : \mathbf{A} \longrightarrow \mathbf{C}$ *arbitrary.*

Exercises.

We shall denote the functions on $\mathcal{O}_2$ (differential forms of degree zero) by lower-case Latin letters, the differentials of degree 1 by lower-case Greek letters and the differentials of degree 2 by capital Greek letters. The notations f_s, f_a etc. will be used for the symmetric and antisymmetric components of f, etc.

It is easy to verify the following list of properties:

$$(40)\qquad \begin{cases} (fg)_s = f_s g_s + f_a g_a \\ (fg)_a = f_s g_a + f_a g_s \quad \text{(See Th. 1.3)}; \end{cases}$$

$$(41)\qquad \begin{cases} (f\omega)_s = f_s\omega_s + f_a\omega_a \\ (f\omega)_a = f_s\omega_a + f_a\omega_s \quad ; \end{cases}$$

$$(42)\qquad \begin{cases} (f\Omega)_s = f_s\Omega_s + f_a\Omega_a \\ (f\Omega)_a = f_s\Omega_a + f_a\Omega_s \quad ; \end{cases}$$

$$(43)\qquad \begin{cases} (\omega\wedge\eta)_s = \omega_s\wedge\eta_s + \omega_a\wedge\eta_a \\ (\omega\wedge\eta)_a = \omega_s\wedge\eta_a + \omega_a\wedge\eta_s. \end{cases}$$

Corollaries:

1) $f, g \in \mathcal{F}_s(\mathcal{O}_2) \Rightarrow fg \in \mathcal{F}_s(\mathcal{O}_2)$;

2) $f \in \mathcal{F}_s(\mathcal{O}_2)$ and $\omega \in \mathcal{D}^1_s(\mathcal{O}_2) \Rightarrow f\omega \in \mathcal{D}^1_s(\mathcal{O}_2)$;

3) $f \in \mathcal{F}_s(\mathcal{O}_2)$ and $\Omega \in \mathcal{D}^2_s(\mathcal{O}_2) \Rightarrow f\Omega \in \mathcal{D}^2_s(\mathcal{O}_2)$;

4) $\omega, \eta \in \mathcal{D}^1_s(\mathcal{O}_2) \Rightarrow \omega\wedge\eta \in \mathcal{D}^2_s(\mathcal{O}_2)$.

From $dx = \frac{1}{2}[dz + d\bar{z}]$ *and* $dy = \frac{1}{2i}[dz - d\bar{z}]$ *one gets:*

$$(44)\qquad \begin{cases} (dx)_s = \frac{1}{4}\left(1 + \frac{\partial\bar{k}}{\partial z}\right)dz + \frac{1}{4}\left(1 + \frac{\partial k}{\partial\bar{z}}\right)d\bar{z} \\ (dy)_s = \frac{1}{4i}\left(1 - \frac{\partial\bar{k}}{\partial z}\right)dz - \frac{1}{4i}\left(1 - \frac{\partial k}{\partial\bar{z}}\right)d\bar{z} \quad \text{and} \end{cases}$$

$$(45)\qquad (dx\wedge dy)_s = \frac{1}{2}\left[1 - \left|\frac{\partial\bar{k}}{\partial z}\right|^2\right]dx\wedge dy = \frac{1}{2}\left[1 + \frac{D(u,v)}{D(x,y)}\right]dx\wedge dy\square$$

12 The double integral.

This section deals with the definition of the double integral on **X**. *As in Section 8, this definition is given by means of the double integral on the orientable surface* $\mathcal{O}_2$.
To define and study the double integral on the N.R.S. **X** *we can use the double integral on either triangles or squares (rectangles) in the euclidian plane.*

By the standard triangle or square in the plane of the variable $t = t_1 + it_2$ *we mean the triangle* δ *or the square* s *given by:*

$$(46)\qquad \begin{cases} \delta := \{t = t_1 + it_2 \mid 0 \leq t_1, t_2 \leq 1,\ \ t_1 + t_2 \leq 1\} \\ \\ s := \{t = t_1 + it_2 \mid 0 \leq t_1, t_2 \leq 1\}. \end{cases}$$

The boundaries of δ and s will have the counterclockwise orientations as their positive orientations.

Definition 11 *A singular triangle (square) or, simply, a triangle (square, respectively) on the manifold* M, *is any function of class* C^1

$$c : \delta \longrightarrow M \ \ (S : s \longrightarrow M, respectively).$$

Definition 12 *A chain of triangles (squares) on* M *is any element of the free group generated by the singular triangles (squares) on* M.

Thus, a chain C of triangles (squares) is a formal sum

$$C = \sum_{i \in I} n_i c_i \ \ (resp.\ C = \sum_{i \in I} n_i S_i),$$

where I is a finite set (depending on C), $n_i \in \mathbf{Z}$ (the ring of real integers) and c_i (S_i, respectively) is a singular triangle (square) on M for every $i \in I$. In the discussion that follows we use triangles; everything can be formulated in terms of squares as well.

To define the double integral on triangles on **X** it is enough to consider only the case where every triangle can be covered by a single parametric disk $\widetilde{D}$ (depending on the triangle) that is evenly covered by p; i.e. its preimage by p consists of disks D and kD, and $D \cap kD = \emptyset$.

If $\widetilde{T}$ is a triangle in $\widetilde{D}$ then $p^{-1}(\widetilde{T})$ consists of triangle $T \subseteq D$ and the image of T by k, $kT \subseteq kD$.

Let $c : \delta \longrightarrow \mathcal{O}_2$ be a singular triangle such that $c(\delta) = T$; then $k \circ c$ is also a singular triangle having as image the triangle $kT \subseteq kD$.

