BOUNDED SOLUTIONS FOR SOME GRADIENT TYPE SYSTEMS

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1. C. Corduneanu [3,Th.1] investigated the existence of a unique bounded solution of the system

$$\frac{d^2u}{dx^2} = \text{grad}_u F(x,u),\ x \in R, \tag{1.1}$$

with $u \in R^n$, $F: R \times R^n \longrightarrow R$, continuous and of class $C^{(2)}$ in u.

In this paper we shall prove the existence of a unique bounded solution of (1.1) under weaker conditions than in [3,Th.1], and also we shall discuss the asymptotic behavior of bounded solution of (1.1) on a half-line. We wish to mention that M. M. Belova [1] had similar ideas, but she did consider only a special case (space L^2), and without concern for existence.

We also prove that the bounded solution of $x'' = f(t,x)$ becomes almost periodic when the differential equation is almost periodic, and a monotonicity condition is verified.

Before we proceed further, we present some results without proof, which help simplify the proofs of our main results. Lemma 1.1 is analogous to Lemma 2 of [2,p.102], Theorem 1.1 is analogous to Theorem 1 of [3], and Lemma 1.2 is a special case of Theorem 1.1. We also need the following definitions for the spaces M and M_0:

$$M = \{x: x \in L_{loc}(R,R^n),\ \sup_{t\in R} \int_t^{t+1} |x(s)|ds = |x|_M < \infty\};$$

$$M_0 = \{x: x \in L_{loc}(R,R^n),\ \int_t^{t+1} |x(s)|ds \longrightarrow 0 \text{ as } |t| \longrightarrow \infty\},$$

where the norm in M_0 is the same as in M. It is easy to see that $M_0 \subset M$.

The definition of the spaces M and M_0, on R_+ instead of R, can be easily formulated by the reader.

Lemma 1.1. If $x \in M$, then for $\alpha > 0$

$$e^{-\alpha t}\int_0^t e^{\alpha s}|x(s)|ds \le \frac{e^{\alpha}}{e^{\alpha}-1}|x|_M, \quad \text{for} \quad t \ge 0,$$

and

$$e^{\alpha t}\int_t^{\infty} e^{-\alpha s}|x(s)|ds \le \frac{e^{\alpha}}{e^{\alpha}-1}|x|_M.$$

Theorem 1.1. Consider equation (1.1) with $u \in R^n$, *and* F *a continuous map from* $R \times R^n$ *into* R, *of class* $C^{(2)}$ *in* u, *such that*

i) $|\mathrm{grad}_u F(x,u)| \le A(r)$, $(x,u) \in R \times B_r$;

ii) $mI \le H(x,u) \le M(r)I$, $(x,u) \in R \times B_r$,

where $B_r = \{u: u \in R^n, |u| \le r, r > 0\}$, $m > 0$, $M(r) > 0$, $A(r) > 0$, *and* $H(x,u) = (\frac{\partial^2 F}{\partial u_j \partial u_i})$, $i,j = 1,2,\ldots,n$. *Then equation (1.1) has a bounded solution on* R, *which is unique, provided* $mr \ge A(0)$.

Lemma 1.2. Consider

$$\frac{d^2u}{dx^2} - Mu = f(x), \ x \in R, \tag{1.2}$$

with $M > 0$, $u \in R^n$, and f is a continuous bounded map from R into R^n. Then (1.2) has a unique bounded solution on R, given by

$$\bar{u}(x) = -\frac{1}{2\sqrt{M}}\left\{e^{\sqrt{M}x}\int_x^{\infty} e^{-\sqrt{M}t} f(t)dt + e^{-\sqrt{M}x}\int_{-\infty}^{x} e^{\sqrt{M}t} f(t)dt\right\}, \tag{1.3}$$

