Classroom Notes

ALMOST PERIODIC DISCRETE PROCESSES

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INTRODUCTION

The discrete processes occur in the investigation of many phenomena, mainly in the case of use of computers. One of the most widely adopted definition of a discrete process can be formulated as follows:

A discrete process is a map from the additive group of the integers Z, into a complete metric space (X,d), such as R, R^n, C, C^n - with the disance function induced by the vector norm.

We shall use two different notations to designate a discrete process. Namely, if $f: Z \to X$ is a discrete process, we shall write instead $\{f(n)\}_{n \in Z}$, or $\{f_n\}_{n \in Z}$, dropping usually the subscript "$n \in Z$", since no confusion can occur (indeed, we are not going to consider in this paper discrete processes defined on a group, other than Z).

Of course, one of the most common sources for the discrete processes is the theory of difference equations, such as

(1) $$x_{n+1} = Ax_n + b_n, \ n \in Z,$$

where $\{x_n\}$ stands for the unknown process, with values in R^m or C^m, A is a square matrix of order m with real or complex entries, and $\{b_n\}$ stands for a given discrete process, with values in the same space as $\{x_n\}$. In practice we deal with solutions of (1) which are defined only on subsets of Z, and therefore, they might be regarded as restrictions of a "complete" process to a subset of its domain of definition.

More sophisticated equations (or systems) than (1) are those described by the relationships

(2) $$x_{n+1} = f(n, x_n), \ n \in Z,$$

where $f: Z \times R^m \to R^m$ (or C^m) is a given map, in general nonlinear in both arguments.

Since our main objective is to provide criteria for the almost periodicity of solutions to equations of the form (1) or (2), we shall first review the basic properties of almost periodic discrete processes. Let us point out that the definition of almost periodic discrete processes has been formulated by A. Walther in [9], shortly after Harald Bohr constructed the theory of almost periodic functions of a continuous argument (see [3] for the basic theory of almost periodic functions, where the discrete almost periodic processes, or sequences, are briefly investigated; also, see [5] for more details in regard to almost periodic discrete processes, as well as some applications to difference equations).

BASICS OF ALMOST PERIODIC DISCRETE PROCESSES

Following H. Bohr, A. Walther [9,10] has formulated the definition of almost periodic discrete processes in the form given below, excepting the fact he considered only scalar valued (R or C) processes.

Definition. Let $\{a_n\}$ be a discrete process with values in (X,d). We shall say $\{a_n\}$ is almost periodic, if to any positive ε, there corresponds a positive integer $N(\varepsilon)$, such that any set consisting of N consecutive integers contains at least one integer p with the property

$$d(a_{n+p},a_n) < \varepsilon, \quad n \in Z. \tag{3}$$

An integer p, with the property shown in the above definition, is called an ε - almost period for $\{a_n\}$, or an ε - translation number.

In order to formulate a property of almost periodic processes, which is equivalent to the definition above, we need the concept of a normal process. Namely, the discrete process $\{a_n\}$ will be called a normal process, if for any sequence $\{m_k\} \subset Z$, there exists a subsequence $\{m_k'\} \subset \{m_k\}$, for which $\{a_{n+m_k'}\}$ converges uniformly with respect to $n \in Z$, as $k \to \infty$. In other words, for every ε positive, there exists $K(\varepsilon)$, and a discrete process $\{\bar{a}_n\}$, such that

$$d(a_{n+m_k'},\bar{a}_n) < \varepsilon, \quad \text{for} \quad k \geq K(\varepsilon),\ n \in Z. \tag{4}$$

The following theorem is proven in [3]. See also [5].

Theorem 1. A necessary and sufficient condition for a discrete process to be almost periodic is that it be normal

Remark. The normality has been recognized for the first time by S. Bochner (in 1927) to be equivalent to the almost periodicity as defined by H. Bohr. Later on, this definition of almost periodicity has been extended by J. von Neuman to the case of functions defined on an abstract group. The theory of almost periodicity on groups developed into a conspicuous chapter of modern harmonic analysis (see W. Maak [6]). The case of discrete processes on Z has been overlooked for a long period of time, but seems now to get new interest, mainly in regard to the applications of the discrete models.

The following results, regarding almost periodic discrete processes with scalar values, can be found in [3] or [5].