In the definition of the double integral on singular triangles on **X**, the following lemma plays **the crucial role**:

Lemma 1 *Let Ω be a symmetric differential form of degree 2 on $\mathcal{O}_2$. Then the pull-backs of Ω by c and by $k \circ c$ are equal:*

$$c^*(\Omega) = (k \circ c)^*(\Omega).$$

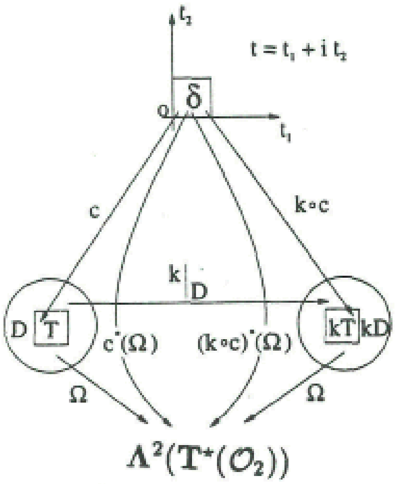

Fig. 4:

Proof: Obviously, we can assume that D lies in the plane of the complex variable z, kD lies in the plane of the variable w, $k : D \longrightarrow kD$ is anticonformal, and Ω is as in (32), the functions A and A' satisfying (34) with $\epsilon = 1$. If

$c(t) = z(t) = x(t) + iy(t)$, $t \in \delta$, *then*

$$c^*(\Omega)(t) := A(z(t)) \, \frac{D(x, y)}{D(t_1, t_2)}(t) \; dt \wedge d\bar{t}.$$

The singular triangle $k \circ c$ is given by :

$$(k \circ c)(t) = u(x(t), y(t)) + iv(x(t), y(t)) = \alpha(t) + i\beta(t).$$

Thus, using (33), we obtain, succesively:

$$(k \circ c)^*(\Omega)(t) = A'(k(z(t))) \, \frac{D(\alpha, \beta)}{D(t_1, t_2)}(t) dt \wedge d\bar{t} =$$

$$= A'(k(z(t))) \, \frac{D(u, v)}{D(x, y)}(z(t)) \, \frac{D(x, y)}{D(t_1, t_2)}(t) \; dt \wedge d\bar{t} =$$

$$= A(z(t)) \, \frac{D(x, y)}{D(t_1, t_2)}(t) \; dt \wedge d\bar{t} = c^*(\Omega)(t).$$

So, $(k \circ c)^*(\Omega) = c^*(\Omega)$. □

Remark: *If we use the functorial properties of "$*$", $(k \circ c)^* = c^* \circ k^*$, the previous proof is implied by:*

$$(k \circ c)^*(\Omega) = c^*(k^*(\Omega)) = c^*(\Omega).$$

Corollary 4 *If Ω is an antisymmetric differential form of degree 2, with the notations of the previous lemma,*

$$(k \circ c)^*(\Omega) = -c^*(\Omega).$$

Proof: Use (34) with $\epsilon = -1$ in the proof of Lemma 1.

Remark:*The equality $h(t)dt = g(t)dt$ we have met in the proof of the Theorem 9 is,for symmetric differential forms of degree 1,the property corresponding to the one given by the previous lemma for the forms of degree 2.*

We remind the definition of the double integral on singular triangles on $\mathcal{O}_2$.

Definition 13 *Let $c : \delta \longrightarrow \mathcal{O}_2$ be a singular triangle lying in the parametric disk D (i.e. $c(\delta) \subseteq D$) and $\Omega : \mathcal{O}_2 \longrightarrow \Lambda^2(\mathbf{T}^*(\mathcal{O}_2))$ a continuous differential form of degree 2. Then, the integral of Ω on c is the complex number*

$$\int\int_c \Omega := \int\int_\delta c^*\Omega.$$

Remark: *If $c(t) = z(t) = x(t) + iy(t)$ and if $\Omega(z) = A(z)\, dz \wedge d\bar{z}$, then*

$$\int\int_\delta c^*\Omega := \int\int_\delta A(z(t))\, \frac{D(x,y)}{D(t_1,t_2)}\, dt \wedge d\bar{t} =$$

$$= \int\int_\delta A(z(t))\, \frac{D(x,y)}{D(t_1,t_2)}\, (-2i) dt_1 dt_2,$$

where $dt_1 dt_2$ is the area element in the plane.

We give now the analogous of Theorem 10.

Theorem 15 *Let Ω be a continuous differential form of degree 2 on $\mathcal{O}_2$. Then the following statements are equivalent:*

1) $\Omega \in \mathcal{D}_s^2(\mathcal{O}_2)$;

2) $\int\int_c \Omega = \int\int_{k \circ c} \Omega$ *for every singular triangle c such that $c(\delta) := T$ can be covered by a single parametric disk;*

3) $\int\int_c \Omega_a = 0$ *for every c of the previous type;*

4) $\Omega_a = 0$.

Proof: The implication "1) $\Rightarrow$ 2)" is a consequance of Lemma 1 and of the definition of the integral on singular triangles.

"2) $\Rightarrow$ 3)". *We have:*

$$\int\int_c \Omega_a + \int\int_c \Omega_s = \int\int_c \Omega = \int\int_{k\circ c} \Omega =$$

$$= \int\int_{k\circ c} \Omega_s + \int\int_{k\circ c} \Omega_a = \int\int_c \Omega_s - \int\int_c \Omega_a;$$

we have used Lemma 1 for $\int\int_{k\circ c}\Omega_s$ *and Corollary 4 for* $\int\int_{k\circ c}\Omega_a$. *Thus,* $\int\int_c \Omega_a = -\int\int_c \Omega_a$.

"3) $\Rightarrow$ 4)" *Assume there exists a point* $P_0 \in \mathcal{O}_2$ *where* Ω_a *is not zero. Assume* z *is a local parameter around* P_0 *and* $\Omega_a(z) = A(z)\, dz \wedge d\bar{z}$. *We may take* $A(P_0) > 0$. *Since* Ω_a *is continuous, there exists a disk* D *with center at* P_0 *such that*

$$\mathcal{R}e\,(A(z)) \geq \frac{1}{2}\, A(P_0),$$

for every $z \in D$. *Let* c *be a triangle in* D. *Then:*

$$\left|\int\int_c \Omega_a\right| = \left|\int\int_\delta c^*(\Omega_a)\right| = \left|\int\int_\delta A(c(t))(-2i)dt_1dt_2\right| =$$

$$= 2\left|\int\int_\delta A(c(t))\; dt_1dt_2\right| \geq 2\mathcal{R}e\int\int_\delta A(c(t))\; dt_1dt_2 = 2\int\int_\delta \mathcal{R}e A(c(t))\; dt_1dt_2 \geq$$

$$\geq 2\int\int_\delta \frac{1}{2}A(P_0)\; dt_1dt_2 = \frac{1}{2}A(P_0) > 0.$$

Thus, $\int\int_c \Omega_a \neq 0$, *a contradiction.*

"4) $\Rightarrow$ 1)". *This is obvious.* ☐

Now we can define the integral on singular triangles and on chains of triangles of the continuous differential forms of degree 2 on the N.R.S. $\mathbf{X}$.

