BOUNDED AND ALMOST PERIODIC SOLUTIONS OF CERTAIN NONLINEAR PARABOLIC EQUATIONS

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To the memory of my father
Costache Corduneanu (1901-1981),
twice thrown in jail by the communist regime of Romania, and then
sent to forced labor camps.

1. In a recent paper [3], we have investigated the almost periodicity of bounded solutions to some classes of nonlinear elliptic equations. The method used in deriving the almost periodicity had been based on the lines method of reducing the partial differential equations to ordinary ones. The results available for ordinary differential equations made possible to establish first the almost periodicity of the approximating equations, and then to carry out this property to the original partial differential equations. The main tool to obtain the almost periodicity, as well as the validity of the approximation scheme, was the Lemma stated and proved in [3].

This paper is dedicated to similar developments, regarding nonlinear parabolic equations. In order to carry out the proof of the approximation theorem which provides the needed connection between the solutions of the partial differential equation and those of the approximating system, we will need a Lemma which can be stated as follows:

Lemma Let x be a differentiable map from R into R_+, such that

$$x'(t) \leq \omega(x(t)),\ t \in R, \tag{I}$$

with ω continuous from R_+ into R, and such that $\omega(x) < 0$ for $x > M > 0$. Then any bounded (on R) solution of (I) must satisfy $x(t) \leq M$, $t \in R$.

Proof. Since $x(t)$ must be bounded on R, there are only two exclusive cases to be examined. First, when $x(t)$ attains its maximum value at a point $t_0 \in R$. Then

$x'(t_o) = 0$, and therefore $x(t_o)$ is such that $\omega(x(t_o)) \geq 0$. This implies obviously $x(t) \leq x(t_o) = M$. Second, there is a sequence $\{t_n\}$, with $t_n \longrightarrow \infty$ (or $t_n \longrightarrow -\infty$), such that $x(t_n) \longrightarrow x_{max}$ as $n \longrightarrow \infty$. If $x(t) \longrightarrow x_{max}$ as $t \longrightarrow \infty$, then we can assume - without loss of generality - that $x'(t_n) \geq 0$ for sufficiently large n. Indeed, the contrary case would mean $x'(t)$ is negative for $t \geq T$, which is impossible. Therefore, for such a sequence $\{t_n\}$ one obtains $\omega(x(t_n)) \geq 0$ for sufficiently large n, which obviously implies $\omega(x_{max}) \geq 0$, and $x_{max} \leq M$. In case $x(t)$ does not tend to x_{max} as $t \longrightarrow \infty$, there is a number x_o, $0 \leq x_o < x_{max}$, such that for a conveniently chosen sequence $\{\bar{t}_n\}$, $\bar{t}_n \longrightarrow \infty$, one has $x(\bar{t}_n) \longrightarrow x_o$ as $n \longrightarrow \infty$. In this case, the sequence $\{t_n\}$ can be chosen in such a manner that $\{x(t_n)\}$ represents a sequence of local maxima for $x(t)$, and therefore $x'(t_n) = 0$ for any n. Again one obtains $\omega(x(t_n)) \geq 0$, from which we get $\omega(x_{max}) \geq 0$. Hence, $x_{max} \leq M$. When the sequence $\{t_n\}$, on which $x(t)$ tends to x_{max}, is such that $t_n \longrightarrow -\infty$, similar arguments hold true. For the case $x(t) \longrightarrow x_{max}$ as $t \longrightarrow -\infty$, it is useful to notice the existence of a sequence $\{t_n\}$, such that $x'(t_n) \longrightarrow 0$.

The proof of the Lemma is now completed.

Remark A sort of dual lemma can be easily obtained from the result above when t is changed into $-t$.

2. Let us consider now the nonlinear parabolic equation

$$u_t = u_{xx} + f(t,x,u), \quad (t,x) \in R\times(0,1), \tag{1}$$

under boundary value conditions

$$u(t,0) = 0, \; u(t,1) = 0, \; t \in R. \tag{2}$$

Since we are interested only in solutions defined in the whole strip $R\times(0,1)$, the initial condition does not play an important role in the forthcoming considerations.

