

CENTRAL COMMUTANT LIFTINGS
IN THE COUPLING APPROACH

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0. INTRODUCTION

In the circle of ideas related with the commutant lifting theorem an important role is played by the central contractive liftings of an operator A . These central liftings can be calculated fairly explicitly, and they have many interesting properties related to minimal entropy, or with approximation theory (cf. [4,6]). Given an operator A which intertwines two contractions, the central lifting is a special operator intertwining the unitary dilations of the two contractions. It is also possible to construct a central lifting of the adjoint A^* . The main result of this paper is to show that the central lifting of A^* coincides with the adjoint of the central lifting of A . This seems to be natural enough, but the constructions of the two central liftings involved are really quite different. Our approach to the problem goes through Arocena's theory of couplings [1,2]. We show in Section 7 that the central liftings correspond in a natural way with central couplings, and the symmetry of the coupling approach yields our main result.

The theory of couplings is a very elegant approach to commutant lifting, and very simple proofs of the commutant lifting theorem were found using it (see for instance [3]). We endeavor in this paper to present a complete account of this approach, developing only the most elementary parts of dilation theory. This gives the paper a largely expository nature. The new results are all in Section 7, while the rest of the paper sets the stage for the proofs. Our presentation tries to emulate the spirit of Professor Froda's elegant exposition style. Professor Alexandru Froda was one of the finest Romanian mathematicians, a very inspiring professor, a gentle dignified man who never compromised on his beliefs in his long academic career.

1. ISOMETRIES AND THEIR UNITARY EXTENSIONS

Consider an isometry V defined on a Hilbert space \mathcal{H} . The von Neumann-Wold decomposition shows that V is the direct sum of a unitary operator and a unilateral shift of some multiplicity. Since a unilateral shift is a restriction of a bilateral shift to an invariant subspace, we see that V itself can be viewed as the restriction of a

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unitary operator to an invariant subspace, i.e., V has a unitary extension. Let U be such an extension acting on a space $\mathcal{K} \supset \mathcal{H}$. We say that U is a *minimal unitary extension* of V if the linear space $\bigcup_{n=1}^{\infty} U^{-n}\mathcal{H}$ is dense in \mathcal{K} . The union is indeed a linear space because $U^{-n-1}\mathcal{H} \supset U^{-n-1}V\mathcal{H} \supset U^{-n}\mathcal{H}$ for $n \geq 1$. Minimal unitary extensions have a very strong uniqueness property expressed in the following well-known result (cf. [5]).

1.1. Proposition. *Assume that $U \in \mathcal{L}(\mathcal{K})$ is a minimal unitary extension of V . Let $W \in \mathcal{L}(\mathcal{M})$ be a unitary operator, and let $B \in \mathcal{L}(\mathcal{H}, \mathcal{M})$ satisfy the relation $BV = WB$. There exists a unique extension $C \in \mathcal{L}(\mathcal{K}, \mathcal{M})$ such that $CU = WC$ and $C|_{\mathcal{H}} = B$. The operator C has the same norm as B . If B is an isometry then C is an isometry as well.*

Proof. Assume that $CU = WC$ and $C|_{\mathcal{H}} = B$. For $x \in U^{-n}\mathcal{H}$, $n \geq 1$, we have

$$Cx = W^{-n}W^nCx = W^{-n}CU^n x = W^{-n}BU^n x,$$

and hence C is uniquely determined on the dense space $\bigcup_{n=1}^{\infty} U^{-n}\mathcal{H}$. This establishes uniqueness. For the existence, we define $C_0 : \bigcup_{n=1}^{\infty} U^{-n}\mathcal{H} \rightarrow \mathcal{M}$ by $C_0 x = W^{-n}BU^n x$, $x \in U^{-n}\mathcal{H}$. It is easy to verify that this definition is consistent, that $\|C_0\| = \|B\|$, and $C_0 Ux = WC_0 x$ for $x \in \bigcup_{n=1}^{\infty} U^{-n}\mathcal{H}$. The unique continuous extension C of C_0 is the required operator. The last two facts in the statement are immediate. \square

It is sometimes convenient to have a concrete description of the minimal unitary extension. This is achieved as follows. The operator $D_{V^*} = I - VV^*$ is a projection; denote its range by \mathcal{D}_{V^*} . One can then identify the space \mathcal{K} of the minimal unitary extension with $\mathcal{H} \oplus \mathcal{D}_{V^*} \oplus \mathcal{D}_{V^*} \oplus \dots$, and in this decomposition U^* has the matrix

$$U^* = \begin{bmatrix} V^* & 0 & 0 & 0 & \cdots \\ D_{V^*} & 0 & 0 & 0 & \cdots \\ 0 & I_{\mathcal{D}_{V^*}} & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{D}_{V^*}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

One recognizes immediately in this matrix the Schäfer form of the Sz.-Nagy unitary dilation theorem for the particular case of an isometry. Let us note for further use the Schäfer form for the minimal isometric dilation of a contraction operator $T \in \mathcal{L}(\mathcal{H})$ (i.e., $\|T\| \leq 1$). It is given by the matrix

$$V_T = \begin{bmatrix} T & 0 & 0 & 0 & \cdots \\ D_T & 0 & 0 & 0 & \cdots \\ 0 & I_{\mathcal{D}_T} & 0 & 0 & \cdots \\ 0 & 0 & I_{\mathcal{D}_T} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where $D_T = (I - T^*T)^{1/2}$ and $\mathcal{D}_T = (D_T\mathcal{H})^-$. The operator V_T is an isometry, and $V_T^*|_{\mathcal{H}} = T^*$ if \mathcal{H} is identified in the obvious way with a part of the domain of V_T .

