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Almost-periodicity of solutions of the abstract wave equation

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Introduction

In the present paper we shall give an abstract version (in terms of operator differential equations) of the well-known result about almost-periodicity in $H^1(\Omega) X L^2(\Omega)$ norm (Ω being a bounded open set in $\mathbb{R}^n$) of the vector-function $\{u(x,t), u_t(x,t)\}, x \in \Omega, t \in \mathbb{R}$, where $u_{tt} = \sum_{i,j=1}^{n} \frac{\partial}{x_i}\left(a_{ij}(x)\frac{\partial u}{x_j}\right)$ (the wave-equation) and u satisfies homogeneous boundary conditions on $\partial\Omega$ (see [1], [2]). We therefore consider a Hilbert space H and then a linear self-adjoint operator L with dense domain $D(L) \subset H$ and with range in H, such that

$$(Lh, Lh) \geq \gamma \|h\|_H^2 \text{ (some } \gamma > 0) \text{ holds true }, \forall\, h \in D(L) \qquad (0.1)$$

The abstract wave equation is the second order equation for H-valued functions $u(\cdot)$:

$$u''(t) = -L^2 u(t), t \in \mathbb{R} \qquad (0.2)$$

We will be concerned here with a class of weak solutions of 0.2); they are defined as functions $u(\cdot), \mathbb{R} \to D(L), u(\cdot) \in C^1(\mathbb{R}; H), Lu \in C(\mathbb{R}; H)$, such that the integral relation

$$\int_{\mathbb{R}} (u'(t), h'(t))_H dt = \int_{\mathbb{R}} (Lu(t), Lh(t))_H dt \qquad (0.3)$$

holds, $\forall\, h \in C_0^1(\mathbb{R}; H), Lh \in C_0(\mathbb{R}; H)(C_0^1(\mathbb{R}; H)$ and $C_0(\mathbb{R}; H)$ are functions with compact support in $\mathbb{R}$).

Supplementary assumptions are made on the operator L as follows: there exist sequences $(e_n)_1^\infty \subset D(L)$ and $(\lambda_n)_1^\infty \subset \mathbb{R}^+$, such that:

$$\begin{aligned} 0 <\lambda_1 < \lambda_2 \cdots < \lambda_n \cdots \to +\infty, (e_n, e_m)_H = \delta_{nm}, (Le_n, Le_m) \\ = \lambda_n \lambda_m \delta_{nm}, n, m = 1, 2, \dots, (Le_n, Lh) \\ = \lambda_n^2 (e_n, h), \forall h \in D(L); \|Lh - \sum_{j=1}^{n} (h, e_j) Le_j\|_H \\ \to 0 \text{ as } n \to \infty, \forall h \in D(L) \end{aligned} \tag{0.4}$$

Our main result which is established in this chapter says that under the above assumptions (0.1) - (0.3) - (0.4) the function $Lu(\cdot)$ is (Bochner) almost-periodic, $\mathbb{R} \to H$ while the function $u'(\cdot)$ is almost-periodic, $\mathbb{R} \to H$.

(this is the result stated as "Teorema 1" in [3]; the proof now given is different from that indicated in [3] - p. 13 and similar to the proof which appears in Amerio-Prouse [1]).

1.

We pass now to give the proof of the above stated results; it consists in several steps. First we have

PROPOSITION 1. *For any $h \in D(L)$ we have that*

$$\|h - \sum_{j=1}^{N} (h, e_j) e_j\| \to 0 \text{ as } N \to \infty, \|h\|^2 = \sum_{j=1}^{\infty} |(h, e_j)|^2 \tag{1.1}$$

In fact, using (0.1) we get

$$\gamma \|h - \sum_{j=1}^{N} (h, e_j) e_j\|^2 \leq \|Lh - \sum_{j=1}^{N} (h, e_j) Le_j\|^2$$

if we use (0.4) we get the first assertion. Next, if we denote $h_N = \sum_{j=1}^{N} (h, e_j) e_j$ we see that $\|h_N\|^2 = (h_N, h_N) = \sum_{j=1}^{N} |(h, e_j)|^2$; as $h_N \to h$ for $N \to \infty$, it follows that $\|h_N\|^2 \to \|h\|^2$, hence $\sum_{j=1}^{\infty} |(h, e_j)|^2 = \|h\|^2$.

