ON CERTAIN CONSTRUCTIONS IN UNIVERSAL ALGEBRA INSPIRED BY AUTOMATA THEORY

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I. INTRODUCTION

We shall discuss in this paper the notion of admissible decomposition of an algebra. This concepts originates in a certain construction from Automata Theory.

Let $\mathcal{A} = (A,F)$ be an algebra, where $A \neq \emptyset$ is the base set of algebra and $F = \{f_o,\dots,f_r\dots\}$ is the set of operations. We refer the reader to [2] for notations and basic facts. If f is an operation from F then n_f will stand for the arity of this operation.

Definition 1. A decomposition of a set A is a collection of non-void subsets of A, $\mathcal{D} = \{B_i \mid i \in I\}$ such that $\cup\{B_i \mid i \in I\} = A$; the sets B_i are the blocks of $\mathcal{D}$. If $\mathcal{A} = (A,F)$ is an algebra then $\mathcal{D}$ is an admissible decomposition (a.d.) if, for any collection of blocks $\{B_1,\dots,B_{n_f}\}$ of $\mathcal{D}$ and any $f \in F$ there is a block B of $\mathcal{D}$ such that $f(B_1,\dots,B_{n_f}) \subseteq B$.

Clearly, if $\mathcal{D}$ is an admissible decomposition such that for any B_i, B_j, with $i \neq j$, $B_i \cap B_j = \emptyset$ then $\mathcal{D}$ is nothing but the partition associated to a congruence. However, in general, for admissible decompositions the block B which contains $f(B_1,\dots,B_{n_f})$ is not uniquely determined by f and $B_1,\dots,B_{n_f}$.

Let $\mathcal{D}$ be an a.d. of the algebra $\mathcal{A} = (A,F)$.

Definition 2. A $\mathcal{D}$ quotient of $\mathcal{A}$ is an algebra $\mathcal{B} = (\mathcal{D},\overline{F})$, where $\overline{f}(B_1,\dots,B_{n_f}) = B$ if B is a $\mathcal{D}$-block such that $f(B_1,\dots,B_{n_f}) \subseteq B$. The set $\overline{F}$ contains the barred version of the elements of F.

An admissible decomposition $\mathcal{D}$ generates many $\mathcal{D}$-quotients because $f(B_1,\dots,B_{n_f})$ can be contained in many $\mathcal{D}$-blocks. We shall denote by $\mathcal{A}/\mathcal{D}$ the set of all $\mathcal{D}$-quotients of $\mathcal{A}$.

The Yoeli Algebra attached to $\mathcal{B} = (\mathcal{D},\overline{F}) \in \mathcal{A}/\mathcal{D}$ is the algebra $Y(\mathcal{A},\mathcal{B}) = (K,\overline{F})$, where $K = \{(a,B) \mid a \in B \in \mathcal{D}\}$ and $\overline{f}((a_1,B_1),\ldots,(a_{n_f})) = (F(a_1,\ldots,a_{n_f}),$ $\overline{f}(B_1,\ldots,B_{n_f}))$. The reader will notice that $f(a_1,\ldots,a_{n_f}) \in \overline{f}(B_1,\ldots,B_{n_f})$, hence this construction is a correct one. If $\pi\colon K \to \mathcal{D}$ is the projection given by $\pi((a,B)) = B$ then $((a,B), (a',B')) \varepsilon \ker \pi$ iff $B = B'$, i.e. if there is a block B containing both a and a'. Thus, we can attach a partition on K, $P_{\mathcal{D}}$, to the decomposition $\mathcal{D}$. A block of $P_{\mathcal{D}}$ is characterized by a block of the decomposition $\mathcal{D}$.

II. SUBDIRECT PRODUCTS AND ADMISSIBLE DECOMPOSITIONS

Let $\{\mathcal{A}_i = (A_i,F) \mid i \in I\}$ be a family of algebras having the same type and suppose that $C \subseteq \Pi\{\mathcal{A}_i \mid i \in I\}$ is a subdirect product of this family. If $p \in C$ we shall denote by $B_{p,i}$ the subset of A_i given by

$$B_{p,i} = \{a \mid a \in A_i,\ \exists\ q \in C,\ q(i) = a,\ q(j) = p(j),\ \text{for } j \neq i\}. \tag{1}$$

Lemma 1. The collection $\mathcal{D} = \{B_{p,i} \mid p \in C\}$ is an admissible decomposition of the algebra $\mathcal{A}_i = (A_i,F)$.

