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AN INTEGRO-DIFFERENTIAL EQUATION FROM THE CAPILLARITY THEORY

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The equation we are going to study in this Note appears in the case of *capillarity in circular glass tubes*.

Denoting by (x_1, x_2, x_3) the Cartesian coordinates in the space and assuming that the tube of radius $R > o$ has its axis colinear with (Ox_3), we shall consider that the surface (S) of the liquid is a rotation surface whose parametric equations are

(1) $\quad x_1 = r\cos\varphi,\ x_2 = r\sin\varphi,\ x_3 = x_3(r)$ with $o \le r \le R,\ o \le \varphi \le 2\pi$

where r and φ are the usual cyllindrical coordinates. The surface (S) has a stable form if the local equilibrium equation of Laplace [1, p.111]

(2) $\quad \rho\, g\, x_3(r) = \sigma(\frac{1}{R_1} + \frac{1}{R_2}),\quad o \le r \le R$

is satisfied at every its point. We use the notation ρ for the density of the liquid, g for the gravitational acceleration, R_1 and R_2 for the principal radii of curvature. The constant $\sigma > o$ is the interfacial tension liquid - air. Using the classical notations in the Differential Geometry for the coefficients of the two fundamental forms of a surface, we have in our case

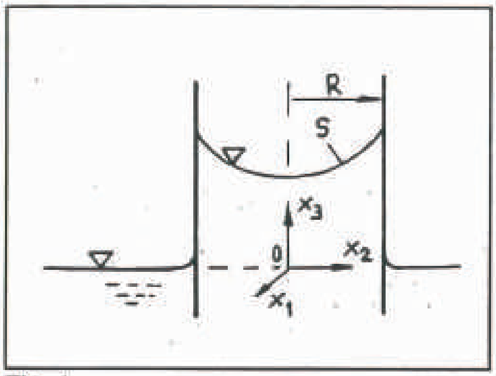

Fig. 1

(3) $$E = 1 + (x_3'(r))^2,\ F = o,\ G = r^2$$

and, respectively,

(4) $$L = \frac{r x_3''(r)}{\sqrt{EG - F^2}},\ M = o,\ N = \frac{r^2 x_3'(r)}{\sqrt{EG - F^2}}.$$

Since the principal curvatures k_1 and k_2 satisfy

(5) $$k_1 + k_2 = \frac{EN + GL - 2FM}{EG - F^2},$$

we obtain after some elementary computations

(6) $$\frac{1}{R_1}+\frac{1}{R_2}=k_1+k_2=\frac{1}{r}\left[\frac{r x_3'(r)}{\sqrt{1+(x_3'(r))^2}}\right]'.$$

From (2), integrating over the interval $[o,r]$, we obtain the integro-differential equation

(7) $$\sigma\frac{r x_3'(r)}{\sqrt{1+(x_3'(r))^2}}=\rho g\int_o^r s x_3(s)ds\,,\quad o\le r\le R$$

to which we associate the boundary condition $x_3'(R)=\beta>o$. We remark that every solution also satisfies the condition $x_3'(o)=o$. Making the substitution $r=Rt$ (t = independent variable) and putting $x_3(Rt)=Rx(t)$, we arrive to the problem

(8) $$\mu\frac{t x'(t)}{\sqrt{1+(x'(t))^2}}=\int_o^t s x(s)ds,\quad o\le t\le 1\ ;\ x'(1)=\beta$$

where $\mu=\sigma/\rho g R^2>o$ is a constant. We shall consider the problem (8) in a more general case, writing it under the form

(9) $$tG(x'(t))=\int_o^t s x(s)ds\,,\quad o\le t\le 1\,;\ x'(1)=\beta>o$$

and imposing conditions to the function $G=G(u)$, which are satfisfied in the particular case when $G(u)=\mu u(1+u^2)^{-\frac{1}{2}}$, described by (8). We also consider the initial value problem

(10) $$tG(x'(t))=\int_o^t s x(s)ds\,,\quad o\le t\le 1\,;\ x(o)=x_o\,.$$

Our aim is to prove that, for every $\beta>o$, there exists $x_o=x_o(\beta)>o$, such that the corresponding solution of (10) satisfies (9), too.