such that

$$\sup|\bar{u}(s)| \le \frac{1}{M}\sup|f(x)|, \ x \in R. \tag{1.4}$$

2. We will state now the first main result of this paper.

Theorem 2.1. Consider

$$\frac{d^2u}{dx^2} = \mathrm{grad}_u F(x,u), \ x \in R, \tag{2.1}$$

with $u \in R^n$, $F: R \times R^n \longrightarrow R$ *continuous and of class* $C^{(2)}$ *in* u, *such that* $\mathrm{grad}_u F(x,0) \in M$, *and* $|\mathrm{grad}_u F(x,0)|_M = K$. *Let* $H(x,u) = (\frac{\partial^2 F}{\partial u_j \partial u_i})$, $i,j = 1,2,\ldots,n$, *and assume there exist a positive number* $m > 0$, *and a positive nondecreasing function* $\alpha(r)$, *such that*

$$mI \le H(x,u) \le \alpha(r)I, \quad (x,u) \in R \times B_r, \tag{2.2}$$

where $B_r = \{u: u \in R^n, |u| \leq r, r > 0\}$ *and* $\alpha(r) \longrightarrow \infty$ *as* $r \longrightarrow \infty$. *Then equation (2.1) has a bounded solution on* R, *which is unique, provided*

$$\frac{\alpha(r)}{r^2} \longrightarrow 0 \quad \text{as} \quad r \longrightarrow \infty. \tag{2.3}$$

Proof. We shall first investigate a special case of (2.1), which is the linear system

$$\frac{d^2u}{dx^2} - \alpha u = f(x), \quad x \in R, \tag{2.4}$$

where $u \in R^n$, $\alpha > 0$ and $f \in M$. We show that equation (2.4) has a unique bounded solution on R. Indeed, according to (1.3) this solution is given by

$$\bar{u}(x) = -\frac{1}{2\sqrt{\alpha}}\left\{e^{\sqrt{\alpha}x}\int_x^{\infty} e^{-\sqrt{\alpha}s} f(s)ds + e^{-\sqrt{\alpha}x}\int_{-\infty}^{x} e^{\sqrt{\alpha}s} f(s)ds\right\}. \tag{2.5}$$

Now, an application of Lemma 1 leads us to the estimate

$$\sup|\bar{u}(x)| \leq \frac{1}{\sqrt{\alpha}} \cdot \frac{e^{\sqrt{\alpha}}}{e^{\sqrt{\alpha}}-1} |f|_M, \quad x \in R. \tag{2.6}$$

Therefore, $\bar{u}(x)$ is bounded on R, that means $\bar{u} \in C(R,R^n)$, which stands for the continuous bounded maps from R into R^n, with the supremum norm. Also, $\bar{u}(x)$ is unique, because if $\bar{v}(x)$ is any other bounded solution of (2.3), then we have

$$\frac{d^2(\bar{u}-\bar{v})}{dx^2} - \alpha(\bar{u}-\bar{v}) = 0. \tag{2.7}$$

The general solution of (2.7) is given by

$$\bar{u}(x) - \bar{v}(x) = C_1 e^{\sqrt{\alpha}x} + C_2 e^{-\sqrt{\alpha}x}, \quad x \in R.$$

Hence, $\bar{u}(x) - \bar{v}(x)$ is bounded if and only if $C_1 = C_2 = 0$. So, $\bar{u}(x) = \bar{v}(x)$, and as a result, $\bar{u}(x)$ is the unique bounded solution of (2.4).

Now, on $B_r \subset C(R,R^n)$ we shall define an operator T by means of the following equation

$$\frac{d^2u}{dx^2} - \alpha u = \operatorname{grad}_u F(x,v) - \alpha v, \tag{2.8}$$

where $\alpha = \alpha(r) > 0$ is as in (2.2). The right hand side of (2.8) is in M, because

$$\begin{aligned}\operatorname{grad}_u F(x,v) - \alpha v &= \operatorname{grad}_u F(x,v) - \operatorname{grad}_u F(x,0) - \alpha v + \operatorname{grad}_u F(x,0)\\ &= \int_0^1 \frac{d}{dt}[\operatorname{grad}_u F(x,tv)]dt - \alpha v + \operatorname{grad}_u F(x,0)\end{aligned}$$

$$= \int_0^1 [H(x,tv)dt]v - \alpha v + \text{grad}_u F(x,0)$$

$$= [\int_0^1 H(x,tv)dt - \alpha I]v + \text{grad}_u F(x,0).$$

Using (2.2) we obtain

$$|\text{grad}_u F(x,v) - \alpha v| \leq (\alpha - m)|v| + |\text{grad}_u F(x,0)|,$$

from which the assertion follows. Hence, equation (2.8) has a unique bounded solution on R, i.e., this solution is in $C(R,R^n)$.