Proof. According to the definition of the subdirect product, for each $a \in A_i$, there is a $p \in C$ such that $p(i) = a$. Therefore, $a \in B_{p,i}$ hence $\mathcal{D} = \{B_{p,i} \mid p \in C\}$ is a decomposition of A_i.

Moreover, if $a_k \in B_{p_k,i}$, for $1 \leq k \leq n_f$ then there exist q_k, $1 \leq k \leq n_f$ such that $a_k = q_k(i)$ and $q_k(j) = p_k(j)$, for $j \neq i$. Thus, $f(a_1,\ldots,a_{n_f}) = f(q_1,\ldots,q_k)(i)$ and $f(q_1,\ldots,q_k)(j) = f(p_1,\ldots,p_k)(j)$, for $j \neq i$, which indicates that $f(a_1,\ldots,a_{n_f}) \in B_{f(p_1,\ldots,p_k),i}$. We obtained the admissibility of the decomposition $\mathcal{D}$.

Lemma 2. The *Yoeli algebra* $Y(\mathcal{A},\mathcal{B})$ attached to the algebra $\mathcal{A} = (A,F)$ and the $\mathcal{D}$-quotient $\mathcal{B}$ is a subdirect product of $\mathcal{A}$ and $\mathcal{B}$. Moreover, $\mathcal{D}$ is the decomposition induced on $\mathcal{A}$ by this subdirect product.

Proof. If $Y(\mathcal{A},\mathcal{B}) = (K,\overline{F})$ it is easy to see that K is a subalgebra of $\mathcal{A} \times \mathcal{B}$. For any $a \in A$ there is a $B \in \mathcal{D}$ such that $a \in B$ (hence $(a,B) \in K$) because $\mathcal{D}$ is a decomposition; similarly, for any $B \in \mathcal{D}$ there is an a such that $(a,B) \in K$, because each $\mathcal{D}$-block is non-void.

For $p = (a,B)$ we have $B_{p,1} = B$, which proves the second part of this lemma. Thus, we obtained the following

Theorem 1. *Any a.d. of an algebra is induced by a subdirect product via the cononical mechanism described by (1).*

We shall discuss now another construction which is inspired by the structural theory of finite automata (see [1],[3]).

Let $M^{\circledast} = \cup\ (M^j | j \geq 0\}$, where M^j is the j^{th} Cartesian power of the set M; M^0 consists of a unique element 1, $M^0 = \{1\}$.

Assume that $A = (A,F)$, $A' = (A',G)$ are two algebras for which there is a partial mapping $\omega : A^{\circledast} \times F \rightarrow G$ such that

i) $\mathrm{dom}\ \omega = \cup\{A^{n_f} \times \{f\} | f \in F\}$ and

ii) if $\omega(a_1,\ldots,a_{n_f},f) = g$ then both f and g have the same arity.

Definition 3. The cascade product of A and A' via the mapping ω is the algebra $A\ \omega\ A' = (A \times A', \tilde{F})$ such that

$$\tilde{f}((a_1,a_1'),\ldots,(a_{n_f},a_{n_f}')) = (f(a_1,\ldots,a_{n_f}),\ \omega(a_1,\ldots,a_{n_f},f)(a_1',\ldots,a_{n_f}')).$$

Here $a_1,\ldots,a_{n_f} \in A$, $a_1',\ldots,a_{n_f}' \in A'$ and for each $f \in F$, $\tilde{f}$ is a n_f- ary operation.

When both A and A' are unary algebras (i.e., essentially finite automata) then $A\ \omega\ A'$ is the cascade product of automata.

We shall use subsequently the concept of derived algebra of an algebra as defined in [2]. If A' is a derived algebra of A we shall say that A covers A'.

Let $\iota_A = \{(a,a) | a \in A\}$ be the diagonal relation on A. Assume that A is a finite set with $|A| = n$. If θ is an equivalence on A, let $m_\theta = \max\limits_{a \in A} |[a]_\theta|$. Suppose that τ is another equivalence on A for which $\theta \cap \tau = \iota_A$. Two distinct elements a', a'' from a θ-class $[a]_\theta$ cannot be τ-equivalent. Therefore, A/τ contains at least m_θ blocks. It is possible to construct actually an equivalence τ_0 which has exactly m_θ blocks by labeling the elements of each θ-class by $1,2,\ldots,m_\theta$; two elements of A will be τ-echivalent if they have the same label.