Using the notation $x'=y$, assuming that y is continuous and taking into account that

(11) $$x(t)=x_o+\int_o^t y(s)ds\,,\quad o\le t\le 1$$

we shall obtain for the function $y=y(t)$, from (10), the equation

(12) $$2tG(y(t))=x_o t^2+2\int_o^t s\int_o^s y(\tau)d\tau\, ds\,,\quad o\le t\le 1$$

or its equivalent form

(13) $$2tG(y(t))=x_o t^2+\int_o^t (t^2-s^2)y(s)ds\,,\quad o\le t\le 1\,.$$

If $x = x(t)$ is a continuously differentiable solution of (10), then $y = x'$ satisfies equation (13). Conversely, if $y = y(t)$ is a continuous solution of (13), then $x = x(t)$ given by (11) satisfies equation (10). In what follows, we assume that the function G satisfies the hypotheses

(H) $G = G(u)$ is defined for $u \geq o$, $G = G(u)$ is continuously differentiable for $u \geq o$, $G(o) = o$, $G'(u) > o$ for $u \geq o$, $G'(u)$ is nonincreasing for $u \geq o$ and $G(u) \longrightarrow \mu$ $(\mu < \infty)$ as $u \longrightarrow \infty$.

Let us denote, for $x_o > o$,

(14) $$K(x_o) = \{y \in C_{[o,1]} : y(t) \geq o,\ 2tG(y(t)) \geq x_o t^2 + \int_o^t (t^2 - s^2) y(s)\,ds\ ,\ o \leq t \leq 1\}$$

Proposition 1. The set of all points $x_o > o$ for which $K(x_o) \neq \varnothing$ is an open interval $I = (o, A)$ with $o < A < 2\mu$.

Proof. Let us first prove that, for every $M > o$, we can choose a sufficiently small $x_o > o$ such that $K(x_o)$ contains a certain function $y = y(t)$ with $o \leq y(t) \leq M$ for $o \leq t \leq 1$. Indeed, let us denote $a = G'(M)$ and let us consider the integral equation

(15) $$2aty(t) = x_o t^2 + \int_o^t (t^2 - s^2) y(s)\,ds\ ,\quad o \leq t \leq 1\ .$$

Putting

(16) $$u(t) = \int_o^t y(s)\,ds\ ,\quad o \leq t \leq 1$$

it follows that $y = u'$. We obtain from (15) the differential equation

(17) $$(atu'(t))' = x_o t + tu(t)$$

and since $u(o) = o$, $u'(o) = o$ we search for $u = u(t)$ a series expansion of the form

(18) $$u(t) = c_2 t^2 + c_3 t^3 + \ldots + c_n t^n + \ldots\ .$$

From (17) and (18), we obtain

(19) $$u(t) = x_o \sum_{n=1}^{\infty} \frac{t^{2n}}{2^{2n}(n!)^2 a^n} = x_o I_o\left(\frac{t}{\sqrt{a}}\right) - x_o\ ,$$

where I_o stands for the modified Bessel function of the first kind of order zero [2, p.181], [3, p.93], given by

(20) $$I_o(t) = \sum_{n=o}^{\infty} \frac{1}{(n!)^2}\left(\frac{t}{2}\right)^{2n}, \quad -\infty < t < \infty .$$

For the solution $y = y(t)$ of (15), we find

(21) $$y(t) = u'(t) = \frac{x_o}{\sqrt{a}} I_o'\left(\frac{t}{\sqrt{a}}\right), \quad o \le t \le 1 .$$

Now, it is clear that for a sufficiently small $x_o > o$ we have $o \le y(t) \le M$ if $o \le t \le 1$. Fixing now a point x_o with this property, we have for the corresponding solution $y = y(t)$ of (15) that

(22) $$G(y(t)) = G(o) + G'(\theta(t))y(t) \text{ with } o \le \theta \le y(t) ;$$

hence, taking into account that $G(o) = o$ and $G'(u)$ is nonincreasing for $u \ge o$, it follows that

