LIBERTAS MATHEMATICA, VOL. XIV (1994)

A TRIBUTE TO ALEXANDRU FRODA (1894—1973)

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This year, 1994, marks the 100th anniversary of the birth of ALEXANDRU FRODA, prominent Romanian mathematician. As a tribute to his memory, some of his former students and collaborators are writing a series of articles for "LIBERTAS MATHEMATICA" about his life and work.

Alexandru Froda was born in Bucharest, Romania, on July 16, 1894. In 1919 he graduated from what was then the Şcoala Naţională de Poduri şi Şosele (in free translation, the School for Civil Engineering, later, the Polytechnic Institute) of Bucharest as a Diplomate Engineer. His calling, however, was to Mathematics, and he pursued his mathemathical studies in the Faculty of Sciences, Mathematics Section, of the University of Bucharest, from which he graduated in 1927, as a licentiate (a degree somewhere between a B.S. and an M.S., but closer to the latter). His mathematical studies were brilliant, as attested by the fact that, at the end of them, he was awarded a very competitive prize.

In 1929, he obtained his Doctor's Degree (doctorat d'état) in Mathematics from the Sorbonne University in Paris, France, with a thesis in Real Analysis. In the thesis he proved a result that stirred surprise among the experts of that time. After his return to Romania, where he earned the titles of Expert Statistician and Expert Actuary, he became a Professor at the Romanian School of Statistics and later, at the Romanian Institute of Statistics and Actuarial Sciences. In 1948 he became a Professor in the Faculty of Mathematics of the Bucharest University, a position which he held until his retirement in 1964 (when he became an Emeritus Professor). From 1948 until his death, he was also a scientific associate in the Institute of Mathematics of the Romanian Academy.

At the University of Bucharest, Alexandru Froda taught undergraduate courses in Modern Algebra and Rational Mechanics, as well as advanced undergraduate courses in Set Theory and Set and Function Theory. [We should, perhaps, mention that in Romania there are no formal courses for graduate students; these students are supposed to do independent study, work in seminars, write projects, etc., after which they must pass their required exams.] His students were very interested in his courses because of their high level and, at the same time, their accessibility. His upper level courses were attended by students who were just interested in the contents thereof, regardless of whether the regulations allowed them to earn any credit for them.

He was very punctilious in his research, as well as in his presentations. Endowed with a very subtle mathematical mind, he never pursued a subject that did not have a deep meaning for him; nor was he eager to publish papers just for the sake of publishing. He always asked, of himself and of those who sought advice from him, "What is the idea behind this problem, or result?" He always looked for depth and rigor in his work and inspired his students to do the same in theirs.

He had a gentle, caring, and sensitive character. He was always careful not to hurt anybody's feelings.

Alexandru Froda died on October 7, 1973 of a kidney malfunctioning.

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The main research field of Alexandru Froda was Real Analysis. He also obtained interesting results in other fields, such as Abstract and Linear Algebra, and I shall briefly refer to some of the latter, without any claim on completeness.

One such result concerns the reduction of a quadratic form to a sum of squares. In the paper "La réduction des formes quadratiques" (Revue de Math. pures et appl., Bucarest, vol. 5, 1960, pp. 229—239), A. Froda presents a method for obtaining the reduction theorem in the real case from the corresponding theorem in the complex case (rather than treating the two cases separately).

Another direction of research is the study of so-called isogonal systems (or sets), a concept introduced by A. Froda (cf. "Systèmes isogonaux d'un espace euclidien," Revue de Math. pures et appl., Bucarest, vol. 6, 1961, pp. 267—281). If V is a finite-dimensional real inner product space and γ is a given real number, a set S consisting of $m > 1$ vectors of V is said to be γ-*isogonal* if, for each pair x, y of vectors from S, the inner product (x,y) equals 1 if $x = y$ and $(x,y) = \gamma$ if $x \neq y$.