Theorem 2. Assume that $\{a_n\}$ and $\{b_n\}$ are scalar valued almost periodic discrete processes, and c is a scalar. Then the following discrete processes are almost periodic:

i) $\{ca_n\}$; ii) $\{a_n + b_n\}$; iii) $\{a_n b_n\}$; iv) $\{a_n/b_n\}$, provided $|b_n| \geq m > 0$ for every $n \in Z$; v) $\{a_{n+k}\}$, where k is a fixed integer.

All the (arithmetic) properties stated in Theorem 2 can be easily obtained if one uses the normality as definition.

Theorem 3. Any almost periodic discrete process is bounded.

In other words, if $\{a_n\}$ is an almost periodic discrete process with values in (X,d), there exist a point $x_o \in X$, and a positive number r, such that $d(a_n,x_o) \leq r$, for any $n \in Z$. For scalar valued processes, the property simply states the existence of a positive number r, such that $|a_n| \leq r$, for any $n \in Z$.

The next property is a convolution type property, and can be stated as follows.

Theorem 4. Let $\{k_n\}$ be a summable sequence, i.e., such that

$$K = \sum_{n\in Z} |k_n| < \infty. \tag{5}$$

Then for any almost periodic discrete process $\{a_n\}$, the process $\{b_n\}$ defined by

$$b_n = \sum_{m\in Z} k_m a_{n-m}, \quad n \in Z, \tag{6}$$

is also almost periodic.

The proof immediately follows from the estimate

$$\sup_{n\in Z} |b_{n+p} - b_n| \leq K \sup_{n\in Z} |a_{n+p} - a_n|, \tag{7}$$

which is a consequence of (5) and (6).

The preservation of almost periodicity by convolution with summable processes is of special interest when investigating infinite systems of linear equations. See, for instance, I.C. Gohberg and I. A. Feldman [4].

Before establishing two more basic properties of almost periodic discrete processes, we shall briefly discuss the connection between the almost periodic functions of a continuous argument, and the almost periodic discrete processes.

Let $f: Z \to C$ (or R) be an almost periodic discrete process. Denote by $\overline{f}: R \to C$ (or R) the map defined as follows:

$$\overline{f}(t) = f_n + (t-n)(f_{n+1}-f_n),\ n \leq t < n-1,\ \forall\, n \in Z. \tag{8}$$

It can be easily seen that $\overline{f}(n) = f_n$, for any $n \in Z$.

<u>Theorem 5</u>. If $\{f_n\}$ is a scalar discrete process, then $\overline{f}(t)$ is Bohr almost periodic if and only if $\{f_n\}$ is almost periodic.

The proof of this theorem can be found in [3]. It is due to A. Walther [10]. See also I. Seynsche [8].

Theorem 5 allows to reduce the proof of certain properties for almost periodic discrete processes, to the similar proof in case of almost periodic functions in Bohr's sense.

<u>Theorem 6</u>. Let $\{a_n\}$ be an almost periodic discrete process with scalar values (R or C). Then the limit

$$\lim_{k\to\infty} \frac{a_{n+1}+a_{n+2}+\dots+a_{n+k}}{k} \tag{9}$$

exists uniformly with respect to $n \in Z$, and is independent of n. This number is called the <u>mean value</u> of the process $\{a_n\}$, and is usually denoted by $M\{a_n\}$.

The proof follows without difficulty if one notices that

$$\frac{1}{k}\int_n^{n+k} \overline{a}(t)\,dt = \frac{a_{n+1}+a_{n+2}+\dots+a_{n+k}}{k} + \frac{a_n-a_{n+k}}{2k} \tag{10}$$

where $\overline{a}(t)$ is the function constructed by linear interpolation, as shown by (8). Since $\overline{a}(t)$ is almost periodic in t, the limit of the left hand side of (1) exists uniformly in n. Due to the boundedness of $\{a_n\}$, the second term in the right hand side of (10) tends uniformly to zero, as $k \to \infty$. Hence, the limit (9) exists uniformly in n, and equals the mean value of the almost periodic function of a continuous variable $\bar{a}(t)$.

Theorem 7. Let $\{a_n\}$ be an almost periodic discrete process, and let $\{A_n\}$ be a discrete process constructed as follows:

(11) A_0 is chosen arbitrarily, and $A_{n+1} - A_n = a_n$ for any $n \in Z$.