(23) $$G(y(t)) \ge G'(y(t))y(t) \ge G'(M)y(t) = a\, y(t), \quad o \le t \le 1 .$$

We derive from (15) and (23) that $y \in K(x_o)$, hence $K(x_o) \ne \varnothing$. Since $G < \mu$, it follows from definition (14) that $K(x_o) = \varnothing$ for $x_o \ge 2\mu$. On another hand, $o < x_o < \overline{x}_o$ implies $K(\overline{x}_o) \subset K(x_o)$, hence $K(\overline{x}_o) \ne \varnothing$ implies $K(x_o) \ne \varnothing$. Consequently, the set of points $x_o > > o$ with $K(x_o) \ne \varnothing$ is an interval with the endpoints o and A, $A < 2\mu$. It remains to show that this interval (denote it by I) is open. To this end, it suffices to prove that $x_o \in I$ implies $K(x_o + \varepsilon) \ne \varnothing$ for a sufficiently small $\varepsilon > o$. Fixing $y \in K(x_o)$ and $M > o$, we denote

(24) $$M_1 = M + \sup\,\{y(t) : o \le t \le 1\}, \quad a_1 = G'(M_1) .$$

If $\alpha = \alpha(t)$ is the solution of the equation

(25) $$2\, a_1 t\, \alpha(t) = \varepsilon t^2 + \int_o^t (t^2 - s^2)\, \alpha(s)\, ds, \quad o \le t \le 1$$

we know that, for a sufficiently small $\varepsilon > o$, it follows $o \le \alpha(t) \le M$ for $o \le t \le 1$. We deduce from

(26) $$G(y(t) + \alpha(t)) = G(y(t)) + G'(\theta(t))\, \alpha(t) ,$$

where $y(t) \le \theta(t) \le y(t) + \alpha(t)$ for $0 \le t \le 1$, that

(27) $$G(y(t) + \alpha(t)) \ge G(y(t)) + G'(M_1)\, \alpha(t) = G(y(t)) + a_1\alpha(t)$$

also holds for $t \in [o,1]$. It follows from (27), (25) and (14) that $y + \alpha \in K(x_o + \varepsilon)$, that is, $K(x_o + \varepsilon) \ne \varnothing$. This concludes the proof. ■

Remark 1. Eq. (21) implies that from

(28) $$o < x_o \le B = sup\left\{ \frac{M\sqrt{G'(M)}}{I_o'(1/\sqrt{G'(M)})} \;:\; M > o \right\}$$

it follows that the corresponding solution $y = y(t)$ of (15) belongs to $K(x_o)$, hence $K(x_o) \neq \emptyset$. In other words, we have $(o,B] \subset I$.

Theorem 1. For $x_o \in I$, the integral equation (13) has a unique solution in $C_{[o,1]}$.

Proof. Let $\bar{y}$ be a fixed element of $K(x_o)$. We denote

(29) $$K(x_o, \bar{y}) = \{y \in K(x_o) : \; y(t) \le \bar{y}(t) \;\text{ for }\; o \le t \le 1\} .$$

Obviously $\bar{y} \in K(x_o, \bar{y})$, hence $K(x_o, \bar{y}) \neq \emptyset$. Moreover, $K(x_o, \bar{y})$ is a closed subset of $C_{[o,1]}$. We define an operator $T : K(x_o, \bar{y}) \longrightarrow C_{[o,1]}$ by

(30) $$(Ty)(t) = G^{-1}\left(\frac{x_o t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)y(s)\,ds\right), \quad o < t \le 1 \quad ; \quad (Ty)(o) = o$$

where G^{-1} is the inverse function of G, with $dom(G^{-1}) = [o,\mu)$. We shall prove that

(31) $$T(K(x_o, \bar{y})) \subset K(x_o, \bar{y}) .$$

Indeed, if y is an element of $K(x_o, \bar{y})$, we have for $t \in (o,1]$

(32) $$G(y(t)) \ge \frac{x_o t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)y(s)\,ds .$$

Applying the function G^{-1}, we obtain $y(t) \ge (Ty)(t)$ for $t \in (o,1]$, but this is also true for $t = o$. Hence, we have $o \le (Ty)(t) \le \bar{y}$ for $t \in [o,1]$. Denoting $Ty = z$, we obtain from (30), for $t \in [o,1]$, that

(33) $$2tG(z(t)) \ge x_o t^2 + \int_o^t (t^2 - s^2)y(s)\,ds \ge x_o t^2 + \int_o^t (t^2 - s^2)z(s)\,ds .$$

Thus $z = Ty$ satisfies both the conditions required by defintion (29), hence the inclusion (31) is true.