We denote $u = Tv$, where $v \in B_r$, and u is the only bounded solution of (2.8). Therefore, $T: B_r \longrightarrow C(R,R^n)$. Now, we show that the operator T is a contraction on B_r, provided r is large enough. At first we note that from (2.2) we have

$$\langle \text{grad}_u F(x,u) - \text{grad}_u F(x,v), u-v \rangle \geq m|u-v|^2,$$

which shows the monotonicity of the right hand side of (2.1). Let $u = Tv$ and $\bar{u} = T\bar{v}$ for $v, \bar{v} \in B_r$. Then (2.8) implies that

$$\frac{d^2(u-\bar{u})}{dx^2} - \alpha(u-\bar{u}) = \text{grad}_u F(x,v) - \text{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v}). \tag{2.9}$$

We show that the right hand side of (2.9) is actually in $C(R,R^n)$ (we know it is in M). We have

$$\text{grad}_u F(x,v) - \text{grad}_u F(x,\bar{v}) = \int_0^1 \frac{d}{dt}[\text{grad}_u F(x,\bar{v}+t(v-\bar{v}))]dt$$

$$= \left[\int_0^1 H(x,\bar{v}+t(v-\bar{v}))dt\right](v-\bar{v}).$$

Therefore, using (2.2) we obtain

$$|\text{grad}_u F(x,v) - \text{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v})| = \left|\left[\int_0^1 H(x,\bar{v}+t(v-\bar{v}))dt - \alpha I\right](v-\bar{v})\right.$$

$$\leq (\alpha - m)|v-\bar{v}|,$$

or

$$\sup_{x\in R}|\text{grad}_u F(x,v) - \text{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v})| \leq (\alpha - m) \sup_{x\in R}|v(x) - \bar{v}(x)|. \tag{2.10}$$

Hence, the right hand side of (2.9) is in $C(R,R^n) \subset M$. Equation (2.9) is a special form of equation (1.2), and therefore, by Lemma 1.2, it has a unique bounded solution on R, that satisfies inequality (1.4). That means

$$\sup_{x\in R}|u(x)-\bar{u}(x)| \le \frac{1}{\alpha} \sup_{x\in R}|\mathrm{grad}_u F(x,v)-\mathrm{grad}_u F(x,\bar{v})-\alpha(v-\bar{v})|,$$

or from (2.10)

$$\sup_{x\in R}|u(x)-\bar{u}(x)| \le \frac{\alpha-m}{\alpha} \sup_{x\in R}|v(x)-\bar{v}(x)|. \tag{2.11}$$

Inequality (2.11) proves that T is a contraction on B_r.

We must prove now that $TB_r \subset B_r$. Indeed the right hand side of (2.8) can be written as

$$\mathrm{grad}_u F(x,v) - \alpha v = - [\alpha I-H(x,\tilde{v}(x))]v(x) + \mathrm{grad}_u F(x,u)|_{u=0}.$$

Since $\alpha I - H(x,\tilde{v}(x))$ is a symmetric matrix, then

$$|[\alpha I-H]v| \le |\alpha I-H||v| \le (\alpha-m)|v| \le (\alpha-m)r.$$

Now applying (1.4), (2.3), (2.6) and above inequality we obtain

$$\sup_{x\in R}|u(x)| \le \frac{\alpha(r)-m}{\alpha(r)} \cdot r + \frac{1}{\sqrt{\alpha(r)}} \cdot \frac{e^{\sqrt{\alpha(r)}}}{e^{\sqrt{\alpha(r)}}-1} |\mathrm{grad}_u F(x,0)|_M,$$

which implies

$$\sup_{x\in R}|u(x)| \le r,$$

provided r is chosen such that

$$\frac{e^{\sqrt{\alpha(r)}}}{e^{\sqrt{\alpha(r)}}-1} \cdot \frac{\sqrt{\alpha(r)}}{r} \le \frac{m}{K},$$

where $K \ge |\mathrm{grad}_u(x,0)|$. This completes the proof of Theorem 2.1, since the left hand side in the last inequality tends to zero as $r \longrightarrow \infty$, on behalf of (2.3).