Let $\widetilde{\Omega} \in \mathcal{D}^2(\mathbf{X})$ *be continuous and let* $\widetilde{c} : \delta \longrightarrow \mathbf{X}$ *be a triangle in* $\mathbf{X}$ *such that* $\widetilde{c}(\delta) = \widetilde{T} \subseteq \widetilde{D}$, *where* $\widetilde{D}$ *is a parametric disk such that* $p^{-1}(\widetilde{D}) = D \cup kD$ *and* $D \cap kD = \emptyset$. *Then,*

$$p^*(\widetilde{\Omega}) := \Omega \in \mathcal{D}^2_s(\mathcal{O}_2).$$

Notice that $\widetilde{c}$ *has two liftings at* $\mathcal{O}_2$:

$c = p|_D^{-1} \circ \widetilde{c}$ and $k \circ c = p|_{kD}^{-1} \circ \widetilde{c}$.

According to Theorem 15,

$$\int\int_c p^*(\widetilde{\Omega}) = \int\int_{k\circ c} p^*(\widetilde{\Omega}).$$

We can state now

Definition 14 *The integral of $\widetilde{\Omega}$ on $\widetilde{c}$ is the commun value of the previous integrals:*

$$\int\int_{\widetilde{c}} \widetilde{\Omega} := \int\int_c p^*(\widetilde{\Omega}) = \int\int_{k\circ c} p^*(\widetilde{\Omega}).$$

Definition 15 *Let $\widetilde{\Omega} \in \mathcal{D}^2(\mathbf{X})$ be continuous and let*

$$\widetilde{C} = \sum_{i\in I} n_i \widetilde{c}_i$$

be a chain of triangles on $\mathbf{X}$, each being as $\widetilde{c}$ from the previous definition. Then, the integral of $\widetilde{\Omega}$ on $\widetilde{C}$ is the number

$$\int\int_{\widetilde{C}} \widetilde{\Omega} := \sum_{i\in I} n_i \int\int_{\widetilde{c}_i} \widetilde{\Omega}.$$

Now, the properties of the integral on chains of $\mathbf{X}$ are those of the integral on chains of $\mathcal{O}_2$. Further details are easy to follow. ◻

13 The differential operators d, d_s and d_a.

In this section we give the canonical decomposition of the differential operator d imposed by the symmetry k of $\mathcal{O}_2$. This decomposition does not exist on non-symmetric Riemann surfaces.

Let Δ be a symmetric open subset of $\mathcal{O}_2$ and let $f \in \mathcal{F}^1(\Delta)$ be a function of class C^1. Then $df \in \mathcal{D}^1(\Delta)$ and, according to Theorem 13, it has two components $(df)_s$ and $(df)_a$. We define the operators

$$\begin{cases} d_s : \mathcal{F}^1(\Delta) \longrightarrow \mathcal{D}^1_s(\Delta) \\ d_a : \mathcal{F}^1(\Delta) \longrightarrow \mathcal{D}^1_a(\Delta) \quad \text{by} \end{cases}$$

$$\text{(47)} \qquad \begin{cases} d_s f := (df)_s = \dfrac{1}{2}\left[df + k^*(df)\right] \\ d_a f := (df)_a = \dfrac{1}{2}\left[df - k^*(df)\right]. \end{cases}$$

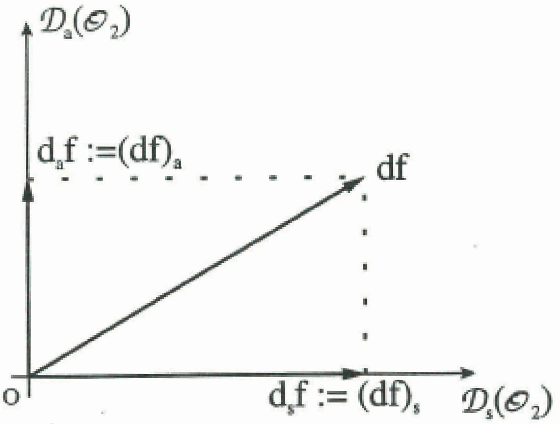

Fig. 5:

It is clear that d_s *and* d_a *are linear operators and one sees that:*

$$d = d_s + d_a.$$

With our old parametric disks D *and* kD*, the restriction to* $D \cup kD$ *of* df *is*

$$df = \begin{cases} \dfrac{\partial f}{\partial z}(z)\, dz + \dfrac{\partial f}{\partial \bar{z}}(z)\, d\bar{z} & , z \in D \\[2ex] \dfrac{\partial f}{\partial w}(w)\, dw + \dfrac{\partial f}{\partial \bar{w}}(w)\, d\bar{w} & , w \in kD. \end{cases}$$

By means of (21) one gets the formulae (where the first equality is classic):

$$(48) \qquad d(k^* f) = k^*(df) = \begin{cases} \dfrac{\partial f}{\partial \bar{w}}(kz)\dfrac{\partial \bar{k}}{\partial z}(z)\, dz + \dfrac{\partial f}{\partial w}(kz)\dfrac{\partial k}{\partial \bar{z}}(z)\, d\bar{z} \\[2ex] \dfrac{\partial f}{\partial \bar{z}}(kw)\dfrac{\partial \bar{k}}{\partial w}(w)\, dw + \dfrac{\partial f}{\partial z}(kw)\dfrac{\partial k}{\partial \bar{w}}(w)\, d\bar{w}. \end{cases}$$