2. COMMUTANT LIFTING

In this section we will review a few facts about a particular form of the commutant lifting problem. The general problem does reduce easily to this particular case, and the reduction is discussed in detail in [8] or [5]. We will fix isometries $V \in \mathcal{L}(\mathcal{H})$, $V' \in \mathcal{L}(\mathcal{H}')$, and we will let $U \in \mathcal{L}(\mathcal{K})$ and $U' \in \mathcal{L}(\mathcal{K}')$ denote the minimal unitary extensions of V and V' , respectively. Assume that $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ is an operator satisfying $CU = U'^*C$. Then the operator $A = P_{\mathcal{H}'}C|_{\mathcal{H}} \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ satisfies the relation $AV = V'^*A$. Indeed, for $x \in \mathcal{H}$,

$$AVx = P_{\mathcal{H}'}CUx = P_{\mathcal{H}'}U'^*Cx = P_{\mathcal{H}'}U'^*P_{\mathcal{H}'}Bx = V'^*Ax,$$

where we used the fact that \mathcal{H}' is invariant for U' so that $P_{\mathcal{H}'}U'^* = P_{\mathcal{H}'}U'^*P_{\mathcal{H}'}$. The commutant lifting theorem is the converse of this fact.

2.1. Theorem. *For every operator $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ such that $AV = V'^*A$ there exists $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ satisfying $CU = U'^*C$ and $P_{\mathcal{H}'}C|_{\mathcal{H}} = A$. Moreover, if $\|A\| \leq 1$ then C can be chosen so that $\|C\| \leq 1$ as well.*

Any operator C satisfying $CU = U'^*C$ and $P_{\mathcal{H}'}C|_{\mathcal{H}} = A$ is called a *lifting* of A . If $\|C\| \leq 1$, then C is called a *contractive lifting*.

Let us observe the inherent symmetry of this version of commutant lifting. If A satisfies $AV = V'^*A$ then A^* satisfies $A^*V' = V^*A^*$. Moreover, C is a (contractive) lifting of A if and only if C^* is a (contractive) lifting of A^* . Another observation which follows from Proposition 1.1 is that an operator $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ such that $CU = U'^*C$ is uniquely determined by the restriction $B = C|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathcal{K}'$, and $\|B\| = \|C\|$. By symmetry, C is also uniquely determined by $B_* = P_{\mathcal{H}'}C : \mathcal{K} \rightarrow \mathcal{H}'$, and $\|C\| = \|C^*|_{\mathcal{H}'}\| = \|B_*\|$. Thus, to construct a (contractive) lifting C of A one only needs to construct a (contractive) operator $B : \mathcal{H} \rightarrow \mathcal{K}'$ satisfying $BV = U'^*B$ and $P_{\mathcal{H}'}B = A$. Symmetrically, it suffices to construct a (contractive) operator $B_* : \mathcal{K} \rightarrow \mathcal{H}'$ satisfying $B_*U = V'^*B_*$ and $B_*|_{\mathcal{H}} = A$. The operator B will also be called a lifting of A , while B_* is clearly an extension of A .

3. CENTRAL DILATIONS

We fix V, V', U, U' as in the preceding section, and we let $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ be a contraction such that $AV = V'^*A$. Among all dilations $B : \mathcal{H} \rightarrow \mathcal{K}'$ of A there is a special one, called the central dilation, whose construction we will sketch here. This construction will provide by the way a proof of Theorem 2.1. We recall that \mathcal{K}' can be identified with $\mathcal{H}' \oplus \mathcal{D}_{V'^*} \oplus \mathcal{D}_{V'^*} \oplus \dots$ such that

$$U'^*(h' \oplus d'_0 \oplus d'_1 \oplus \dots) = V'^*h' \oplus \mathcal{D}_{V'^*}h' \oplus d'_0 \oplus d'_1 \oplus \dots$$

for $h' \in \mathcal{H}'$, $d'_j \in \mathcal{D}_{V'^*}$, $j \geq 0$. Any dilation $B : \mathcal{H} \rightarrow \mathcal{K}'$ must have the form $Bh = Ah \oplus X_0h \oplus X_1h \oplus \dots$, where $X_j \in \mathcal{L}(\mathcal{H}, \mathcal{D}_{V'^*})$ satisfy the relations