Remark 1. In Theorem 2.1, if α is bounded, then one can define an operator T from $C(R,R^n)$ to $C(R,R^n)$ by means of equation (2.8).

Remark 2. The scalar case of this theorem has been dealt with by C. Corduneanu in [4], where the first derivative can also occur in the right hand side.

Theorem 2.2. Consider

$$\frac{d^2u}{dx^2} = \mathrm{grad}_u F(x,u), \; x \in R_+, \tag{2.12}$$

with $u \in R^n, F: R_+ x R^n \longrightarrow R$ *continuous, and of class* $C^{(2)}$ *in* u, *such that* $\text{grad}_u F(x,0) \in M_0$. *Let* $H(x,u)$ *be the Hessian of* F, *and suppose there exist a number* $m > 0$, *and a function* $\alpha(r) > 0$, *such that*

$$mI \le H(x,u) \le \alpha(r)I, \quad (x,u) \in R_+ x B_r, \tag{2.13}$$

where $B_r = \{u: u \in C(R_+, R^n), |u|_C \le r\}$. *Then equation (2.12) has a unique bounded solution on* R_+, *such that* $u(0) = u_0$, *provided* $\alpha(r)$ *verifies (2.3). Furthermore, this solution tends to zero as* $x \longrightarrow \infty$.

Proof. As before, at first we shall consider the special case of (2.12) which is the linear system

$$\frac{d^2u}{dx^2} - \alpha u = f(x), \quad x \in R_+, \tag{2.14}$$

with $f \in M_0, \alpha > 0$, and $u(0) = u_0$. A particular solution of (2.14) is given by

$$\bar{u}(x) = -\frac{1}{2\sqrt{\alpha}}\left\{e^{\sqrt{\alpha}x}\int_x^\infty e^{-\sqrt{\alpha}s}f(s)ds + e^{-\sqrt{\alpha}x}\int_0^x e^{\sqrt{\alpha}s}f(s)ds\right\}, \; x \in R_+. \tag{2.15}$$

Since $f \in M_0 \subset M$, then by Lemma 1.1, this solution is bounded on R_+ and therefore, $\bar{u} \in C(R_+, R^n)$. We show that $\lim_{x\to\infty} |\bar{u}(x)| = 0$, or equivalently,

$\lim_{x\to\infty} e^{\sqrt{\alpha}x}\int_x^\infty e^{-\sqrt{\alpha}s}|f(s)|ds = 0$ and $\lim_{x\to\infty} e^{-\sqrt{\alpha}x}\int_0^x e^{\sqrt{\alpha}s}|f(s)|ds = 0$. From Lemma 1.1,

there exists $K > 0$ such that

$$e^{\sqrt{\alpha}x}\int_x^\infty e^{-\sqrt{\alpha}s}|f(s)|ds \le K \sup_{t\ge x}\int_t^{t+1}|f(s)|ds,$$

and since $f \in M_0$, then

$$\lim_{x\to\infty} e^{\sqrt{\alpha}x}\int_x^\infty e^{-\sqrt{\alpha}s}|f(s)|ds = 0.$$

Also, by Lemma 1.1, we have for $0 < x_0 < x$

$$e^{-\sqrt{\alpha}x}\int_{x_0}^x e^{\sqrt{\alpha}s}|f(s)|ds \le \frac{e^{\sqrt{\alpha}}}{e^{\sqrt{\alpha}}-1}\sup_{t\ge x_0}\int_t^{t+1}|f(s)|ds,$$

and $f \in M_0$ implies

$$\lim_{x\to\infty} e^{-\sqrt{\alpha}x} \int_0^x e^{\sqrt{\alpha}s} |f(s)|ds = 0.$$

Therefore, $\lim_{x\to\infty}|\bar{u}(x)| = 0$.