Thus, the concept of isogonality is a generalization of that of orthonormality. A γ-isogonal system is said to be *incomplete* or *complete* (in V) according to whether it can be properly imbedded or not in a γ-isogonal system. Properties of γ-isogonal systems are studied, for example: (i) If a γ-isogonal incomplete system S is a basis of V, there is one and only one more vector x of norm 1 in $V-S$ such that $S\bigcup\{x\}$ is still isogonal; and the extended system is then complete. (ii) If the dimension of V is n, a γ-isogonal system S which is a basis of V is incomplete in V iff $\gamma = -\frac{1}{n}$. (iii) Let S be a γ-isogonal system in V, basis of V, and complete in V, where $\dim V = n$. Let V be a proper subspace of another finite-dimensional real inner product space W (the inner product in V being inherited from W). Then S is incomplete in W iff $\gamma > -\frac{1}{n}$; and S is complete in W iff $\gamma < -\frac{1}{n}$.

To extend the results from the real to the complex case, A. Froda needed to define the angle of two nonzero vectors in a finite-dimensional complex inner product space V. This is treated in his paper, "Sur l'angle complexe, orienté, de deux vecteurs d'un espace unitaire" (Rendiconti Accad. Naz. dei Lincei, Series VIII, vol. XXX, 1961, pp. 845—853). The angle $\theta(z,t)$ of two nonzero vectors $z,t \in V$, *orientated* from z to t, is defined in the following way:

$$\theta(z,t) = \text{Arc cos}\,\frac{\text{Re}(z,t)}{\|z\|\cdot\|t\|} + i\text{Arc sin}\,\frac{\text{Im}(z,t)}{\|z\|\cdot\|t\|},$$

where the determinations are such that $0 \leq \text{Re}\,\theta(z,t) \leq \pi$ and $-\frac{\pi}{2} \leq \text{Im}\,\theta(z,t) \leq \frac{\pi}{2}$ (here $\|\ \|$ represents norm). This definition satisfies certain natural conditions, (again, when z and t are nonzero vectors), such as: (i) $\theta(z,t) = \overline{\theta(t,z)}$ (complex conjugate); (ii) $\theta(t,z) + \theta(-t,z) = \pi$; (iii) if α and β are any positive real numbers, then $\theta(\alpha z,\beta t) = \theta(z,t)$; (iv) for z and t of norm 1, $\text{Re}(z,t)$ equals 1, resp. 0, resp. -1, iff $\text{Re}\theta(z,t)$ equals 0, resp. $\frac{\pi}{2}$, resp. π, and analogous statements for the imaginary parts. The study of *isogonal* systems in finite-dimensional complex inner product spaces is carried out in the paper, "Properties of Isogonal Systems of a Unitary Space" (Revue de Math. pures et appl., Bucarest, vol. 7, 1962, pp. 69—80).

I will mention one more result of Alexandru Froda, namely, from Number Theory (cf. "Rational Triangles" (in Romanian), Com. Acad., no. 12, 1955). He was led to this problem by his studies concerning sets of distances in the Euclidean space.

He calls a triangle, say, in the Euclidean plane, *rational* if all its sides have rational length. And he shows that a triangle is rational iff its sides a, b, c can be represented in the parametric form

$$a = \rho[(m^2+n^2)q - mnp], \quad b = \epsilon\rho(m^2-n^2)q, \quad c = \epsilon\rho n(2mq-np),$$

where m, n, p, q are integers with $n > 0$, $q > 0$, m and n are relatively prime, so are p and q, $\epsilon = \pm 1$, ρ is a rational number, and the following conditions hold: $-1 < \frac{p}{2q} < 1$ and $\frac{m}{n} > 1$ if $\epsilon = 1$, $-1 < \frac{m}{n} < \frac{p}{q} < 1$ if $\epsilon = -1$.

To conclude, let me also mention the book "Algebra Superioară" ("Higher Algebra", Edit. Acad., Bucharest, 1958, 454 pp.) written by A. Froda. It contains the basic topics in Algebra for the Mathematics students, and it was for many years an important tool in the professional education of these students.