$$X_0V = \mathcal{D}_{V'^*}A, X_{j+1}V = X_j, \quad j \geq 0.$$

Let us consider first the possible choices for X_0 . Since we want a contractive dilation, the operator $A_0 : \mathcal{H} \rightarrow \mathcal{H}' \oplus \mathcal{D}_{V'^*}$ defined by $A_0h = Ah \oplus X_0h$ must be

a contraction. Thus we must have $X_0 = YD_A$, where $D_A = (I - A^*A)^{1/2}$, and $Y : \mathcal{D}_A = (D_A\mathcal{H})^- \rightarrow \mathcal{D}_{V^*}$ is a contraction. Observe that the values of Y are uniquely determined on the space $\mathcal{F} = (D_A V\mathcal{H})^-$ because

$$YD_A Vh = X_0 Vh = D_{V^*} Ah, \quad h \in \mathcal{H}.$$

Such contractions Y do exist since

$$\begin{aligned} \|D_A Vh\|^2 &= \|Vh\|^2 - \|AVh\|^2 \\ &= \|h\|^2 - \|V^*Ah\|^2 \\ &= \|h\|^2 - \|Ah\|^2 + \|D_{V^*} Ah\|^2 \geq \|D_{V^*} Ah\|^2, \end{aligned}$$

but there may be some freedom in the definition of Y on $\mathcal{D}_A \ominus \mathcal{F}$. For the central lifting one sets simply $Y|_{\mathcal{D}_A \ominus \mathcal{F}} = 0$. Let us denote by Y_0 this central choice; it is 'central' because it corresponds with the center of a certain hidden ball of possible choices.

These considerations can be extended to the successive choices of X_1, X_2, \dots . If one makes the 'central' choice at each step, the resulting operator B is called the central lifting of A . We mention without proof the explicit formulas for calculating the X_j corresponding with the central lifting. We need an additional operator $Z_0 : \mathcal{D}_A \rightarrow \mathcal{D}_A$ which is uniquely determined by the following conditions:

$$Z_0 D_A Vh = D_A h, \quad Z_0|_{\mathcal{D}_A \ominus \mathcal{F}} = 0.$$

Then for the central lifting we have (see for instance relation (1.15) in [6]):

$$X_j = Y_0 Z_0^j D_A, \quad j = 0, 1, 2, \dots$$

It will be useful later to describe the central lifting in a 'coordinate free' manner. To do this, assume that $U' \in \mathcal{L}(\mathcal{K}')$ is any minimal unitary extension of V' with $\mathcal{K}' \supset \mathcal{H}'$, and note that the subspaces \mathcal{H}' , $\mathcal{D}_0 = U'^* \mathcal{D}_{V^*} = U'^*(\mathcal{H}' \ominus V'\mathcal{H}')$, $\mathcal{D}_1 = U'^{*2} \mathcal{D}_{V^*}, \dots$ are pairwise orthogonal and they span \mathcal{K}' . The central dilation B is given by the formula

$$Bh = Ah + \sum_{j=0}^{\infty} U'^{*j+1} Y_0 Z_0^j D_A h, \quad h \in \mathcal{H}.$$

Now, the above considerations show how to construct the central lifting B of A , and the central lifting uniquely extends to a contractive lifting $C_1 \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ of A . The same construction applied to A^* yields the central lifting of A^* , which corresponds to an extension $B_* : \mathcal{K} \rightarrow \mathcal{H}'$ of A satisfying $B_* U = V'^* B_*$. This extension in turn determines uniquely a lifting $C_2 \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ of A . The main result of this paper is as follows.

3.1. Theorem. *The lifting C_1 of A obtained as an extension of the central lifting coincides with the lifting C_2 whose adjoint C_2^* is an extension of the central lifting of A^* .*

The proof of this result given in Section 7 will use the coupling approach to commutant lifting, an approach which exploits to a greater extent the inherent symmetry of the problem.

4. CONTRACTIONS AND PROJECTIONS

The main point of this section is a simple technical result asserting that any contraction between Hilbert spaces can be viewed as the restriction of an orthogonal projection by embedding the two spaces into a larger space.

4.1. Proposition. *Let $\mathcal{H}, \mathcal{H}'$ be Hilbert spaces, and $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ a contraction. There exist a Hilbert space \mathcal{H}_0 , and isometries $\Phi : \mathcal{H} \rightarrow \mathcal{H}_0, \Phi' : \mathcal{H}' \rightarrow \mathcal{H}_0$ with the following properties:*

- (i) $\Phi\mathcal{H} + \Phi'\mathcal{H}'$ is dense in \mathcal{H}_0 , and
- (ii) $\Phi'A = P_{\Phi'\mathcal{H}'}\Phi$.

Observe that if $U : \mathcal{H}_0 \rightarrow \mathcal{H}_1$ is unitary, we can replace $\mathcal{H}_0, \Phi,$ and Φ' by $\mathcal{H}_1, U\Phi,$ and $U\Phi'$, respectively. Except for this trivial kind of change, $\mathcal{H}_0, \Phi,$ and Φ' are easily seen to be unique.