If $b > o$ is $sup\{\bar{y}(t) : \; o \le t \le 1\}$, then for every $y \in K(x_o, \bar{y})$, we have

(34) $$o \le \frac{x_o t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)y(s)\,ds \le G(b) < \mu , \quad o < t \le 1 .$$

Denoting by $L = L(b) = 1/G'(b)$ the Lipschitz constant of the function G^{-1} with respect to the interval $[o, G(b)]$, we can write

(35) $$|(Ty)(t) - (Tz)(t)| \le \frac{L}{2t}\int_o^t (t^2 - s^2)|y(s) - z(s)|\,ds , \quad o < t \le 1$$

for every pair of functions y and z of $K(x_o, \bar{y})$. Using the notation $\|\cdot\|$ for the norm in the space $C_{[o,1]}$, we obtain from the preceding inequality

$$(36) \qquad |(Ty)(t)-(Tz)(t)| \le L\,\|y-z\|\frac{t^2}{3}\,, \quad o \le t \le 1 \qquad .$$

By iterating this (upper bounding) inequality we find

$$(37) \qquad |(T^n y)(t)-(T^n z)(t)| \le \frac{L^n\,\|y-z\|}{3^2\cdot 5^2\cdot \ldots \cdot (2n-1)^2}\,\frac{t^{2n}}{2n+1}\,, \quad o \le t \le 1$$

true for every $n \ge 2$. This fact shows us that the sequence of successive approximations $\{T^n y_o\}_{n\ge o}$, where y_o is an arbitrary element of $K(x_o, \bar{y})$, is uniformly convergent on $[o,1]$ to a function $y \in K(x_o, \bar{y})$, which is a fixed point of operator T and a solution of equation (13). We remark that this sequence is nonincreasing and tends to the solution $y = y(t)$ of (13) and we have $y(t) \le \bar{y}(t)$ on $[o,1]$. The unicity of the solution may be proved by a standard argument : if $z = z(t)$ is another solution of Eq.(13) in $C_{[o,1]}$ and we have $o \le z(t) \le b_1$ for $o \le t \le 1$ and if we denote $b_2 = max\,\{b_1, b\}$ and $L_2 = 1/G'(b_2) =$ the Lipschitz constant of the function G^{-1} on the interval $[o, G(b_2)]$, we obtain

$$(38) \qquad |y(t)-z(t)| \le L_2\,\|y-z\|\frac{t^2}{3}\,, \quad o \le t \le 1 \qquad .$$

Iterating this inequality, we deduce an (upper bounding) inequality similar to to those given in (37). Hence $z(t) \equiv y(t)$ and this completes the proof. ■

Remark 2. Since the element $\bar{y}$ was taken (in the above proof) arbitrarily in $K(x_o)$, we have for the solution y of (13)

$$(39) \qquad y(t) \le \bar{y}(t) \ \text{for} \ o \le t \le 1, \ \forall\, \bar{y} \in K(x_o) \ .$$

Remark 3. The following equivalence is true :

$$(40) \qquad x_o > o,\ K(x_o) \ne \emptyset \iff \text{equation (13) has a unique solution in } C_{[o,1]}.$$

For $x_o \in I$, we denote by $y(t,x_o)$ the corresponding solution of equation (13).

Proposition 2. Assume that $x_o, \bar{x}_o \in I$. Then

(*i*) $y(t,x_o) > o$ for $o < t \le 1$,

(*ii*) $y'(t,x_o) > o$ for $o \le t \le 1$,

(*iii*) $x_o < \bar{x}_o$ implies $y(t,x_o) < y(t,\bar{x}_o)$ for $o \le t \le 1$,

(*iv*) the mapping $x_o \longrightarrow y(\cdot, x_o)$ is locally Lipschitzian from I to $C_{[o,1]}$.