The problem we want to solve in this paper can be formulated as follows: Assume $u(t,x)$ is a solution of (1), (2), and is bounded in the strip $R\times(0,1)$. Under what conditions can we obtain the almost periodicity of $u(t,x)$ in t, uniformly with respect to $x \in [0,1]$, if $f(t,x,u)$ is almost periodic in t?

Using the lines method of approximation, we shall give a positive answer to the above problem. Let us notice that the problem considered in this paper has been dealth with by Carla Vaghi [5], under somewhat different assumptions.

We shall assume that the function $f(t,x,u)$ occuring in (1) is continuous from

$R \times [0,1] \times R$ into R, together with its partial derivative f_u. Moreover, it will be assumed that an inequality of the form

$$f_u \leq \lambda < \pi^2, \tag{3}$$

holds true in the whole domain of definition for f, with λ fixed.

To the equation (1), with boundary value conditions (2), we shall associate the system of ordinary differential equations with n unknown functions $u_k(t), k=1,2,..,n$,

$$\frac{du_k}{dt} = (n+1)^2(u_{k+1} - 2u_k + u_{k-1}) + f(t,x_k,u_k), \tag{4}$$

where

$$u_o(t) = u_{n+1}(t) = 0. \tag{5}$$

Of course, conditions (5) are dictated by the boundary value conditions (2), while $u_k(t)$, $k = 1,2,...,n$, stands for an approximation of $u(t,x_k)$, where $x_k = k/(n+1)$, $k = 1,2,...,n$.

Using obvious vector and matrix notations, the system (4) can be rewritten in the more concise form

$$\frac{du}{dt} = (n+1)^2 A_n u + f_n(t,u), \tag{6}$$

where A_n stands for the tridiagonal matrix of order n, with -2 on the main diagonal, and 1 on the diagonals bordering the main one. As pointed out in [2], A_n is a stable matrix, and the greatest eigenvalue of the matrix $(n+1)^2 A_n$ tends to $-\pi^2$ as $n \longrightarrow \infty$.

If we assume $f(t,x,u)$ to be almost periodic in t, uniformly with respect to $(x,u) \in [0,1] \times [-N,N]$, where N is an arbitrary positive number, then the vector function $f_n(t,u)$ is almost periodic in t, in the sense of Bohr, uniformly with respect to u in any bounded set of R^n (see [1] for basic properties of almost periodic functions, depending uniformly upon parameters). It is then a natural question whether the system (6) has almost periodic solutions. The next Section of this paper is devoted to this problem.

3. Let us consider now the system (6), under assumptions specified above, and prove that any bounded (on R) solution is almost periodic. Indeed, if one writes system (6) for $t + \tau$ instead of t, and then subtract term by term, one obtains for $v(t) = u(t+\tau) - u(t)$ the system

$$\frac{dv}{dt} = (n+1)^2 A_n v + f_n(t+\tau,u(t+\tau)) - f_n(t,u(t)). \tag{7}$$

If one multiplies scalarly by v both sides of the system (7), one obtains after obvious calculations

$$(8) \qquad \frac{1}{2}\frac{d}{dt}\|v\|^2 = (n+1)^2 \langle A_n v, v\rangle + \langle f_n(t+\tau, u(t+\tau)) - f_n(t+\tau, u(t)), v\rangle$$
$$+ \langle f_n(t+\tau, u(t)) - f_n(t, u(t)), v\rangle.$$

As noticed in the preceding Section, the greatest eigenvalue of the symetric matrix $(n+1)^2 A_n$ tends to $-\pi^2$ as $n \longrightarrow \infty$. Therefore, we can write

$$(9) \qquad \langle (n+1)^2 A_n v, v\rangle \le -(\pi^2-\varepsilon)\|v\|^2, \quad \varepsilon > 0 \text{ and small}, \quad n \text{ large enough.}$$

On behalf of condition (3) one can write the inequality

$$(10) \qquad \langle f_n(t+\tau, u(t+\tau)) - f_n(t+\tau, u(t)), v\rangle \le \lambda\|v\|^2.$$

From (8), (9), and (10) one derives

$$(11) \qquad \frac{d}{dt}\|v\|^2 \le -(\pi^2-\lambda-\varepsilon)\|v\|^2 + \|v\| \sup\|f_n(t+\tau, u) - f_n(t, u)\|,$$

where the supremum is taken for $t \in R$, and for u in a bounded set (sufficiently large, to contain inside the graph of $u(t)$). Of course, for large enough n, the quantity in the paranthesis is positive, and an application of the Lemma yields the following estimate for v:

$$(12) \qquad \|v(t)\| = \|u(t+\tau) - u(t)\| \le (\pi^2-\lambda-\varepsilon)^{-1} \sup\|f_n(t+\tau, u) - f_n(t, u)\|,$$

where the supremum is taken as described above.