Proof. The fact that $\|A\| \leq 1$ implies that the operator $\tilde{A} = \begin{bmatrix} I_{\mathcal{H}} & A^* \\ A & I_{\mathcal{H}'} \end{bmatrix}$ is nonnegative definite on $\mathcal{H} \oplus \mathcal{H}'$. Denote by \mathcal{H}_0 the completion of $\mathcal{H} \oplus \mathcal{H}'$ under the seminorm $\xi \rightarrow (\tilde{A}\xi, \xi)^{1/2}, \xi \in \mathcal{H} \oplus \mathcal{H}'$, and let $J : \mathcal{H} \oplus \mathcal{H}' \rightarrow \mathcal{H}_0$ be the canonical 'inclusion' (which might not be one-to-one). Finally, set $\Phi h = J(h \oplus 0)$ and $\Phi' h' = J(0 \oplus h')$ for $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$. Observe that $(\tilde{A}(h \oplus 0), h \oplus 0) = \|h\|^2$ for $h \in \mathcal{H}$, and hence Φ is an isometry. Analogously, Φ' is isometric. Condition (i) is obviously satisfied. To verify condition (ii) we must only prove that $(\Phi'Ah, \Phi'h') = (\Phi h, \Phi'h')$ or, equivalently, $(Ah, h') = (\Phi h, \Phi'h')$ for all $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$. This follows immediately from the fact that $(\Phi h, \Phi'h') = (\tilde{A}(h \oplus 0), 0 \oplus h')$. □

It will be convenient to have a symbol for the space \mathcal{H}_0 constructed in Proposition 4.1. In order to simplify notation, we will simply view \mathcal{H} and \mathcal{H}' as subspaces of \mathcal{H}_0 , and we will denote $\mathcal{H}_0 = \mathcal{H} \vee_A \mathcal{H}'$.

4.2. Proposition. *Let $\mathcal{K}, \mathcal{K}'$ be Hilbert spaces, $\mathcal{H} \subset \mathcal{K}$ and $\mathcal{H}' \subset \mathcal{K}'$ closed subspaces, and $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ a contraction. Then the closed linear span of \mathcal{H} and \mathcal{H}' in $\mathcal{K} \vee_C \mathcal{K}'$ is exactly $\mathcal{H} \vee_A \mathcal{H}'$, where $A = P_{\mathcal{H}'}C|_{\mathcal{H}}$.*

Proof. For $h \in \mathcal{H}$ we have indeed $P_{\mathcal{H}'}h = P_{\mathcal{H}'}P_{\mathcal{K}'}h = P_{\mathcal{H}'}Ch = Ah$. □

4.3. Proposition. *Assume that $U \in \mathcal{L}(\mathcal{K})$ and $U' \in \mathcal{L}(\mathcal{K}')$ are unitary operators, and $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ a contraction such that $CU = U'^*C$. Then there exists a unique unitary operator W on $\mathcal{K} \vee_C \mathcal{K}'$ such that $W|_{\mathcal{K}} = U$ and $W|_{\mathcal{K}'} = U'^*$.*

Proof. Uniqueness is immediate. To prove existence it is enough to show that $\|Uk + U'^*k'\| = \|k + k'\|$ for $k \in \mathcal{K}$ and $k' \in \mathcal{K}'$. Indeed,

$$\begin{aligned} \|Uk + U'^*k'\|^2 &= \|Uk\|^2 + \|U'^*k'\|^2 + 2\Re(Uk, U'^*k') \\ &= \|k\|^2 + \|k'\|^2 + 2\Re(P_{\mathcal{K}'}Uk, U'^*k') \\ &= \|k\|^2 + \|k'\|^2 + 2\Re(CUk, U'^*k') \\ &= \|k\|^2 + \|k'\|^2 + 2\Re(U'^*Ck, U'^*k') \\ &= \|k\|^2 + \|k'\|^2 + 2\Re(Ck, k') \\ &= \|k\|^2 + \|k'\|^2 + 2\Re(P_{\mathcal{K}'}k, k') = \|k + k'\|^2. \end{aligned}$$

□

Sometimes a more concrete realization of $\mathcal{H} \vee_A \mathcal{H}'$ is needed, and the following result provides such a realization. Recall that $D_A = (I - A^*A)^{1/2}$ and $\mathcal{D}_A = (D_A\mathcal{H})^-$.

4.4. Proposition. *There exists a unitary operator $\Psi: \mathcal{H} \vee_A \mathcal{H}' \rightarrow \mathcal{H}' \oplus \mathcal{D}_A$ such that $\Psi(h + h') = (Ah + h') \oplus D_A h$ for $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$.*

Proof. Observe that

$$\begin{aligned} \|h + h'\|^2 &= \|h\|^2 + \|h'\|^2 + 2\Re(Ah, h') \\ &= \|D_A h\|^2 + (\|Ah\|^2 + \|h'\|^2 + 2\Re(Ah, h')) \\ &= \|(Ah + h') \oplus D_A h\|^2 \end{aligned}$$

for $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$. This makes it possible to define Ψ on $\mathcal{H} + \mathcal{H}'$, and then extend it by continuity to an isometry on $\mathcal{H} \vee_A \mathcal{H}'$. Since the range of Ψ is clearly dense, Ψ is in fact unitary. □

5. COUPLINGS AND LIFTINGS

We now return to the notation of Section 3. Thus, $V \in \mathcal{L}(\mathcal{H})$, $V' \in \mathcal{L}(\mathcal{H}')$ are isometries with minimal unitary extensions $U \in \mathcal{L}(\mathcal{K})$, $U' \in \mathcal{L}(\mathcal{K}')$, and $A \in \mathcal{L}(\mathcal{H}, \mathcal{H}')$ is a contraction such that $AV = V'^*A$.