A necessary and sufficient condition for $\{A_n\}$ to be almost periodic is that it be bounded.

The proof of this theorem is a consequence of the following formula, which follows from the definitions of $\bar{a}(t)$ and $\{A_n\}$:

$$\int_0^x \bar{a}(t)dt = A_{[x]+1} + \int_{[x]}^x \bar{a}(t)dt - \frac{a_0 + a_{[x]}}{2}, \tag{12}$$

where x stands for a positive number, while $[x]$ means the greatest integer in x (similar formula holds true for negative x).

From (12) one sees that the integral of $\bar{a}(t)$ is bounded if and only if $\{A_n\}$ is bounded. Indeed, the last two terms in the right hand side of (12) are dominated in modulus by $\sup|a_n|$, $n \in Z$, hence bounded. If $\{A_n\}$ is bounded, then the integral of the almost periodic function $\bar{a}(t)$ is bounded on the real axis, and therefore, it is almost periodic. Using again (12) and Theorem 5, one obtains the almost periodicity of $\{A_n\}$.

Remark. For Theorems 6 and 7, proofs which do not rely on the corresponding properties for the continuous case are available (see, for instance, [9], [10]). Nevertheless, the simplicity of the above proofs, based on Theorem 5, is too tempting to be overlooked.

Before concluding this section, let us point out that further properties of almost periodic discrete processes can be derived from the general theory of almost periodic functions on groups (see [3] or [6]).

Most properties of almost periodic discrete processes hold true for asymptotically almost periodic discrete processes. For the definition and the basic facts concerning the asymptotically almost periodic discrete processes we send the reader to [5], [1].

DIFFERENCE EQUATIONS: ALMOST PERIODICITY OF THEIR SOLUTIONS

As mentioned in the Introduction, our main objective is to provide some criteria of almost periodicity for the discrete process defined by means of difference equations, such as (1) or (2).

Since almost periodic discrete processes generalize the periodic ones, the occurence of almost periodic solutions to such equations is a phenomenon of higher frequency than the occurence of periodic solutions. Unfortunately, much less attention has been paid by researchers to the case of existence of almost periodic discrete processes defined by difference equations, than to the periodic case.

Papers like [2], [7], or the monograph [5], display a good deal of results regarding periodicity. Only [5] has some results related to the existence of almost periodic discrete solutions, and those are basically shaped on stability schemes.

Let us consider now the equation (1), assuming the almost periodicity of the discrete process $\{b_n\}$, which takes its values in the complex vector space of m dimensions - C^m. Since A is a square matrix of order m, with complex entries, it is natural to look for solutions of (1) which are discrete processes with values in C^m. Of course, only bounded solutions of (1) - if any - can be discussed in regard to their almost periodicity, since any almost periodic discrete process is bounded on Z. The analogous problem for continuous almost periodic functions, concerning the equation $x' = Ax + f(t)$, $t \in R$, has been first investigated by Bohr and Neugebauer (see, for instance, [3]), and conducted to a classical result: any bounded solution is almost periodic, provided $f(t)$ is almost periodic. It turns out that this result holds true for the equations of the form (1), which we shall establish below.

<u>Theorem 8</u>. Let $\{b_n\}$ be an almost periodic discrete process, A - a square matrix of order m, and assume $\{x_n\}$ is a discrete process satisfying the equation (1). Then $\{x_n\}$ is almost periodic, if and only if it is bounded.

<u>Proof</u>. We shall follow the method of proof used in [3] for the continuous case. It is well known that the matrix A is equivalent to a matrix of upper triangular form. In other words, there exists a square nonsingular matrix of order m, say T, such that $T^{-1}A\,T = B$ has the form

(13)
$$B = \begin{bmatrix} \lambda_1 & b_{12} & b_{13} & \cdots\cdots & b_{1m} \\ 0 & \lambda_2 & b_{23} & \cdots\cdots & b_{2m} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots\cdots & \lambda_m \end{bmatrix},$$

where λ_k, $k = 1,2,\ldots,m$, are the eigenvalues of A (or B). If in the equation (1) one operates the change of variable $x_n = Ty_n$, then one obtains