Theorem 2. *Let θ be a congruence of the algebra $A = (A,F)$. There exist two algebras B and C such that $B = A/\theta$ and A is a derived algebra of a cascade product $B\ \omega\ C$.*

Proof. Let τ be an equivalence on A such that $\theta \cap \tau = \iota_A$. We shall consider the family of operations $G = \{ < [a_1]_\theta,\dots,[a_{n_f}]_\theta, f > | a_1,\dots,a_{n_f} \in A,\ f \in F\}$ on the set A/τ, where

$$< [a_1]_\theta,\dots,[a_{n_f}]_\theta, f > ([a'_1]_\tau,\dots,[a'_{n_f}]_\tau) =$$

$[f([a_1]_\theta \cap [a'_1]_\tau,\dots,[a_{n_f}]_\theta\ [a'_{n_f}]_\tau)]_\tau$, if $[a_j]_\theta \cap [a'_j]_\tau \neq \emptyset$ for $1 \leq j \leq n_f$ and

arbitrary, otherwise.

We have used here the fact that if $[a_j]_\theta \cap [a'_j]_\tau \neq \emptyset$ then this set can contain at most one element.

The mapping ω: $(A/\theta)^{\circledast} \times F \to G$ is given by $\omega([a_1]_\theta,\dots,[a_{n_f}]_\theta,f) = < [a_1]_\theta,\dots,[a_{n_f}]_\theta, f >$; the algebra $\mathcal{C}$ is defined by $\mathcal{C} = (A/\tau,G)$.

Let $W \subseteq A/\theta \times A/\tau$ be the set $W = \{([a]_\theta,[a']_\tau | [a]_\theta \cap [a']_\tau \neq \emptyset\}$; we claim that W is a subalgebra of $\mathcal{B}\ \omega\ \mathcal{C}$.

Indeed, assume that $([a_1]_\theta,[a'_1]_\tau),\dots,([a_{n_f}]_\theta,[a'_{n_f}]_\tau)$ are n_f arbitrary elements of $\mathcal{B}\ \omega\ \mathcal{C}$. We have

$\tilde{f}(([a_1]_\theta,[a'_1]_\tau),\dots,([a_{n_f}]_\theta,[a'_{n_f}]_\tau)) = ([f(a_1,\dots,a_{n_f})]\ _\theta$,

$\omega([a_1]_\theta,\dots,[a_{n_f}],f)([a'_1]_\tau,\dots,[a'_{n_f}]_\tau) =$

$= ([f(a_1,\dots,a_{n_f})]_\theta,[f(a''_1,\dots,a''_{n_f})]_\tau)$, where $\{a''_j\} = [a_j]_\theta \cap [a'_j]_\tau$

for $1 \leq j \leq n_f$. Due to the fact $(a_j,a''_j) \in \theta$ for $1 \leq j \leq n_f$

we obtain $(f(a_1,\dots,a_{n_f}),\ f(a''_1,\dots,a''_{n_f})) \in \theta$, which indicates that

$f(a''_n,\dots,a''_{n_f}) \in [f(a_1,\dots,a_{n_f})]_\theta \cap [f(a''_1,\dots,a''_{n_f})]_\tau$. Thus,

$\tilde{f}(([a_1]_\theta,[a'_1]_\tau),\dots,([a_{n_f}]_\theta,[a'_{n_f}]_\tau)) \in W$, which points out that W is a subalgebra of $\mathcal{B}\ \omega\ \mathcal{C}$.

Moreover, the mapping ψ: $W \to A$ given by $\psi([a]_\theta,[a']_\tau) = a$ is a surjective homomorphism. Eventually, we shall remark that if $\mathcal{A}$ is a finite algebra then we can choose τ such that the base set of $\mathcal{C}$ will contain $m = \max\{|[a]_\theta|\ | a \in A\}$ blocks.

Lemma 3. Let $\mathcal{D}$ be an a.d. of the algebra $\mathcal{A} = (A.F)$. If $\theta_{\mathcal{D}}$ is the congruence related to the partition $P_{\mathcal{D}}$ of the Yoeli algebra $Y(\mathcal{A},\mathcal{B})$ then $Y(A,B)/\theta_{\mathcal{D}}$ is isomorphic to $\mathcal{B}$.