5.1. Definition. An operator $T \in \mathcal{L}(\mathcal{H} \vee_A \mathcal{H}')$ is called a *coupling* for (V, V'^*, A) if the following conditions are satisfied:

- (i) $\|T\| \leq 1$;
- (ii) $T|_{\mathcal{H}} = V$; and
- (iii) $T^*|_{\mathcal{H}'} = V'$.

The following result shows how a contractive lifting of A gives rise to a coupling.

5.2. Proposition. *Let $C \in \mathcal{L}(\mathcal{K}, \mathcal{K}')$ be a contractive lifting of A , and let $W \in \mathcal{L}(\mathcal{K} \vee_C \mathcal{K}')$ be the unitary operator determined by $W|_{\mathcal{K}} = U$ and $W|_{\mathcal{K}'} = U'^*$. Then the operator $T = P_{\mathcal{H} \vee_A \mathcal{H}'} W|_{\mathcal{H} \vee_A \mathcal{H}'}$ is a coupling for (V, V'^*, A) .*

Proof. Clearly T is a contraction. If $h \in \mathcal{H}$ then $Wh = Uh \in \mathcal{H}$, and therefore $Th = P_{\mathcal{H} \vee_A \mathcal{H}'} Wh = Uh$. Finally observe that \mathcal{K}' is a reducing space for W and $W^*|_{\mathcal{K}'} = U'$. The fact that $Th' = U'^*h'$, $h' \in \mathcal{H}'$, follows as above. □

Conversely, a coupling can be used to yield liftings of A .

5.3. Proposition. *Assume that T is a coupling for (V, V'^*, A) , and $W \in \mathcal{L}(\mathcal{K}_0)$ is a unitary operator such that $\mathcal{K}_0 \supset \mathcal{H} \vee_A \mathcal{H}'$, and $P_{\mathcal{H} \vee_A \mathcal{H}'} W|_{\mathcal{H} \vee_A \mathcal{H}'} = T$. Then there exist reducing subspaces \mathcal{K} and \mathcal{K}' for W such that*

- (i) $W|_{\mathcal{K}}$ is a minimal unitary extension of V ;
- (ii) $W^*|_{\mathcal{K}'}$ is a minimal unitary extension of V' ; and
- (iii) $P_{\mathcal{K}'} W|_{\mathcal{K}}$ is a lifting of A .

Proof. Observe that for $h \in \mathcal{H}$ we have $P_{\mathcal{H} \vee_A \mathcal{H}'} Wh = Th = Vh$. We must therefore have $P_{\mathcal{H} \vee_A \mathcal{H}'} Wh = Wh$, and hence $Wh = Vh$. We conclude that W is a unitary extension of V , and hence the reducing subspace \mathcal{K} for W generated by \mathcal{H} satisfies (i). The existence of \mathcal{K}' satisfying (ii) is proved analogously. If we set $U = W|_{\mathcal{K}}$, $U' = W^*|_{\mathcal{K}'}$, and $C = P_{\mathcal{K}'}|_{\mathcal{K}}$, then we have $CU = U'^*C$ because $P_{\mathcal{K}'}$ commutes with W . Since clearly $\|C\| \leq 1$, it remains to show that $P_{\mathcal{K}'}C|_{\mathcal{H}} = A$. This however is obvious because $A = P_{\mathcal{H}'}|_{\mathcal{H}}$, $P_{\mathcal{H}'}P_{\mathcal{K}'} = P_{\mathcal{H}'}$, and $\mathcal{H} \subset \mathcal{K}$. \square

Observe that the passage from a coupling to a lifting is not unique since there may be many, essentially different, unitary operators W such that $P_{\mathcal{H} \vee_A \mathcal{H}'} W|_{\mathcal{H} \vee_A \mathcal{H}'} = T$. Let us describe one particular way in which we can obtain a lifting from a given coupling.

5.4. Proposition. *Assume that T is a coupling for (V, V'^*, A) , and $V_T \in \mathcal{L}(\mathcal{K}_1)$ is the minimal isometric dilation of T ; thus $\mathcal{K}_1 \supset \mathcal{H} \vee_A \mathcal{H}'$ and $T^* = V_T^*|_{\mathcal{H} \vee_A \mathcal{H}'}$. Then there exists a reducing subspace \mathcal{K}' for V_T such that*

- (i) $V_T^*|_{\mathcal{K}'}$ is a minimal unitary extension of V' ; and
- (ii) $P_{\mathcal{K}'}|_{\mathcal{H}}$ is a lifting of A .