The general solution of (2.14) is of the form

$$u(x) = C_1 e^{-\sqrt{\alpha}x} + C_2 e^{\sqrt{\alpha}x} + \bar{u}(x),$$

and $u(x)$ is bounded on R_+ if and only if $C_2 = 0$. So

$$u(x) = C_1 e^{-\sqrt{\alpha}x} + \bar{u}(x),\ x \in R_+, \tag{2.16}$$

where $C_1 = u_0 - \bar{u}(0)$. Therefore, $\lim_{x\to\infty}|u(x)| = 0$, and $u(x)$ is bounded on R_+, that means $u \in C_0(R_+,R^n)$.

Now on $B_r \subseteq C_0(R_+,R^n) = \left\{u \in C(R_+,R^n):\ u(\infty) = 0\ |u|_C \le r,\right\}$ we shall define an operator T by means of the following equation:

$$\frac{d^2u}{dx^2} - \alpha u = \text{grad}_u F(x,v) - \alpha v,\ u(0) = u_0, \tag{2.17}$$

where $v \in C_0(R_+,R^n)$. At first, we show that the right hand side of (2.17) is an M_0. We know that

$$|\text{grad}_u F(x,v)-\alpha v| \le (\alpha-m)|v| + |\text{grad}_u F(x,0)|,$$

or

$$\int_x^{x+1} |\text{grad}_u F(t,v)-\alpha v|dt \le (\alpha-m) \sup_{t\in[x,x+1]} |v| + \int_x^{x+1} |\text{grad}_u F(t,0)|dt.$$

Since $v \in C_0(R_+,R^n)$ and $\text{grad}_u F(x,0) \in M_0$, then

$$\lim_{x\to\infty} \int_x^{x+1} |\text{grad}_u F(t,v)-\alpha v|dt = 0.$$

This shows that the right hand side of (2.17) is in M_0. Therefore, (2.17) is a form of (2.14), and hence (2.17) has a bounded solution u on R_+, which tends to zero as $x \longrightarrow \infty$.

We denote $u = Tv$, where $v \in C_0(R_+,R^n)$ and u is the only bounded solution of (2.17) which tends to zero. Therefore, $T\colon B_r \longrightarrow C_0(R_+,R^n)$. We show that the operator T is a contraction on B_r. Let $u = Tv$ and $\bar{u} = T\bar{v}$ for $v,\bar{v} \in C_0(R_+,R^n)$. Then (2.17) implies that

$$\frac{d^2(u-\bar{u})}{dx^2} - \alpha(u-\bar{u}) = \operatorname{grad}_u F(x,v) - \operatorname{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v}),\ u(0) - \bar{u}(0) = 0. \quad (2.19)$$

Using the same argument as in Theorem 2.1, we get

$$|\operatorname{grad}_u F(x,v) - \operatorname{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v})| \leq (\alpha-m)|v-\bar{v}|, \quad (2.20)$$

which shows that the right hand side of (2.19) is in $C_0 \subset M_0$. Therefore, $u(x) - \bar{u}(x)$ is bounded on R_+, and $\lim_{x\to+\infty}|u(x)-\bar{u}(x)| = 0$.

It has been shown in [4] that the bounded solution of (2.14), with $u(0) = 0$, and $f \in C(R_+,R^n)$, verifies

$$\sup_{x\in R_+} |u(x)| \leq \frac{1}{\alpha} \sup_{x\in R_+} |f(x)|. \quad (2.21)$$

Applying the above estimate to our problem, and using the fact that the right side of (2.19) is in $C_0(R_+,R^n)$, we observe that

$$\sup_{x\in R_+} |u-\bar{u}| \leq \frac{1}{\alpha} \sup_{x\in R_+} |\operatorname{grad}_u F(x,v) - \operatorname{grad}_u F(x,\bar{v}) - \alpha(v-\bar{v})|,$$

or using (2.20), (2.21)

$$\sup_{x\in R_+} |u-\bar{u}| \leq \frac{\alpha-m}{\alpha} \sup_{x\in R_+} |v-\bar{v}|. \quad (2.22)$$

Equation (2.22) proves that T is a contraction on $B_r \subset C_0(R_+,R^n)$. We can use the same argument as in Theorem 2.1, to show that $TB_r \subset B_r$. The proof is complete.