Next, we prove

PROPOSITION 2. *For any $h \in D(L)$ the equality*

$$\|Lh\|^2 = \sum_{j=1}^{\infty} \lambda_j^2 |(h, e_j)|^2 \tag{1.2}$$

holds true.

In fact, if $h_N = \sum_{j=1}^N (h,e_j)e_j$ and $g_N = h - h_N$, we obtain

$$\|Lg_N\|^2 = (Lh - Lh_N, Lh - Lh_N) = \|Lh\|^2 - \left(Lh, \sum_{j=1}^N (h,e_j)Le_j\right)$$
$$- \left(\sum_{j=1}^N (h,e_j)Le_j, Lh\right) + \left(\sum_{j=1}^N (h,e_j)Le_j, \sum_{j=1}^N (h,e_j)Le_j\right)$$
$$= \|Lh\|^2 - \sum_{j=1}^N \overline{(h,e_j)}(Lh, Le_j) - \sum_{j=1}^N (h,e_j)(Le_j, Lh) + \sum_{j=1}^N |(h,e_j)|^2\lambda_j^2$$
$$= \|Lh\|^2 - \sum_{j=1}^N \overline{(h,e_j)}\lambda_j^2(h,e_j) - \sum_{j=1}^N (h,e_j)\lambda_j^2(e_j,h) + \sum_{j=1}^N |(h,e_j)|^2\lambda_j^2$$
$$= \|Lh\|^2 - \sum_{j=1}^N |(h,e_j)|^2\lambda_j^2$$

Also we see that $Lg_N = Lh - Lh_N = Lh - \sum_{j=1}^N (h,e_j)Le_j$ and from (0.4) we derive: $\|Lg_N\|^2 \to 0$ as $N \to \infty$ which means, in view of the above computation, that $\sum_{j=1}^\infty |(h,e_j)|^2\lambda_j^2 = \|Lh\|^2$.

We can now consider the demonstration of the main result. Assume therefore the relation (0.3) and then take test-functions $h(t)$ of the special form: $h(t) = \zeta(t)e_n$ where $\zeta(t) \in C_0^1(\mathbb{R})$. We get accordingly

$$\int_{\mathbb{R}} (u'(t), e_n)\zeta'(t)dt = \int_{\mathbb{R}} (Lu(t), Le_n)\zeta(t)dt, n = 1, 2, \ldots \tag{1.3}$$

With the notation $(u(t), e_n) = u_n(t)$ and using again (0.4) we obtain

$$\int_{\mathbb{R}} u_n'(t)\zeta'(t)dt = \int_{\mathbb{R}} \lambda_n^2 u_n(t)\zeta(t)dt\ ,\ \forall n \in \mathbb{N}, \forall \zeta(\cdot) \in C_0^1(\mathbb{R})$$

From the continuity of the function $\lambda_n^2 u_n(t)$ we derive: (elementary "distribution theory"): u_n'' exists and $u_n''(t) = -\lambda_n^2 u_n(t), \forall t \in \mathbb{R}, n \in \mathbb{N}$. Therefore we get

$$u_n(t) = (\cos \lambda_n t)u_n(0) + \frac{1}{\lambda_n}(\sin \lambda_n t)u_n'(0) \tag{1.4}$$

Next, from Proposition 1, we have that

$$u(t) = \sum_{j=1}^\infty u_j(t)e_j \quad \text{in H- norm, for any } t \in \mathbb{R} \tag{1.5}$$

Actually, as we see below, the series (1.5) is H-uniformly convergent on $\mathbb{R}$; the same holds for the series $\sum_{j=1}^\infty u_j(t)Le_j$ as well for $\sum_{j=1}^\infty u_j'(t)e_j$.