Proof. Observe that $V_T^*|_{\mathcal{H}'} = V'$ is isometric, and hence \mathcal{H}' is contained in the range of V_T . Therefore $V_T \mathcal{H}' \supset V_T V' \mathcal{H}' = V_T V_T^* \mathcal{H}' = \mathcal{H}'$ and thus $V_T^n \mathcal{H}' \subset V_T^{n+1} \mathcal{H}'$, $n \geq 0$. Denote by \mathcal{K}' the closure of $\bigcup_{n=1}^{\infty} V_T^n \mathcal{H}'$. It is clear that $V_T \mathcal{K}' = \mathcal{K}'$ so that \mathcal{K}' is reducing for V_T , and $V_T|_{\mathcal{K}'}$ is unitary. Condition (i) follows immediately, and (ii) is proved like the analogous statement in Proposition 5.3. \square

6. THE CENTRAL COUPLING

We will use the notation of Section 5. We begin with a slightly different characterization of couplings.

6.1. Lemma. *Let T be a contraction on $\mathcal{H} \vee_A \mathcal{H}'$. Then T is a coupling for (V, V'^*, A) if and only if*

- (a) $T|_{\mathcal{H}} = V$; and
- (b) $T|_{V' \mathcal{H}'} = V'^*|_{V' \mathcal{H}'}$.

Proof. Assume first that T is a coupling. Condition (ii) in Definition 5.1 is exactly condition (a), and $P_{\mathcal{H}'} T|_{\mathcal{H}'} = V'^*$. In particular for $h' \in \mathcal{H}'$, $P_{\mathcal{H}'} T V' h' = h'$. Thus $\|P_{\mathcal{H}'} T V' h'\| = \|h'\| \geq \|T V' h'\|$, and therefore $P_{\mathcal{H}'} T V' h' = T V' h'$. Condition (b) in the statement follows.

Conversely, assume that (a) and (b) are satisfied. We must verify condition (iii) in Definition 5.1. Fix a unit vector $h' \in \mathcal{H}'$ and observe that $(T^* h', V' h') = (h', T V' h') = \|h'\|^2 = 1$. Since $\|T^* h'\| \leq 1$, the converse of the Schwarz inequality implies that $T^* h' = V' h'$. \square

The reader will have no difficulty formulating the dual version of Lemma 6.1.

Let us denote by \mathcal{M} the closed linear span of \mathcal{H} and $V' \mathcal{H}'$ in $\mathcal{H} \vee_A \mathcal{H}'$, and by \mathcal{N} the closed linear span of $V \mathcal{H}$ and \mathcal{H}' . The preceding lemma tells us that all couplings T coincide on \mathcal{M} and, symmetrically, all adjoints T^* coincide on \mathcal{N} . We can now give a description of all couplings (cf. [5]).

6.2. Proposition.

- (i) There exists a unique coupling T_0 for (V, V'^*, A) such that $T_0|_{\mathcal{M}^\perp} = 0$. Moreover, T_0 is a partial isometry with kernel \mathcal{M}^\perp and range \mathcal{N} .
- (ii) For each contraction $\Gamma \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$, the operator $T = T_0 + \Gamma P_{\mathcal{M}^\perp}$ is a coupling. Conversely, every coupling can be written uniquely under this form.

Proof. The uniqueness of T_0 is obvious from Lemma 6.1. To prove that T_0 exists and is a partial isometry it suffices to show that the map $h + V'h' \rightarrow Vh + h'$ is well defined and isometric for $h \in \mathcal{H}$ and $h' \in \mathcal{H}'$. This is a simple calculation:

$$\begin{aligned} \|h + V'h'\|^2 &= \|h\|^2 + \|V'h'\|^2 + 2\Re(Ah, V'h') \\ &= \|Vh\|^2 + \|h'\|^2 + 2\Re(V'^*Ah, h') \\ &= \|Vh\|^2 + \|h'\|^2 + 2\Re(AVh, h') \\ &= \|Vh + h'\|^2, \end{aligned}$$

where we used the fact that $A = P_{\mathcal{H}}|_{\mathcal{H}}$. Note by the way that the range of T is \mathcal{N} . Now, if T is another coupling, we know from Lemma 6.1 that $(T - T_0)|_{\mathcal{M}} = 0$ and $(T^* - T_0^*)|_{\mathcal{N}} = 0$. It follows that $T - T_0$ is uniquely determined by $\Gamma = (T - T_0)|_{\mathcal{M}^\perp}$, and Γ is a contraction with range contained in \mathcal{N}^\perp . Conversely, the formula $T = T_0 + \Gamma P_{\mathcal{M}^\perp}$, where $\Gamma \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$ is a contraction, defines a contraction such that $T|_{\mathcal{M}} = T_0|_{\mathcal{M}}$, hence a coupling. \square

The coupling T_0 constructed above is called the *central coupling* for (V, V'^*, A) . The idea again is that T_0 is the center of a ball corresponding with the center $\Gamma = 0$ of the unit ball of $\mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. The symmetry of the construction makes the following result obvious.