Thus the restriction to D *of* $d_s f$ *is given by*

$$(49) \qquad (d_s f)(z) := (df)_s(z) = \frac{1}{2}[df(z) + k^*(df)(z)] =$$

$$= \frac{1}{2}\left[\frac{\partial f}{\partial z}(z) + \frac{\partial f}{\partial \bar{w}}(kz)\frac{\partial \bar{k}}{\partial z}(z)\right] dz + \frac{1}{2}\left[\frac{\partial f}{\partial \bar{z}}(z) + \frac{\partial f}{\partial w}(kz)\frac{\partial k}{\partial \bar{z}}(z)\right] d\bar{z}.$$

An obvious formula exists for $(d_a f)(z)$*: In the previous paranthesis exchange the "+" sign by "-".*

From (18) and (41) one sees that for f symmetric,

$$d_s f = df,$$

that is the restrictions of d_s and d to the space of symmetric functions are equal.
Remark: *From (41) and the properties of k^* one gets directly*

$$d_s f = df_s \quad \text{and} \quad d_a f = df_a.$$

Let $\omega \in \mathcal{D}^1(\Delta)$ be a differential of class C^1. Suppose ω is given on $D \cup kD$ as in (19).

The exterior differential (or derivative) $d\omega$ of ω is defined (on $D \cup kD$) by:

$$d\omega := \begin{cases} dA(z) \wedge dz + dB(z) \wedge d\bar{z} \\ \\ dA'(w) \wedge dw + dB'(w) \wedge d\bar{w} \end{cases} = \begin{cases} \alpha(z)\, dz \wedge d\bar{z} \\ \\ \alpha'(w)\, dw \wedge d\bar{w}, \quad \text{where} \end{cases}$$

$$\begin{cases} \alpha(z) = \dfrac{\partial B}{\partial z}(z) - \dfrac{\partial A}{\partial \bar{z}}(z) \\ \\ \alpha'(w) = \dfrac{\partial B'}{\partial w}(w) - \dfrac{\partial A'}{\partial \bar{w}}(w). \end{cases}$$

According to (35) we define the exterior symmetric and antisymmetric differentials of ω by

$$\begin{cases} d_s\omega := (d\omega)_s = \dfrac{1}{2}\left[d\omega + k^*(d\omega)\right] \\ \\ d_a\omega := (d\omega)_a = \dfrac{1}{2}\left[d\omega - k^*(d\omega)\right]. \end{cases} \tag{50}$$

It is easy to see that the restriction to D of $d_s\omega$ is given by:

$$(d_s\omega)(z) = \frac{1}{2}\left[\frac{\partial B}{\partial z}(z) - \frac{\partial A}{\partial \bar{z}}(z) + \left(\frac{\partial B'}{\partial w}(kz) - \frac{\partial A'}{\partial \bar{w}}(kz)\right)\frac{D(u,v)}{D(x,y)}(z)\right] dz \wedge d\bar{z}.$$

Remark: *Even for differential forms*

$$d_s\omega = d\omega_s \quad \text{and} \quad d_a\omega = d\omega_a.$$

Now we can state:

Theorem 16 *The following properties hold:*

$$a) \begin{cases} f \in \mathcal{F}_s(\Delta) \Rightarrow df \in \mathcal{D}^1_s(\Delta) \quad \text{and} \quad df = d_s f; \\ \\ f \in \mathcal{F}_a(\Delta) \Rightarrow df \in \mathcal{D}^1_a(\Delta) \quad \text{and} \quad df = d_a f; \end{cases}$$

$$b) \begin{cases} \omega \in \mathcal{D}^1_s(\Delta) \Rightarrow d\omega \in \mathcal{D}^2_s(\Delta) \quad \text{and} \quad d\omega = d_s\omega; \\ \\ \omega \text{ [illegible] } \mathcal{D}^1_a(\Delta) \Rightarrow d\omega \in \mathcal{D}^2_a(\Delta) \quad \text{and} \quad d\omega = d_a\omega. \end{cases}$$

Proof: a) $f \in \mathcal{F}_s(\Delta) \Leftrightarrow k^* f = f$. Thus

$$k^*(df) = d(k^* f) = df.$$

According to Definition 7, df is symmetric. We have also: $d_s f = (df)_s = df$.

$f \in \mathcal{F}_a(\Delta) \Leftrightarrow k^* f = -f$. One gets:

$$k^*(df) = d(k^* f) = d(-f) = -df$$

and, according to the same definition, $df \in \mathcal{D}^1_a(\Delta)$. As a consequence, $d_a f = df$.

b) $\omega \in \mathcal{D}^1_s(\Delta) \Leftrightarrow k^*\omega = \omega \Rightarrow$
$k^*(d\omega) = d(k^*\omega) = d\omega \Rightarrow d\omega \in \mathcal{D}^2_s(\Delta)$ (Definition 10).
The second part of b) can be proved similarly □

14 Harmonic Differentials.

In this section we see how the symmetric harmonic differential forms of degree 1 on $\mathcal{O}_2$ represent the harmonic differential forms on $\mathbf{X}$. In the case of compact N.R.S. we compute the dimension of the space of harmonic differentials in function of the topological genus of the surface.
Let us denote by $\mathbf{H} = \mathbf{H}(\mathcal{O}_2)$ the vector space of harmonic differential forms of degree 1 on $\mathcal{O}_2$.

Thus, $\omega \in \mathbf{H}$ if and only if for every parametric disk $D \subset \mathcal{O}_2$ there exists an harmonic function $f : D \longrightarrow \mathbf{C}$ such that $\omega|_D = df$.

We shall use the operator

$$\Lambda : \mathcal{D}^1(\mathcal{O}_2) \longrightarrow \mathcal{D}^1(\mathcal{O}_2)$$

defined locally by

$$(\Lambda\omega)|_D = A(z)\, dz - B(z)\, d\bar{z} \tag{51}$$

if $\omega|_D = A(z)\, dz + B(z)\, d\bar{z}$. [See the operator "" in [14] p.169].*

Theorem 17 *The operator Λ has the following properties:*

a) It is a linear involution of $\mathcal{D}^1(\mathcal{O}_2)$:

$$\Lambda \circ \Lambda = \mathcal{I} = \text{the identity of } \mathcal{D}^1(\mathcal{O}_2);$$

b)

$$\begin{cases} \Lambda\omega \in \mathcal{D}^1_s(\mathcal{O}_2) \Leftrightarrow \omega \in \mathcal{D}^1_a(\mathcal{O}_2); \\ \Lambda\omega \in \mathcal{D}^1_a(\mathcal{O}_2) \Leftrightarrow \omega \in \mathcal{D}^1_s(\mathcal{O}_2). \end{cases}$$

Proof. It is obvious that Λ is linear and $\Lambda \circ \Lambda = \mathcal{I}$. As a consequence, Λ is an automorphism of $\mathcal{D}^1(\mathcal{O}_2)$.