From the inequality (12), in which τ is an arbitrary real number, one obtains the almost periodicity of the solution $u(t)$ of (6).

The uniqueness of the bounded solution (on R) of the system (6) can be obtained by means of the Lemma, provided we write the inequality verified by $v(t) = u(t) - \bar{u}(t)$, where $u(t)$ and $\bar{u}(t)$ stand for two bounded (on R) solutions of (6).

In order to prove the existence of a bounded solution (on R) to the system (6), we need a variant of the Lemma, which will provide the needed estimates in applying a compactness argument. This variant is concerned with inequality (I) on the positive half-axis, the initial condition $x(0) = 0$ being attached. The conclusion of the Lemma remains valid, the estimate $x(t) \le M$ being verified only for $t \ge 0$.

Let $T < 0$ be an arbitrary negative number, and let $u = u(t;T)$ be the solution of (6), with $u(T;T) = 0$. It can be easily seen that $u(t;T)$ is defined for $t \ge T$, and is bounded there. Moreover, a common upper bound can be found by using the variant of the Lemma specified above. More precisely, one obtains

$$(13) \qquad \|u(t;T)\| \le 2(\pi^2-\lambda)^{-1} \sup\|f_n(t,0)\|, \quad t \ge T,$$

provided we choose n sufficiently large. The inequality (13), together with the equation (6), constitute the basis on which we can conclude about the compactness of the family $\{u(t;T) | T < 0\}$ on any compact interval of R (assuming $u(t;T)$ is set 0 for $t < T$). Therefore, the existence of a solution to (6), bounded on the entire real axis, is assured.

Let us summarize the above conducted discussion in the following

Theorem 1. *Let $f(t,x,u)$ be a continuous map from $R\times[0,1]\times R$ into R, together with its derivative f_u, and let condition (3) be verified. Assume further that $f(t,x,0)$ is bounded on $R\times[0,1]$. Then the approximating system (6), with conditions (5), has a unique bounded solution on R. If, moreover, $f(t,x,u)$ is almost periodic in t, uniformly with respect to $(x,u) \in [0,1]\times[-N,N], N > 0$ arbitrary, then the unique bounded solution is almost periodic.*

Remark 1 The rank n of the approximating system has to be chosen sufficiently large.

Remark 2 The almost periodicity of the bounded solution of (6) could be obtained also by means of Amerio's criterion (separated solutions; see [1], for instance) or from a recent result of Pankov [4] concerning the differential inequalities with monotonicity conditions.

4. Let us go back now to the equation (1), with boundary conditions (2), and assume that $u(t,x)$ is a bounded solution of this problem. We want to prove that this solution can be uniformly approximated in the strip $R\times[0,1]$, by means of the bounded solutions of the systems (6), for large values of n.

Since $u_k(t)$, $k = 1,2,\ldots,n$, have been constructed in such a manner to provide approximations for the functions $u(t,x_k)$, $k = 1,2,\ldots,n$, it is necessary to find adequate estimates for the quantities

$$\varepsilon_k(t) = u(t,x_k) - u_k(t), \quad k = 1,2,\ldots,n. \tag{14}$$

If we denote $\varepsilon(t) = \mathrm{col}(\varepsilon_1(t), \varepsilon_2(t),\ldots,\varepsilon_n(t))$, then the following equation can be easily derived from (1), taken for $x = x_k$, $k = 1,2,\ldots,n$, and (4):

$$\frac{d\varepsilon}{dt} = (n+1)^2 A_n \varepsilon + f_n(t,u+\varepsilon) - f_n(t,u) + r(t), \tag{15}$$

where $\varepsilon_o(t) = \varepsilon_{n+1}(t) = 0$, and $r(t)$ is the column vector whose coordinates are given by

$$r_k(t) = u_{xx}(t,x_k) - (n+1)^2[u(t,x_{k+1}) - 2u(t,x_k) + u(t,x_{k-1})], \tag{16}$$

$$k = 1,2,\ldots,n.$$

In (15), u stands for the vector function representing the only bounded solution of the system (6).