(14)
$$y_{n+1} = By_n + T^{-1}b_n,\ n \in Z.$$

The system (14) is of the same form as (1), with $\{T^{-1}b\}_n$ an almost periodic discrete process, but reduces considerably the difficulty in discussing the almost periodicity of its solutions. More precisely, the general case of an arbitrary matrix A is now reduced to the scalar case. Indeed, the last (scalar) equation of the system (14) is of the form

(15)
$$z_{n+1} = \lambda z_n + c_n,\ n \in Z,$$

where λ is any complex number, and $\{c_n\}$ is a scalar almost periodic discrete process. All we need to prove is that any bounded solution $\{z_n\}$ of (15) is almost periodic. It will imply that from the last equation of (14) we can derive the almost periodicity of the m-th coordinate of the process $\{y_n\}$. Then, substituting $y_n^{(m)}$ in the $(m-1)$-th equation of (14), we obtain again an equation of the form (15) for $y_n^{(m-1)}$, and so on. Therefore, we have to discuss only (15). Three distinct cases have to be considered: 1) $|\lambda| < 1$; 2) $|\lambda| > 1$; 3) $|\lambda| = 1$.

Case 1. Let us consider (15) for the value $n+p$ of the index, and then subtract side by side. One obtains

(16)
$$z_{n+p+1} - z_{n+1} = \lambda(z_{n+p}-z_n) + (c_{n+p}-c_n),$$

and since $\{z_n\}$ is a bounded process by assumption, one derives from (16)

(17)
$$\sup_{n\in Z} |z_{n+p} - z_n| \le (1-|\lambda|)^{-1} \sup_{n\in Z} |c_{n+p} - c_n|,$$

if taking into account that $\sup |z_{n+p} - z_n|$ is the same as $\sup |z_{n+p+1} - z_{n+1}|$, for $n \in Z$. But (17) shows that any $(1-|\lambda|)\varepsilon$ - almost period of $\{c_n\}$ is an ε-almost period for $\{z_n\}$. Moreover, one sees that for periodic $\{c_n\}$, one obtains periodic $\{z_n\}$, of the same period.

Case 2. One starts again from (16), which leads to

$$\sup_{n \in Z} |z_{n+p} - z_n| \leq (|\lambda|-1)^{-1} \sup_{n \in Z} |c_{n+p} - c_n|. \tag{18}$$

From (18) one derives the almost periodicity of the process $\{z_n\}$.

Case 3. In this case one can obviously write $\lambda = \exp(-i\alpha)$, for some real α. Multiplying both sides of (15) by $\exp(i(n+1)\alpha)$, one obtains

$$\exp(i(n+1)\alpha) z_{n+1} = \exp(in\alpha) z_n + \exp(i\alpha)[\exp(in\alpha) c_n], \tag{19}$$

which can be rewritten as

$$Z_{n+1} - Z_n = \tilde{c}_n, \; n \in Z, \tag{20}$$

with obvious notations. The equation (20) allows to conclude about the almost periodicity of the discrete process $\{Z_n\}$, according to Theorem 7 of this paper. Since $z_n = Z_n \exp(in\alpha)$, $n \in Z$, one sees that $\{z_n\}$ is almost periodic, being the product of two almost periodic discrete processes.

The proof of Theorem 8 is thus accomplished.

Remark. In cases 1 and 2, which correspond to eigenvalues of A with modulus different of 1, one can obtain as shown above the inequality

$$\sup_{N \in Z} |z_n| \leq M \sup_{n \in Z} |c_n|, \tag{21}$$

where $M > 0$ is a number which depends only on A. In particular, if we assume all the eigenvalues of A have moduli different of 1, then the uniqueness of the almost periodic solution is implied, and for that unique solution one obtains the estimate

$$\sup_{n \in Z} |x_n| \leq K \sup_{n \in Z} |b_n|, \tag{22}$$

with K a positive constant which depends on A only. The inequality (22) easily follows from the scalar inequalities (21).

In case 3 it is not possible to obtain inequalities like (17) or (18). Moreover, it is not possible to derive periodicity for the solution, even though we assume the periodicity of $\{b_n\}$. But periodicity is assured in cases 1 and 2, which correspond to the absence of eigenvalues with modulus equal to 1 to the matrix A.