Now, if ω is given by (19), then

$$(\Lambda\omega)|_{D\cup kD} = \begin{cases} A(z)\,dz - B(z)\,d\bar{z}, & z \in D \\ A'(w)\,dw - B'(w)\,d\bar{w}, & w \in kD. \end{cases}$$

and by means of (21) we get :

$$k^*(\Lambda\omega)|_{D\cup kD} = \begin{cases} -B'(kz)\dfrac{\partial \bar{k}}{\partial z}(z)dz + A'(kz)\dfrac{\partial k}{\partial \bar{z}}(z)d\bar{z}, & z \in D \\ -B(kw)\dfrac{\partial \bar{k}}{\partial w}(w)dw + A(kw)\dfrac{\partial k}{\partial \bar{w}}(w)d\bar{w}, & w \in kD \end{cases} =$$

$=-\Lambda\left(k^*\omega|_{D\cup kD}\right)$.

Thus, we have the formula:

$$k^*(\Lambda\omega) = -\Lambda(k^*\omega),$$

for every $\omega \in \mathcal{D}^1(\mathcal{O}_2)$.
Now, if $\omega \in \mathcal{D}^1_a(\mathcal{O}_2)$, $k^*\omega = -\omega$ and

$$k^*(\Lambda\omega) = -\Lambda(k^*\omega) = \Lambda\omega.$$

This means that $\Lambda\omega \in \mathcal{D}^1_s(\mathcal{O}_2)$.
Conversely, if $\Lambda\omega \in \mathcal{D}^1_s(\mathcal{O}_2)$, $k^*(\Lambda\omega) = \Lambda\omega = -\Lambda(k^*\omega) = \Lambda(-k^*\omega)$ and Λ being injective, $\omega = -k^*\omega$ that is $\omega \in \mathcal{D}^1_a(\mathcal{O}_2)$, etc.□

By means of Λ the harmonic differentials can be characterized as follows:

Theorem 18 *The following statements are equivalent:*
a) $\omega \in \mathbf{H}$;
b) $d\omega = d(\Lambda\omega) = 0$.

Proof: "a) $\Rightarrow$ b)". If $\omega|_D = A(z)\,dz + B(z)\,d\bar{z}$ then

$$(d\omega)|_D = \left(\frac{\partial B}{\partial z}(z) - \frac{\partial A}{\partial \bar{z}}(z)\right)\ dz \wedge d\bar{z},$$

and ω is closed (i.e. $d\omega = 0$) if and only if $\dfrac{\partial B}{\partial z} = \dfrac{\partial A}{\partial \bar{z}}$.

Now, suppose $\omega \in \mathbf{H}$. Then there exists an harmonic function $f : D \longrightarrow \mathbf{C}$ such that

$$\omega|_D = \frac{\partial f}{\partial z}(z)\,dz + \frac{\partial f}{\partial \bar{z}}(z)\,d\bar{z}.$$

In this case

$$\frac{\partial B}{\partial z}(z) - \frac{\partial A}{\partial \bar{z}}(z) = \frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{\partial^2 f}{\partial \bar{z} \partial z} = 0$$

and thus $(d\omega)|_D = 0$.

On the other hand,

$$(\Lambda\omega)|_D = \frac{\partial f}{\partial z}(z)\, dz - \frac{\partial f}{\partial \bar{z}}(z)\, d\bar{z}$$

and, in this case,

$$d(\Lambda\omega)|_D = \left(-\frac{\partial^2 f}{\partial z \partial \bar{z}} - \frac{\partial^2 f}{\partial \bar{z} \partial z}\right)\ dz \wedge d\bar{z} = -8\Delta f\ \ dz \wedge d\bar{z} = 0$$

because $\Delta f = 0$; *(*$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}}$ *is the Laplace's operator).*

"b) $\Rightarrow$ *a)".Suppose that* $d\omega = d(\Lambda\omega) = 0$. *Let* D *be a parametric disk on* $\mathcal{O}_2$. *According to Poincaré's lemma (*$d\omega = 0$ *) there exists* $f : D \longrightarrow \mathbf{C}$
(of class C^2 *) such that* $\omega = df$.

$$(\Lambda\omega)|_D = \frac{\partial f}{\partial z}(z)\, dz - \frac{\partial f}{\partial \bar{z}}(z)\, d\bar{z} \quad \text{and} \quad 0 = d(\Lambda\omega)|_D = -8\Delta f\, dz \wedge d\bar{z}$$

implies $\Delta f = 0$ *that is* f *is harmonic.*□

We denote

$$\mathbf{H}_s := \mathbf{H} \cap \mathcal{D}^1_s(\mathcal{O}_2) \quad \textit{and} \quad \mathbf{H}_a := \mathbf{H} \cap \mathcal{D}^1_a(\mathcal{O}_2).$$

The following direct sum decomposition is obvious:

$$\mathbf{H} = \mathbf{H}_s \bigoplus \mathbf{H}_a.$$

Theorem 19 *If the compact N.R.S.* $\mathbf{X}$ *has topological genus* g *(i.e.* $\mathbf{X}$ *is homeomorphic with the sphere with* g *crosscaps attached), then the subspaces* $\mathbf{H}_s$ *and* $\mathbf{H}_a$ *have the same dimension :*

$$dim(\mathbf{H}_s) = dim(\mathbf{H}_a) = g - 1.$$

Proof. Since $\mathbf{X}$ *is nonorientable,* $g \geq 1$. *From the Hurwitz-Kerékjártó formula concerning the ramification index of the nonconstant morphisms of Riemann surfaces, orientable or not ([9],p.160), the topological genus of* $\mathcal{O}_2$ *is*

$$g' = g - 1.$$

A classic theorem of Hodge ([14] p.205-206) says that $\mathbf{H}$ *is finite dimensional and*

$$dim(\mathbf{H}) = 2g'.$$

Let $\{\omega_1, \omega_2, ..., \omega_m\}$ *be a base of* $\mathbf{H}_s$ *and* $\{\omega_{m+1}, \omega_{m+2}, ..., \omega_{m+n}\}$ *a base of* $\mathbf{H}_a$.