Proof. (*i*) It follows from Theorem 1 that $y(t,x_o) \ge o$ for $o < t \le 1$. We remark that $y(o,x_o) = o$. Assume that there exists $t_1 > o$ such that $y(t_1,x_o) = o$. From the equation

$$(41) \qquad o = G(y(t_1,x_o)) = \frac{x_o t_1}{2} + \frac{1}{2t_1}\int_o^{t_1}(t_1^2 - s^2)\,y(s,x_o)\,ds \ ,$$

we arrive to a contradiction since the last term is positive. Thus, $y(t,x_o) > o$ for $o < t \le 1$. (*ii*) Starting from

$$(42) \qquad y(t,x_o) = G^{-1}\Big(\frac{x_o t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)\,y(s,x_o)\,ds\Big)\ , \quad o < t \le 1$$

we easily deduce the existence and the continuity, for $t \in (o,1]$, of the derivative $y'(t,x_o) = y_t'(t,x_o)$. From Eq.(13), we obtain

$$(43) \qquad G'(y(t,x_o))\,y'(t,x_o) = \frac{x_o}{2} + \int_o^t y(s,x_o)\,ds - \frac{1}{2t^2}\int_o^t (t^2 - s^2)\,y(s,x_o)\,ds$$

also for $t \in (o,1]$. If we denote

$$(44) \qquad g(t) = 2t^2\int_o^t y(s,x_o)\,ds - \int_o^t (t^2 - s^2)\,y(s,x_o)\,ds \quad , \quad o \le t \le 1$$

we have $g(t) > o$ for $t \in (o,1]$, because $g(o) = o$ and $g'(t) > o$ for $t \in (o,1]$. Using (43) and (44), we can now write

$$(45) \qquad y'(t,x_o) > \frac{x_o}{2G'(y(t,x_o))} \ge \frac{x_o}{2G'(o)} > o \ , \quad o < t \le 1 \ .$$

Making $t \longrightarrow o_+$ in (43), we have

$$(46) \qquad y_+'(o,x_o) = \lim_{t \to o_+} y'(t,x_o) = \frac{x_o}{2G'(o)} \ .$$

(*iii*) Denoting $u(t) = y(t,\bar{x}_o) - y(t,x_o)$, we have $u(o) = o$ and $u_+'(o) = (\bar{x}_o - x_o)/2G'(o) > > o$. This implies $u(t) > o$ for $t =$ sufficiently small. Assume that $u(t) > o$ on (o,t_1) and $u(t_1) = o$. We obtain from (13)

$$(47) \qquad o = (\bar{x}_o - x_o)t_1^2 + \int_o^{t_1}(t_1^2 - s^2)\,u(s)\,ds \ ,$$

what is impossible. Thus $u(t) > o$ on $(o,1]$, that is, $y(t,x_o) < y(t,x_o)$ on $(o,1]$.
(*iv*) Let $[c,d]$ be a compact interval enclosed in I. For every $x_o \in [c,d]$ and $t \in [o,1]$ we have $o \le y(t,x_o) \le y(1,d)$. Then we get from (42)

$$(48) \qquad o \le \frac{x_o t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)\,y(s,x_o)\,ds \le G(y(1,d)) < \mu \quad , \quad o < t \le 1 \ .$$

Taking x_o, $\bar{x}_o \in [c,d]$, keeping for $u = u(t)$ the same meaning as above, denoting by L^* the Lipschitz constant of G^{-1} on the interval $[o, G(y(1,d))]$ and making use of (42), we obtain

$$u(t) \le \frac{L^*}{2}\left(|\bar{x}_o - x_o| + \int_o^t u(s)\,ds\right), \quad o \le t \le 1 . \tag{49}$$