From (1.4) we obtain: $|u_n(t)| \le |u_n(0)| + \frac{1}{\lambda_n}|u_n'(0)|$ and

$$|u_n(t)|^2 \le 2(|u_n(0)|^2 + \frac{1}{\lambda_n^2}|u_n'(0)|^2), \forall t \in \mathbb{R} \tag{1.6}$$

The series $\sum_1^\infty |u_n(0)|^2 = \sum_1^\infty |(u(0), e_n)|^2$ is convergent to $\|u(0)\|^2$ (Proposition 1).

Also, using (0.1) and (0.4) we find that $\gamma\|e_n\|^2 \le (Le_n, Le_n) = \lambda_n^2$, hence $\lambda_n^2 \ge \gamma, n = 1, 2, \ldots$.

Therefore the series $\sum_1^\infty \frac{1}{\lambda_n^2}|u_n'(0)|^2$ is $\le \sum_1^\infty \frac{1}{\gamma}|(u'(0), e_n)|^2 = \frac{1}{\gamma}\|u'(0)\|^2$, hence it is also convergent. This gives uniform convergence on $\mathbb{R}$ of the series $\sum_{j=1}^\infty |u_j(t)|^2$.

Next, we estimate the expression

$$\|\sum_N^{N+p} u_j(t)Le_j\|^2$$

which is also (in view of (0.4)).

$$\left(\sum_N^{N+p} u_j(t)Le_j, \sum_N^{N+p} u_j(t)Le_j\right) = \sum_N^{N+p} |u_j(t)|^2\lambda_j^2 \tag{1.7}$$

From (1.6) it follows that

$$\lambda_j^2|u_j(t)|^2 \le 2(\lambda_j^2|u_j(0)|^2 + |u_j'(0)|^2).$$

Then we use Proposition 2, and we get

$$\|Lu(0)\|^2 = \sum_{j=1}^\infty \lambda_j^2|(u(0), e_j)|^2 = \sum_{j=1}^\infty \lambda_j^2|u_j(0)|^2, \tag{1.8}$$

again a convergent series.

This way we have established uniform convergence on $\mathbb{R}$ of the series $\sum_1^\infty u_j(t)Le_j$.

Finally, $u_j'(t) = -\lambda_j \sin(\lambda_j t)u_j(0) + \cos(\lambda_j t)u_j'(0)$ and accordingly $|u_j'(t)| \le \lambda_j|u_j(0)| + |u_j'(0)|$ and $|u_j'(t)|^2 \le 2(\lambda_j^2|u_j(0)|^2 + |u_j'(0)|^2)$ and we use same arguments to establish uniform convergence on

$\mathbb{R}$ of the series $\sum_1^\infty u_j'(t)e_j$.

It remains only to see that

$$Lu(t) = \sum_1^\infty u_j(t)Le_j \quad \text{and} \quad u'(t) = \sum_1^\infty u_j'(t)e_j \tag{1.9}$$

We know that $u(t) = \sum_1^\infty u_j(t)e_j$ and that L is closed.

From the convergence of $\sum_1^\infty u_j(t)Le_j$ we derive that $Lu(t) = \sum_1^\infty u_j(t)Le_j$; hence Lu is almost-periodic, $\mathbb{R} \to H$.

Finally we know that both series: $\sum_1^\infty u_j(t)e_j$ and $\sum_1^\infty u_j'(t)e_j$ are uniformly convergent. We get that $u'(t) = \sum_1^\infty u_j'(t)e_j$, and is again an almost-periodic function.

References

[1] L. Amerio-G. Prouse: Almost-periodic functions and functional equations, Van Nostrand Reinhold, New-York-Toronto-Melbourne, 1971.

[2] S. Zaidman: Solutions presque-périodiques des équations hyperboliques, Ann. scient. Ec. Norm. Sup., 3° série, t. 79, 1962, p. 151-198.

[3] S. Zaidman: Distribuzioni quasi-periodiche e applicazioni, in the book "Teoria delle Distribuzioni", Cremonese, Roma, 1961.

Note. This article appears in detailed form for the first time. It gives complete proof to a statement in [3] (Teorema 1).