Then $\{\omega_1, \omega_2, ..., \omega_{m+n}\}$ *is a base of* $\mathbf{H}$ *and thus*

$$m + n = 2g'.$$

According to Theorem 17, $\Lambda\omega_j$ *is antisymmetric for* $1 \le j \le m$ *and symmetric for* $m + 1 \le j \le m + n$.
Λ *being an automorphism of* $\mathbf{H}$, $\{\Lambda\omega_j | 1 \le j \le m + n\}$ *is also a base. Now,* $m \le n$ *because* $\{\Lambda\omega_j | 1 \le j \le m\}$ *is a set of linear independent elements of* $\mathbf{H}_a$ *and* $n \le m$ *because* $\{\Lambda\omega_j | m + 1 \le j \le m + n\}$ *is a set of linear independent elements of* $\mathbf{H}_s$. *Thus* $m = n$ *and finally*

$$m = n = g - 1. \square$$

Let us denote by $\mathbf{H}(\mathbf{X})$ *the subspace of* $\mathcal{D}^1(\mathbf{X})$ *that consists of all harmonic differential forms on* $\mathbf{X}$.

Theorem 20 (Hodge Theorem) *If* $\mathbf{X}$ *is compact and has the topological genus* g *then*

$$dim(\mathbf{H}(\mathbf{X})) = g - 1.$$

Proof. The restriction to $\mathbf{H}(\mathbf{X})$ *of the pull-back mapping*

$$p^* : \mathcal{D}^1(\mathbf{X}) \longrightarrow \mathcal{D}^1_s(\mathcal{O}_2)$$

is an isomorphism of $\mathbf{H}(\mathbf{X})$ *onto* $\mathbf{H}_s$. $\square$

15 The de Rham cohomology.

In the case of symmetric Riemann surfaces, the de Rham vector space of the differential forms of degree 1 has a natural decomposition as a direct product. The subspace of symmetric elements is cannonically isomorphic with the de Rham vector space of the covered surface. In this section we deal with this problem.
Let us consider the following vector spaces:

$$\mathcal{F}^2(\mathcal{O}_2) := \{f \in \mathcal{F}(\mathcal{O}_2) \mid f \quad \text{of class} \quad C^2\};$$

$$\mathcal{D}^{1,1}(\mathcal{O}_2) := \{\omega \in \mathcal{D}^1(\mathcal{O}_2) \mid \omega \ \text{of class} \ C^1\};$$

$$\mathcal{D}^{2,0}(\mathcal{O}_2) := \{\Omega \in \mathcal{D}^2(\mathcal{O}_2) \mid \Omega \ \text{of class} \ C^0\}.$$

Let us take the following morphisms of vector spaces:

$$d = d_1 : \mathcal{F}^2(\mathcal{O}_2) \longrightarrow \mathcal{D}^{1,1}(\mathcal{O}_2)$$

and

$$d = d_2 : \mathcal{D}^{1,1}(\mathcal{O}_2) \longrightarrow \mathcal{D}^{2,0}(\mathcal{O}_2),$$

where the first homomorphism d is the usual differentiation of functions and the second one is the exterior differentiation of forms of degree 1.

It is known that

$$\mathcal{I}m\,(d_1) \subseteq Ker\,(d_2)$$

and that the de Rham's cohomology vector space is the factor space

$$\mathbf{R}^1(\mathcal{O}_2) := \frac{Ker\,(d_2)}{\mathcal{I}m\,(d_1)}\,.$$

Further on, if $\mathbf{X}$ *is compact and has the genus* g *then* $\mathcal{O}_2$ *is compact and has the genus* $g' = g - 1$ *and* $\mathbf{R}^1(\mathcal{O}_2)$ *has the dimension ([14] p. 206)*

$$dim(\mathbf{R}^1(\mathcal{O}_2)) = 2g'.$$

Other notations and relations :

$$\mathcal{E} = \mathcal{I}m\,(d_1) = \{df \mid f \in \mathcal{F}^2(\mathcal{O}_2)\} =$$

= *the vector space of exact differential forms of degree 1;*

$$\mathcal{K} = Ker\,(d_2) = \{\omega \in \mathcal{D}^{1,1}(\mathcal{O}_2) \mid d\omega = 0\} =$$

= *the vector space of closed differential forms of degree 1.*

It is obvious that

$$\mathcal{E} = \mathcal{E}_s \bigoplus \mathcal{E}_a,$$

where:
a) $\mathcal{E}_s = \{d_s f \mid f \in \mathcal{F}^2(\mathcal{O}_2)\}$;
b) $\mathcal{E}_a = \{d_a f \mid f \in \mathcal{F}^2(\mathcal{O}_2)\}$.

$$\mathcal{E}_s = \mathcal{D}^1_s(\mathcal{O}_2) \cap \mathcal{D}^{1,1}(\mathcal{O}_2) \cap \mathcal{E} \quad \text{and} \quad \mathcal{E}_a = \mathcal{D}^1_a(\mathcal{O}_2) \cap \mathcal{D}^{1,1}(\mathcal{O}_2) \cap \mathcal{E}.$$

We have also

$$\mathcal{K} = \mathcal{K}_s \bigoplus \mathcal{K}_a$$

(with obvious meanings for the two factors).