We shall make now one more assumption concerning the solution $u(t,x)$ which will guarantee the fact that $r_k(t)$, $k = 1,2,\dots,n$, can be done arbitrarily small on R, provided n is chosen large enough. Namely, we will assume the existence of a continuity modulus for u_{xx}, say $\omega(\delta)$, such that

$$|u_{xx}(t,x) - u_{xx}(t,\xi)| \leq \omega(|x-\xi|),\ t \in R,\ x,\xi \in [0,1], \tag{17}$$

where $\omega(\delta)$ is continuous at the right of the origin, $\omega(0) = 0$, and

$$\sqrt{n}\ \omega(\tfrac{1}{n+1}) \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{18}$$

As pointed out in [2], condition (1) holds true when u_{xx} is Hölder continuous, with index greater than 1/2.

In order to estimate the error in the approximating procedure described above, we start with equation (15). Multiplying scalarly by ε, one obtains after similar calculations to those leading to the inequality (11)

$$\frac{1}{2}\frac{d}{dt}\|\varepsilon\|^2 \leq -(\pi^2-\lambda-\eta)\|\varepsilon\|^2 - \|\varepsilon\|\ \sup\|r(t)\|, \tag{19}$$

where the supremum has to be taken on R, and η represents a small positive number. Of course, n is assumed large enough, such that the greatest eigenvalue of the matrix $(n+1)^2A_n$ does not exceed $-\pi^2 + \eta$.

From (16), (17), and (18) one obtains

$$\sup\|r(t)\| \leq \sqrt{n}\ \omega(\tfrac{1}{n+1}), \quad \text{hence} \quad \sup\|r(t)\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \tag{20}$$

Consequently, from (19) one derives by means of the Lemma

$$\|\varepsilon(t)\| \leq (\pi^2-\lambda-\eta)^{-1}\sqrt{n}\ \omega(\tfrac{1}{n+1}),\ t \in R, \tag{21}$$

and (20) guarantees the convergence of the approximation procedure, i.e.,

$$\sup_{t\in R}\|\varepsilon(t)\| \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{22}$$

It remains only to prove that the solution $u(t,x)$, which can be approximated as described above by almost periodic functions, is itself almost periodic in t, uniformly with respect to x, $x \in [0,1]$.

First, let us point out that $u(t,x)$ is almost periodic in t, for fixed x, any time x is a rational number in $(0,1)$. Indeed, any such number is an $x_k = k/(n+1)$, for conveniently chosen n and k. But x_k is also representable as $(pk)/(p(n+1))$, for any natural p. Hence, $u(t,x_k)$ is the limit of a sequence of almost periodic functions, uniformly convergent (on behalf of the convergence of the approximation procedure) on R. The almost periodicity of $u(t,x)$, of course in case $f(t,x,u)$ is almost periodic in the sense described in Section 2, will be now a simple consequence of its uniform continuity in x, uniformly with respect to $t \in R$.

Let us point out that a (classical) solution of (1), with boundary value conditions (2), in case of almost periodic $f(t,x,u)$, is necessarily bounded in the strip $R\times[0,1]$, as well as its first and second derivatives in x, and therefore it is also uniformly continuous. Indeed, for $x = 0$ one has $u(t,0) = 0$, which implies $u_t(t,0) = 0$, $t \in R$. Consequently, letting $x = 0$ in (1), one obtains $u_{xx}(t,0) + f(t,0,0) = 0$. This shows that $u_{xx}(t,0)$ is bounded on R, which combined with (17) provide the boundedness of $u_{xx}(t,x)$ in the whole strip $R\times[0,1]$. But $u(t,0) = 0 = u(t,1)$ means that for each t, there exists $x_t \in (0,1)$, such that $u_x(t,x_t) = 0$. Therefore, $u_x(t,x) = u_x(t,x) - u_x(t,x_t) = (x-x_t)u_{xx}(t,x^*)$. But $|x-x_t| < 2$, which proves the boundedness of $u_x(t,x)$. Similarly one obtains the boundedness of $u(t,x)$, and since $u_x(t,x)$ is bounded, there results that $u(t,x)$ satisfies a Lipschitz condition in x (hence, it is uniformly continuous in x, uniformly with respect to $t \in R$). Of course, the above conclusion stands in case we assume only the boundedness of $f(t,0,0)$ on R.