The Gronwall-Bellman inequality implies

$$|y(t,\bar{x}_o) - y(t,x_o)| \le \frac{L^*}{2}|\bar{x}_o - x_o|\, e^{\frac{L^*}{2}}, \quad o \le t \le 1 \tag{50}$$

what concludes the proof. ■

Remark 4. The function $y = y(t,x_o)$ is continuous for $t \in [o,1]$, $x_o \in I$. Moreover, it may be proved the existence of the partial derivative $\frac{\partial y}{\partial x_o}(t,x_o) = \eta(t,x_o)$, which satisfies the equation

$$G'(y(t,x_o))\,\eta(t,x_o) = \frac{t}{2} + \frac{1}{2t}\int_o^t (t^2 - s^2)\,\eta(s,x_o)\,ds , \quad o < t \le 1 . \tag{51}$$

Remark 5. For a fixed $t \in (o,1]$, the image of I by the function $y = y(t,x_o)$ is an interval enclosed in (o,∞) with its left endpoint $= o$. In what follows, our aim is to prove that the image of I by $y = y(1,x_o)$ is the whole halfaxis (o,∞).

We again consider the main problem of this paper, namely : to find a function $x = x(t)$ satisfying the conditions

$$t\,G(x'(t)) = \int_o^t s\,x(s)\,ds , \quad o \le t \le 1 ; \ x'(1) = \beta > o . \tag{52}$$

Theorem 2. Under the hypothesis (H), for every $\beta > o$, the problem (52) has a unique solution in $\mathbf{C}^1_{[o,1]}$.

Proof. We recall that, with notation $y = x'$, the initial value problem (10) may be reduced to the integral equation (13). For solving the problem (52), it must be proved the existence of $x_o \in I$ such that the corresponding solution of (13) satisfies condition $y(1,x_o) = \beta$. We know that the set $\{y(1,x_o) : x_o \in I\}$ is an interval with its left endpoint $= o$. To complete the proof, it suffices to prove that $\{y(1,x_o) : x_o \in I\} = (o,\infty)$. Indeed, let us suppose that there exists a constant $C > o$ such that $y(1,x_o) \le C$ for every $x_o \in I$. Then we have $o \le y(t,x_o) \le C$ for $t \in [o,1]$ and $x_o \in I$. In other words, the set $\{y(t,x_o) : x_o \in I\}$ of all the solutions of (13) is uniformly bounded on the interval $[o,1]$. On another hand, we deduce from Proposition 2 and from (43) that

$$o < y'(t,x_o) \le \frac{\frac{A}{2} + C}{G'(C)} , \quad o \le t \le 1 , \quad x_o \in I , \tag{53}$$

that is, the set of the derivatives is also uniformly bounded on $[o,1]$. The Arzelà-Ascoli theorem implies that the function set $\{y(t,x_o) : x_o \in I\}$ is relatively compact in $C_{[o,1]}$. Now, let $(x_o^n)_{n\geq 1}$ be an increasing sequence, convergent to A. The sequence of functions $(y(t,x_o^n))$ is increasing and pointwise convergent on $[o,1]$ to a function $y = \bar{y}(t)$, with $o \leq \bar{y}(t) \leq C$ on $[o,1]$. But this convergence is uniform and it therefore follows that $y = \bar{y}(t)$ is continuous on $[o,1]$. Making $n \longrightarrow \infty$ in the equation

$$(54) \qquad 2tG(y(t,x_o^n)) = x_o^n t^2 + \int_o^t (t^2 - s^2) y(s,x_o^n)\, ds \quad , \quad o \leq t \leq 1 \,, \quad n \geq 1$$

we obtain

$$(55) \qquad 2tG(\bar{y}(t)) = At^2 + \int_o^t (t^2 - s^2)\bar{y}(s)ds \quad , \quad o \leq t \leq 1 \,,$$

what implies $K(A) \neq \emptyset$, that is, $A \in I$. This contradiction shows that the assumption that $y(1,x_o) \leq C$ for every $x_o \in I$ is false. We conclude this proof with the remark that the mapping $x_o \longrightarrow y(1,x_o)$ is an increasing bijection from $I = (o,A)$ to (o,∞). ■

Remark 6. In connection with the problem (52), let us still notice that its solution $x = x(t)$ also satisfies the additional condition $x'(o) = o$ (i.e., $y(o,x_o)) = o$) which has — by the way — a physical meaning.

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