It is obvious that

$$\mathcal{E}_s \subseteq \mathcal{K}_s \text{ and } \mathcal{E}_a \subseteq \mathcal{K}_a.$$

Definition 16 *De Rham's symmetric and antisymmetric vector spaces of* $\mathcal{O}_2$ *are the vector spaces:*

$$\mathbf{R}^1_s(\mathcal{O}_2) := \frac{\mathcal{K}_s}{\mathcal{E}_s} \quad \textit{resp.} \quad \mathbf{R}^1_a(\mathcal{O}_2) := \frac{\mathcal{K}_a}{\mathcal{E}_a}.$$

The theorems that follow give some deeper properties of the de Rham's vector spaces of $\mathcal{O}_2$.

Theorem 21 $\mathbf{R}^1(\mathcal{O}_2) \cong \mathbf{R}^1_s(\mathcal{O}_2) \times \mathbf{R}^1_a(\mathcal{O}_2)$.

Proof. Let $\omega = \omega_s + \omega_a \in \mathcal{K}$. Then:
$0 = d\omega = d\omega_s + d\omega_a; \Rightarrow d\omega_s = d\omega_a = 0$ (according to Theorems 13 and 16).
Thus, $\omega_s \in \mathcal{K}_s$ and $\omega_a \in \mathcal{K}_a$ and thereby we can consider the classes

$$\widehat{\omega_s} \in \mathbf{R}^1_s(\mathcal{O}_2) \quad \text{and} \quad \widehat{\omega_a} \in \mathbf{R}^1_a(\mathcal{O}_2).$$

The function

$$\Gamma : \mathbf{R}^1(\mathcal{O}_2) \longrightarrow \mathbf{R}^1_s(\mathcal{O}_2) \times \mathbf{R}^1_a(\mathcal{O}_2),$$
$$\widehat{\omega} \longrightarrow \Gamma(\widehat{\omega}) := (\widehat{\omega_s}, \widehat{\omega_a})$$

is well-defined and it is an isomorphism of vector spaces.

($\omega' \in \widehat{\omega} \Leftrightarrow \omega' = \omega + df = \omega_s + df_s + \omega_a + df_a \Rightarrow \omega'_s = \omega_s + df_s$ and $\omega'_a = \omega_a + df_a; \Rightarrow \widehat{\omega'_s} = \widehat{\omega_s}$ and $\widehat{\omega'_a} = \widehat{\omega_a}$ because $df_s \in \mathcal{E}_s$ and $df_a \in \mathcal{E}_a$. Thus Γ is well-defined, etc.).

Theorem 22 *If the N.R.S.* $\mathbf{X}$ *is compact of genus* g *then the spaces* $\mathbf{R}^1_s(\mathcal{O}_2)$ *and* $\mathbf{R}^1_a(\mathcal{O}_2)$ *have the same dimension:*

$$dim\mathbf{R}^1_s(\mathcal{O}_2) = dim\mathbf{R}^1_a(\mathcal{O}_2) = g - 1.$$

Proof. It is known ([14],p.206) that

$$dim\ \mathbf{R}^1(\mathcal{O}_2) = dim\ \mathbf{H}(\mathcal{O}_2) = 2g' = 2(g-1),$$

where, as in the previous section, $g' = g - 1$ is the topological genus of $\mathcal{O}_2$.

Let $\{\omega_1, \omega_2, ..., \omega_{g-1}\}$ and $\{\eta_1, \eta_2, ..., \eta_{g-1}\}$ be bases for $\mathbf{H}_s = \mathbf{H}_s(\mathcal{O}_2)$ respectively $\mathbf{H}_a = \mathbf{H}_a(\mathcal{O}_2)$. We prove that

$$\{\widehat{\omega_1}, \widehat{\omega_2}, ..., \widehat{\omega_{g-1}}\} \text{ and } \{\widehat{\eta_1}, \widehat{\eta_2}, ..., \widehat{\eta_{g-1}}\}$$

are bases in $\mathbf{R}^1_s(\mathcal{O}_2)$ respectively $\mathbf{R}^1_a(\mathcal{O}_2)$. It is enough to prove the statement for one of the sets.
Let $a_1, a_2, ..., a_{g-1}$ be complex numbers such that

$$\sum_{j=1}^{g-1} a_j \widehat{\omega_j} = \widehat{0}.$$

This means that the symmetric differential $\sum_{j=1}^{g-1} a_j\omega_j$ is cohomologous to 0 i.e. there exists a function f of class C^2 such that

$$\sum_{j=1}^{g-1} a_j\omega_j = df.$$

Obviously f is harmonic (and symmetric). But, on a compact Riemann surface, an harmonic function is constant; thus $df = 0$ and consequently $\sum_{j=1}^{g-1} a_j\omega_j = 0$.
The set $\{\omega_1, \omega_2, ..., \omega_{g-1}\}$ being linearly independent, all the coefficients a_j must be equal to zero.
In other words, $\{\widehat{\omega_1}, \widehat{\omega_2}, ..., \widehat{\omega_{g-1}}\}$ is linearly independent and thus

$$(i) \quad dim\mathbf{R}^1_s(\mathcal{O}_2) \geq g-1.$$

In the same way we obtain:

$$(ii) \quad dim\mathbf{R}^1_a(\mathcal{O}_2) \geq g-1.$$

Since $dim\mathbf{R}^1_s(\mathcal{O}_2) + dim\mathbf{R}^1_a(\mathcal{O}_2) = dim\ \mathbf{R}^1(\mathcal{O}_2) = 2(g-1)$, in (i) and (ii) only equalities are possible. □

We denote by $\mathbf{R}^1(\mathbf{X})$ the de Rham cohomology vector space of $\mathbf{X}$.

Theorem 23 $\mathbf{R}^1(\mathbf{X}) \cong \mathbf{R}^1_s(\mathcal{O}_2)$.

Proof. Let us denote the subspaces of exact an closed differential forms on $\mathbf{X}$ by

$$\mathcal{E}(\mathbf{X}) = \{d\tilde{f} \mid \tilde{f} : \mathbf{X} \longrightarrow \mathbf{C},\ \tilde{f} \text{ of class } C^2\},$$

respectively

$$\mathcal{K}(\mathbf{X}) = \{\tilde{\omega} \in \mathcal{D}^1(\mathbf{X}) \mid \tilde{\omega} \text{ of class } C^1 \text{ and } d\tilde{\omega} = 0\}.$$

Then

$$\mathbf{R}^1(\mathbf{X}) := \frac{\mathcal{K}(\mathbf{X})}{\mathcal{E}(\mathbf{X})}.$$

If $d\tilde{f} \in \mathcal{E}(\mathbf{X})$ then

$$p^*(d\tilde{f}) = d(p^*\tilde{f}) \in \mathcal{E}_s$$

because $p^*\tilde{f} \in \mathcal{F}^2(\mathcal{O}_2)$, it is symmetric and of class C^2.