3. Let us now consider the general form of equation (2.1) which is

$$x'' = f(t,x),\ t \in R. \quad (3.1)$$

We shall prove that any bounded solution of (3.1) is a almost periodic, if the differential equation (3.1) is almost periodic, and a monotonicity condition holds true. To do this, we need the following Lemma [3]:

Lemma 3.1. Let $x = x(t)$ be a $C^{(2)}$- map from R into $[0,\infty)$, such that

$$x'' \geq w(x),\ t \in R, \quad (3.2)$$

with $w(x)$ continuous on $[0,\infty)$, and satisfying

$$w(x) > 0 \quad \text{for} \quad x > M > 0. \quad (3.3)$$

If it is known that $x(t)$ is bounded on R, then $x(t) \leq M$, for $t \in R$.

Theorem 3.1. *Consider the differential system (3.1), where $f(t,x)$ is almost periodic on t, uniformly with respect to x in any bounded set of R^n. If $x(t)$ is a bounded solution of (3.1), then $x(t)$ is almost periodic under the condition:*

$$\langle f(t,x)-f(t,y),x-y\rangle \geq k\|x-y\|^2, \tag{3.4}$$

where k is a positive constant.

Proof. Since $x''(t+\tau) = f(t+\tau,x(t+\tau))$, we have

$$\frac{d^2}{dt^2}[x(t+\tau)-x(t)] = f(t+\tau,x(t+\tau))-f(t,x(t))$$

$$= [f(t+\tau,x(t+\tau))-f(t+\tau,x(t))] + [f(t+\tau,x(t))-f(t,x(t))]. \tag{3.5}$$

Let $u(t) = x(t+\tau) - x(t)$. Then from (3.5) we derive

$$\langle u'',u\rangle = \langle [f(t+\tau,x(t+\tau))-f(t+\tau,x(t))],u\rangle + \langle [f(t+\tau,x(t))-f(t,x(t))],u\rangle. \tag{3.6}$$

We know that

$$\langle u'',u\rangle = \frac{d}{dt}\langle u',u\rangle - \langle u',u'\rangle \quad \text{and} \quad \langle u',u\rangle = \frac{1}{2}\frac{d}{dt}\|u\|^2,$$

and therefore

$$\langle u'',u\rangle = \frac{1}{2}\frac{d^2}{dt^2}\|u\|^2 - \|u'\|^2. \tag{3.7}$$

Now using (3.4), (3.6) and (3.7), we have

$$\frac{1}{2}\frac{d^2}{dt^2}\|u\|^2 - \|u'\|^2 \geq k\|u\|^2 + \langle [f(t+\tau,x(t))-f(t,x(t))],u\rangle$$

or

$$\frac{d^2}{dt^2}\|u\|^2 \geq 2k\|u\|^2 - 2\|u\| \sup_{t\in R}\|f(t+\tau,x(t))-f(t,x(t))\|.$$

Then using Lemma 3.1, with $w(r) = 2kr^2 - 2r \sup_{t\in R}\|f(t+\tau,x(t))-f(t,x(t))\|$, we obtain

$$\|u\| \leq k^{-1}\sup_{t\in R}\|f(t+\tau,x)-f(t,x)\|, \quad \|x\| \leq r = \sup_{t\in R}\|x(t)\|,$$

or

$$\|x(t+\tau)-x(t)\| \leq k^{-1}\sup_{t\in R}\|f(t+\tau,x)-f(t,x)\|,$$

which shows that $x(t)$ is almost periodic. The proof is complete.

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REFERENCES

[1] Belova, M. M. "Asymptotic Properties of Bounded Solutions of Systems of Nonlinear Second-Order Differential Equations," *Differential Equations*, Vol. 12, No. 11, pp. 1354-1358, (1976).

[2] Coppel, W. A. *Stability and Asymptotic Behavior of Differential Equations*, Heath, Boston, (1965).

[3] Corduneanu, C. "Bounded and Almost Periodic Solution of Certain Nonlinear Elliptic Equations," *Tôhoku Mathematical Journal*, Vol 32, No. 2, pp. 265-278, (1980).

[4] Corduneanu, C. "On the Existence of Bounded Solutions for a Class of Second-Order Differential Equations (Romanian)," *St. Cerc. Mat. (Jaşi). t. X*(1959), pp. 27-34; MR 22(A) #5790