6.3. Proposition. Let T_0 be the central coupling for (V, V'^*, A) . Then T_0^* is the central coupling for (V', V^*, A^*) .

Observe now that one has a simple path for proving the commutant lifting theorem (this is the proof due to Arocena, Cotlar and Sadoski). Namely, starting from (V, V'^*, A) one constructs a coupling, for instance the central coupling T_0 . Next one finds a unitary W such that $T_0 = P_{\mathcal{H} \vee_A \mathcal{H}'} W|_{\mathcal{H} \vee_A \mathcal{H}'}$, and finally Proposition 5.3 yields a lifting for A .

7. CENTRAL COUPLINGS AND CENTRAL LIFTINGS

In this section we will see how the central coupling for (V, V'^*, A) gives rise to the central lifting of A . Let $U' \in \mathcal{L}(\mathcal{K}')$ be a minimal unitary extension of V' . we recall that the central lifting $B : \mathcal{H} \rightarrow \mathcal{K}'$ of A is given by the formula

$$Bh = Ah + \sum_{j=0}^{\infty} U'^{j+1} Y_0 Z_0^j D_A h, \quad h \in \mathcal{H},$$

where $Y_0 : \mathcal{D}_A \rightarrow \mathcal{D}_{V'^*}$ and $Z_0 : \mathcal{D}_A \rightarrow \mathcal{D}_A$ are determined by $Y_0 D_A V h = D_{V'^*} A h$, $Z_0 D_A V h = D_A h$, $h \in \mathcal{H}$, and by the fact that they are zero on $\mathcal{D}_A \ominus \mathcal{F}$, $\mathcal{F} = (\mathcal{D}_A V \mathcal{H})^\perp$.

Interestingly, the operators Y_0 and Z_0 also play a role in the description of the central coupling T_0 for (V, V^*, A) . In the following result Y_0 is regarded as an operator with values in \mathcal{H}' .

7.1. Lemma. *Let $\Psi : \mathcal{H} \vee_A \mathcal{H}' \rightarrow \mathcal{H}' \oplus D_A$ be the unitary operator determined by the formula $\Psi(h + h') = (Ah + h') \oplus D_A h$, $h \in \mathcal{H}$, $h' \in \mathcal{H}'$. Then the operator $\tilde{T} = \Psi T_0 \Psi^*$ is given by the matrix $\tilde{T} = \begin{bmatrix} V'^* & 0 \\ Y_0^* & Z_0^* \end{bmatrix}$.*

Proof. The operator \tilde{T} is completely determined by the following three conditions: $\tilde{T}^* \Psi V h = \Psi h$ for $h \in \mathcal{H}$, $\tilde{T}^* \Psi h' = \Psi V' h'$ for $h' \in \mathcal{H}'$, and $\tilde{T}^* = 0$ on the orthogonal complement of $\tilde{\mathcal{N}} = \Psi \mathcal{N} = \Psi(V\mathcal{H} \vee \mathcal{H}')$. The second condition means simply that $\tilde{T}^*(h' \oplus 0) = V' h' \oplus 0$, so that $\tilde{T}^* = \begin{bmatrix} V' & Y \\ 0 & Z \end{bmatrix}$ for some Y and Z . Observe that $\tilde{\mathcal{N}} = \Psi(\mathcal{N})$ is the linear span of $\{AVh \oplus D_A Vh : h \in \mathcal{H}\}$ and $\mathcal{H}' \oplus \{0\}$, and hence $\tilde{\mathcal{N}} = \mathcal{H}' \oplus \mathcal{F}$. Thus to conclude the proof it suffices to show that $Y|_{\mathcal{F}} = Y_0|_{\mathcal{F}}$ and $Z|_{\mathcal{F}} = Z_0|_{\mathcal{F}}$. We have

$$\begin{aligned} Ah \oplus D_A h &= \Psi h = \tilde{T}^* \Psi V h = \tilde{T}^*(AVh \oplus D_A Vh) \\ &= \tilde{T}^*(V'^* Ah \oplus D_A Vh) = (V' V'^* Ah + Y D_A Vh) \oplus Z D_A Vh, \end{aligned}$$

whence $Z D_A Vh = D_A h = Z_0 D_A Vh$, and $Y D_A Vh = Ah - V' V'^* Ah = D_{V'^*} Ah = Y_0 D_A Vh$, as desired. \square

We are now ready to relate the central lifting with the central coupling.