Conversely, if f is symmetric and of class C^2 then

$$df \in \mathcal{D}^1_s(\mathcal{O}_2) \cap \mathcal{D}^{1,1}(\mathcal{O}_2) \text{ and } p_{\#}(df) = d\tilde{f}.$$

Thus the restriction of p^* to $\mathcal{E}(\mathbf{X})$ maps this space isomorphicaly onto $\mathcal{E}_s$.

Let $\tilde{\omega} \in \mathcal{K}(\mathbf{X})$; $d(p^*\tilde{\omega}) = p^*(d\tilde{\omega}) = p^*(0) = 0$ that is $p^*\tilde{\omega}$ lies in $\mathcal{K}_s$.

As in the previous step, the restriction

$$p^* : \mathcal{K}(\mathbf{X}) \longrightarrow \mathcal{K}_s$$

is an isomorphism.

Now, the canonical map induced by p^*,

$$\frac{\mathcal{K}(\mathbf{X})}{\mathcal{E}(\mathbf{X})} \longrightarrow \frac{\mathcal{K}_s}{\mathcal{E}_s},$$

is an isomorphism. □

Corollary 5 (de Rham Theorem) *If the N.R.S.* $\mathbf{X}$ *is compact and has the genus* g, *its de Rham cohomology vector space has the dimension* $g-1$:

$$dim(\mathbf{R}^1(\mathbf{X})) = g-1.$$

16 The Stokes's Formula.

In this section we shall present the Stokes's formula on N.R.S. This problem points one important application of the differential operator d_s.
Let us consider $\omega \in \mathcal{D}^{1,1}(\mathcal{O}_2)$ *and the following three equalities:*

$$1). \quad \int_{\partial c} \omega = \iint_c d\omega\,;$$

$$2). \quad \int_{\partial c} \omega_a = \iint_c d_a\omega\,;$$

$$3). \quad \int_{\partial c} \omega_s = \iint_c d_s\omega\,,$$

where $c : \delta \longrightarrow \mathcal{O}_2$ *or* $c : s \longrightarrow \mathcal{O}_2$ *is a singular triangle or square of class* C^1 *on* $\mathcal{O}_2$.

The equality 1) is the classical Stokes's theorem on orientable Riemann surfaces and 2) and 3) are consequences of 1) with ω *replaced by* ω_a *and* ω_s, *respectively, because* $d_a\omega = d\omega_a$ *and* $d_s\omega = d\omega_s$.

Let us have a look at the first equality.
The line integrals $\int_{\partial c} \omega$ *and* $\int_{ko\partial c} \omega$ *as well as the double integrals* $\iint_c d\omega$ *and* $\iint_{koc} d\omega$ *are, in general, far away of each other; they are equal if and only if* $\int_{\partial c} \omega_a = 0$ *(in which case* $\iint_c d\omega_a = 0$ *too).*

With other words the equality 1) cannot be transposed on $\mathbf{X}$.

The integral $\int_{\partial c} \omega_a$ *from equality 2) is connected with the integral* $\int_{ko\partial c} \omega_a$ *by:*

$$\int_{ko\partial c} \omega_a = -\int_{\partial c} \omega_a.$$

The double integrals have a similar property.

Even if the behaviour of the equality 2) is much more regular than that of 1), it cannot be "projected" on the N.R.S. $\mathbf{X}$.

In the last case

$$\int_{\partial c} \omega_s = \int_{ko\partial c} \omega_s \text{ and } \iint_c d_s\omega = \iint_{koc} d_s\omega.$$

With other words, the integral $\int_{\partial c} \omega_s$ *does* **not** *depend on* ∂c *but on the "class"* $\{\partial c, k \circ \partial c\}$ *only or, equivalently, it depends only on the curve*

$$\partial \tilde{c} = p \circ \partial c = p \circ (k \circ \partial c)$$

on $\mathbf{X}$.

A similar remark is valid for the double integral $\iint_c d_s \omega$.

Thus the equality 3) *can be "projected" by* p *giving Stokes's formula on* $\mathbf{X}$.

More exactly, let $\tilde{\omega} \in \mathcal{D}^1(\mathbf{X})$ *be continous and let* $\tilde{c}$ *be a singular triangle or square of class* C^1 *on* $\mathbf{X}$. *In this context we can formulate Stokes's theorem:*

Theorem 24 $\int_{\partial \tilde{c}} \tilde{\omega} = \iint_{\tilde{c}} d\tilde{\omega}$.

Proof. By the definitions of the integrals it follows:

$$\int_{\partial \tilde{c}} \tilde{\omega} = \int_{\partial c} p^*(\tilde{\omega}) \quad \text{and} \quad \iint_{\tilde{c}} d\tilde{\omega} = \iint_c p^*(d\tilde{\omega}).$$

But it is well-known that $p^*(d\tilde{\omega}) = d(p^*\tilde{\omega})$ *and* $p^*(\tilde{\omega}) \in \mathcal{D}^1_s(\mathcal{O}_2)$; *according to case* 3) *at the begining of the section and to the equality* $d_s \omega = d\omega$ *for* ω *symmetric,*

$$\int_{\partial c} p^*(\tilde{\omega}) = \iint_c p^*(d\tilde{\omega}). \quad \square$$

Stokes's formula for chains of singular triangles or squares is a corollary to the previous theorem.

Let $\tilde{C} = \sum_{i \in I} n_i \tilde{c}_i$ *be such a chain where each* $\tilde{c}_i$ *is of class* C^1. *Then,*

Corollary 6 $\int_{\partial \tilde{C}} \tilde{\omega} = \iint_{\tilde{C}} d\tilde{\omega}$.

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