The inequality (22) plays an important role when trying to extend the investigation of almost periodicity to the nonlinear case of systems of the form

$$x_{n+1} = Ax_n + f(n;x),\ n \in Z, \tag{23}$$

where f is almost periodic in the first argument. In (23), x stands for the process $\{x_n\}$. Equations of the form (23) have been investigated in [5], under the assumption of uniform asymptotic stability for the corresponding homogeneous equation $x_{n+1} = Ax_n$, case in which the uniqueness of the almost periodic process is assured.

Let us quote now a result from [5], which deals with the nonlinear system (2). For the proof, we send the reader to the book [5].

Theorem 9. Consider the system (2), with $\{x_n\}$ and f taking their values in R^m, and such that f is periodic with respect to the first argument. If there exists a bounded solution $\{\tilde{x}_n\}$ of (2) which is uniformly stable, then (2) has an almost periodic solution.

Let us formulate now the definition of uniform stability for the solution $\{\tilde{x}_n\}$ of the system (2). It means that to every $\varepsilon > 0$, there corresponds $\delta(\varepsilon) > 0$, such that $|x_{n_o} - \tilde{x}_{n_o}| < \delta(\varepsilon)$ implies $|x_n - \tilde{x}_n| < \varepsilon$ for $n \geq n_o$. It should be noted that the solution $\{\tilde{x}_n\}$ itself is not necessarily almost periodic.

We shall prove now a theorem concerning the almost periodicity of bounded solutions for a system of the form (2), under the assumption that the right hand side of the system is almost periodic in the first argument, and satisfies a certain monotonicity condition with respect to the second argument.

Theorem 10. Consider the system (2), and assume that f is almost periodic from Z to R^m (or to C^m), uniformly with respect to the second argument in any bounded set. Moreover, let f verify the condition

$$\langle f(n,x) - f(n,y), x - y\rangle \geq k|x - y|^2,\ k > 1, \tag{24}$$

for any $n \in Z$, and any $x,y \in R^m$ (or C^m). If (2) has a bounded solution on Z, then this solution is almost periodic.

Proof. Let $\{x_n\}$ be the bounded solution of the system (2). Writing the system

(2) for $n + p$ instead of n, and subtracting side by side, one obtains

$$x_{n+p+1} - x_{n+1} = f(n+p,x_{n+p}) - f(n+p,x_n) + f(n+p,x_n) - f(n,x_n),\ n \in Z. \tag{25}$$

By scalar multiplication of both sides in (25) by $x_{n+p} - x_n$, one obtains taking (24) into account:

$$\langle x_{n+p+1} - x_{n+1}, x_{n+p} - x_n \rangle \geq k|x_{n+p} - x_n|^2 + \langle f(n+p,x_n) - f(n,x_n), x_{n+p} - x_n \rangle. \tag{26}$$

Using Schwarz inequality, one can strengthen the inequality (26) as follows:

$$|x_{n+p+1} - x_{n+1}||x_{n+p} - x_n| \geq k|x_{n+p} - x_n|^2 - |f(n+p,x_n) - f(n,x_n)||x_{n+p} - x_n|. \tag{27}$$

Taking now the supremum with respect to $n \in Z$, one obtains from (27):

$$\sup_{n\in Z} |x_{n+p} - x_n|^2 \geq k \sup_{n\in Z} |x_{n+p} - x_n|^2 - \sup_{n\in Z} |x_{n+p} - x_n| \sup_{n\in Z} |f(n+p,x_n) - f(n,x_n)|. \tag{28}$$

Excepting the case in which $\{x_n\}$ is periodic, with period p, (28) implies

$$\sup_{n\in Z}|x_{n+p} - x_n| \leq (k-1)^{-1} \sup_{n\in Z} |f(n+p,x_n) - f(n,x_n)|. \tag{29}$$

Taking into account the hypothesis of almost periodicity of f in the first argument, (29) shows that $\{x_n\}$ is an almost periodic discrete process. If f is periodic in the first argument, with period p, then the second term in the right hand side of (28) is zero, and since $k > 1$ one sees that (28) is possible only if $x_{n+p} = x_n$ for any $n \in Z$.

The proof of Theorem 10 is now complete.

Remark. Condition (24) leads immediately to the conclusion that (2) has at most one bounded solution on Z.

To conclude the discussion, it seems rather interesting to point out the fact that the conclusion of Theorem 10 remains valid if (2) is replaced by the "higher order" system $x_{n+k} = f(n,x_n)$, with $k \in Z$ fixed.

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