7.2. Theorem. *Let V_{T_0} be the minimal isometric dilation of T_0 , and let \mathcal{K}' be the reducing subspace for V_{T_0} such that $V_{T_0}^*|_{\mathcal{K}'}$ is a minimal unitary extension of V' . Then $P_{\mathcal{K}'}|_{\mathcal{H}}$ is the central lifting of A .*

Proof. It is easy to reformulate the result in terms of the operator \tilde{T} . Let $\tilde{\mathcal{K}} = (\mathcal{H}' \oplus D_A) \oplus D_{\tilde{T}} \oplus D_{\tilde{T}} \oplus \dots$ be the space of the minimal isometric dilation \tilde{V} of \tilde{T} . We identify \mathcal{H}' with $(\mathcal{H}' \oplus \{0\}) \oplus \{0\} \oplus \dots$, and denote by \mathcal{K}' the reducing subspace for \tilde{V} generated by \mathcal{H}' . As we saw in the proof of Proposition 5.4, $U' = \tilde{V}^*|_{\mathcal{K}'}$ is a minimal unitary extension of V' , so that \mathcal{K}' is the orthogonal sum of \mathcal{H}' and of the spaces $D_j = \tilde{V}^{j+1} D_{V'^*}$ for $j \geq 0$. We must show that

$$P_{\mathcal{K}'}(Ah \oplus D_A h) = Ah + \sum_{j=0}^{\infty} \tilde{V}^{j+1} Y_0 Z_0^j D_A h$$

for all $h \in \mathcal{H}$ or, equivalently, that

$$P_{D_j}(Ah \oplus D_A h) = \tilde{V}^{j+1} Y_0 Z_0^j D_A h$$

for $h \in \mathcal{H}$ and $j \geq 0$. This can be rewritten as

$$(Ah \oplus D_A h, \tilde{V}^{j+1} h') = (\tilde{V}^{j+1} Y_0 Z_0^j D_A h, \tilde{V}^{j+1} h')$$

for all $h \in \mathcal{H}$, $j \geq 0$, and $h' \in \mathcal{D}_{V^*}$. Since \tilde{V} is an isometry, the last condition is equivalent to

$$(Ah \oplus D_A h, \tilde{V}^{j+1} h') = (Y_0 Z_0^j D_A h, h').$$

(Here, of course, $Ah \oplus D_A h$ is a short notation for the vector $(Ah \oplus D_A h) \oplus 0 \oplus 0 \oplus \dots$ in $\tilde{\mathcal{K}}$, and h' stands for $(h' \oplus 0) \oplus 0 \oplus 0 \oplus \dots$.) In order to verify these equalities we note that

$$\tilde{V}((h' \oplus 0) \oplus 0 \oplus 0 \oplus \dots) = \tilde{T}(h' \oplus 0) \oplus D_T h' \oplus 0 \oplus \dots = (0 \oplus Y_0^* h') \oplus \dots,$$

and generally

$$\tilde{V}^{j+1}((h' \oplus 0) \oplus 0 \oplus 0 \oplus \dots) = (0 \oplus Z_0^{*j} Y_0^* h') \oplus \dots,$$

where the \dots in the right hand side denote terms whose exact formulas we do not need. We have therefore

$$\begin{aligned} (Ah \oplus D_A h, \tilde{V}^{j+1} h') &= ((Ah \oplus D_A h) \oplus 0 \oplus 0 \oplus \dots, \tilde{V}^{j+1}((h' \oplus 0) \oplus 0 \oplus 0 \oplus \dots)) \\ &= (D_A h, Z_0^{*j} Y_0^* h') = (Y_0 Z_0^j D_A h, h'), \end{aligned}$$

as claimed. \square

The main result of the paper is now easily proved.

Proof of Theorem 3.1. Denote by $W \in \mathcal{L}(\mathcal{K}_0)$ the minimal unitary dilation of T_0 , and let the spaces \mathcal{K} and \mathcal{K}' be as in Proposition 5.3. In particular, $C = P_{\mathcal{K}'}|_{\mathcal{K}}$ is a lifting of A . We claim that $C|_{\mathcal{H}}$ is the central lifting of A . This follows indeed from Theorem 7.2 because V_{T_0} can be viewed as a restriction of W to an invariant subspace containing both \mathcal{H} and \mathcal{K}' . Thus C is the unique extension to \mathcal{K} of the central lifting of A , i.e., $C = C_1$ with the notation of Section 3. Observe next that W^* is the minimal unitary dilation of T_0^* , and T_0^* is the central coupling for (V', V^*, A^*) . The preceding argument shows that $C^*|_{\mathcal{H}'}$ is the central lifting of A^* , and hence $C = C_2$. \square

It is worthwhile to observe that the argument of Theorem 7.2 yields something for arbitrary, not necessarily central, couplings. To be precise, assume that T is any coupling for (V, V^*, A) , and $\tilde{T} = \Psi T \Psi^*$. Then $\tilde{T} = \begin{bmatrix} V'^* & 0 \\ Y^* & Z^* \end{bmatrix}$ for some operators Y and Z . The operator Y must have range contained in \mathcal{D}_{V^*} because \tilde{T} is a contraction. The minimal isometric dilation V_T of T yields a lifting $B : \mathcal{H} \rightarrow \mathcal{K}'$ of A as in Proposition 5.4. The argument of Theorem 7.2 yields the following formula for B :

$$Bh = Ah + \sum_{j=0}^{\infty} U'^{*j+1} Y Z^j D_A h, \quad h \in \mathcal{H}.$$

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