Now let $u(t,x)$ the solution of (1), under boundary value conditions (2), and assume $f(t,x,u)$ is almost periodic in t, uniformly with respect $(x,u) \in [0,1]\times[-N,N]$, for any $N > 0$. As seen above, $u(t,x)$ is bounded and uniformly continuous. Since $u(t,x)$ is almost periodic in t for every rational $x \in [0,1]$, from the inequality

$$(23) \quad |u(t+\tau,x) - u(t,x)| \leq |u(t+\tau,x) - u(t+\tau,x_k)| + |u(t+\tau,x_k) - u(t,x_k)| + |u(t,x_k) - u(t,x)|,$$

where x_k stands for the closest rational of the form $k/(n+1)$, with given n, to x, one derives the almost periodicity in t, uniformly with respect to x.

Let us summarize now the discussion conducted in this Section.

Theorem 2. Assume $f(t,x,u)$ *satisfies the conditions of Theorem 1. Then there exists at most one bounded solution* $u(t,x)$ *of the equation (1), with boundary value conditions (2), and it can be uniformly approximated on* R *by means of bounded solutions of the system (6), provided (17) and (18) hold true. When* $f(t,x,u)$ *is almost periodic in* t, *then any bounded solution (if any) satisfying (17) and (18) is almost periodic in* t, *uniformly with respect to* $x \in [0,1]$.

Remark 1 The uniqueness of the bounded solution is the consequence of the convergence scheme described in Section 3.

Remark 2 Under somewhat different conditions, and using the concept of almost periodicity in the sense of Stepanov instead of Bohr's almost periodicity, the existence is proven (for variational inequalities) in [4].

5. A few final remarks will be concerned with the periodic case, as well as with possible generalizations of the problems discussed above.

First, in case of a periodic $f(t,x,u)$ with respect to t, the system (6) has a unique periodic solution of the same period as $f(t,x,u)$. Indeed, this clearly follows form the inequality (12) of Section 3. Therefore, any bounded solution of (1) will result periodic, with the same period as $f(t,x,u)$. In other words, the periodic case appears as a special case of the almost periodic one.

Second, the condition (3) for the derivative f_u means that this derivative has to take values outside the spectrum of the operator $L(u) = -u_{xx}$, with the boundary conditions (2). Since the spectrum of L consists of those numbers which can be represented as $n^2\pi^2$, $n = 1,2,\ldots$, it would be interesting to see whether such results can be carried out to the case in which the derivative f_u is such that

(24) $$n^2\pi^2 < \lambda \leq f_u \leq \mu < (n+1)^2\pi^2.$$

While the periodic case seems to be treatable by means of series expansion for the solution, it seems more difficult to find out what should be done in case of almost periodic solutions.

Third, the method based on the qualitative inequality (I) can be used under more general circumstances. For instance, multidimensional elliptic operators could be substituted to u_{xx} in the right hand side of (1).

REFERENCES

[1] Corduneanu, C. *Almost Periodic Functions*, John Wiley, New York, 1968.

[2] Corduneanu, C. "Some applications of Rothe's method to parabolic and related equations," *Applied Nonlinear Analysis*, Academic Press, New York, 1979, pp. 111-121.

[3] Corduneanu, C. "Bounded and almost periodic solutions of certain nonlinear elliptic equations," Tôhoku Math. Journal, 32 (1980), pp. 265-278.

[4] Pankov, A. A. "Bounded and almost periodic solutions of evolutionary variational inequalities," Math. USSR Sbornik, 36 (1980), pp. 519-533.

[5] Vaghi, Carla "Soluzioni limitate,o quasi-periodiche,di un'equazione di tipo parabolico non lineare," Boll. Unione Mat. Italiana, (4),1 (1968), pp. 559-580.

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