Proof. Assume that $Y(\mathcal{A},\mathcal{B}) = (K,\overline{F})$. The projection $\pi\colon K \to \mathcal{D}$ previously defined is a surjective homomorphism and we have $\ker \pi = \theta_{\mathcal{D}}$. Therefore, $\mathcal{B}$ is isomophic to $Y(\mathcal{A},\mathcal{B})/\ker \pi$, i.e. to $Y(\mathcal{A},\mathcal{B})/\theta_{\mathcal{D}}$.

Theorem 3. Let $\mathcal{D}$ *be an a.d. of the algebra* $A = (A,F)$. *For every algebra* $\mathcal{B} = A/\mathcal{D}$ *there exists an algebra* C *such that* A *is a derived algebra of a certain cascade product* $\mathcal{B} \,\omega\, C$.

Proof. If $\mathcal{B} = (\mathcal{D},\overline{F}) \in A/\mathcal{D}$ we can consider the Yoeli algebra $Y(\mathcal{A},\mathcal{B}) = (K,\overline{F})$ and the partition $P_{\mathcal{D}}$. It is easy to see that the equivalence $\theta_{\mathcal{D}}$ generated by $P_{\mathcal{D}}$ is a congruence of $Y(\mathcal{A},\mathcal{B})$; the class $[(a,B)]_{\theta_{\mathcal{D}}}$ has the same cardinality as the set B. Using Theorem 2 we can find an algebra $\mathcal{C}$ such that $Y(\mathcal{A},\mathcal{B})$ is a derived algebra of the cascade product $Y(\mathcal{A},\mathcal{B})/\theta_{\mathcal{D}} \,\omega\, \mathcal{C}$.

According to Lemma 3, $Y(\mathcal{A},\mathcal{B})/\theta_{\mathcal{D}}$ is isomorphic to $\mathcal{B}$, hence $Y(\mathcal{A},\mathcal{B})$ is a derived algebra of $\mathcal{B} \,\omega\, \mathcal{C}$. Since $\mathcal{A}$ is a derived algebra of $Y(\mathcal{A},\mathcal{B})$ (being a homomorphic image of this algebra) it follows that $\mathcal{A}$ is a derived algebra of $\mathcal{B} \,\omega\, \mathcal{C}$.

Remark 1. If the algebra A is finite, we have noticed that $[(a,B)]_{\theta}$ contains $|B|$ elements. Therefore, we can choose τ such that τ is an equivalence on the base set K of $Y(\mathcal{A},\mathcal{B})$ and K/τ contains a number of elements equal to the largest number of elements contained by a class $[(a,B)]_{\theta_{\mathcal{D}}}$, i.e. by the largest block of $\mathcal{D}$.

III. A COVERING PROPERTY FOR FINITE CONTRACTIVE ALGEBRAS

We shall consider now a class of algebra which generalizes a certain property of finite unary algebras.

Definition 4. An algebra $A = (A,F)$ is said to be contractive if for any $f \in F$, the set $f(\underbrace{A,\ldots,A}_{n_f})$ is strictly included in A.

For instance, any unary algebra $\mathcal{A} = (A,\{f_1,\ldots,f_n\})$ in which no operation $f_j\colon A \to A$ is surjective is a contractive algebra.

Any right-null semigroup is also contractive, etc.

Lemma 4. If $\mathcal{A}$ is a contractive algebra then the family of sets $\mathcal{D} = \{W_a | W_a = A \setminus \{a\},\ a \in A\}$ is an a.d. of $\mathcal{A}$.

Proof. $\mathcal{D}$ is clearly a decomposition. Moreover, we have for $f \in F$ and $W_{a_1}, \dots, W_{a_{n_f}} \in \mathcal{D}$:

$$f(W_{a_1}, \dots, W_{a_{n_f}}) \subseteq f(A, \dots, A) \subsetneq A.$$

Therefore, there is an element a $A \setminus f(W_{a_1}, \dots, W_{a_{n_f}})$ which depends only on f such that $f(W_{a_1}, \dots, W_{a_{n_f}}) \subseteq W_a$. We proved that $\mathcal{D}$ is an a.d. on $\mathcal{A}$.

In the family of algebras $\mathcal{A}/\mathcal{D}$ there is an algebra $\mathcal{B} \in \mathcal{A}/\mathcal{D}$, $\mathcal{B} = (\mathcal{D}, \overline{F})$ such that each $\overline{f} \in \overline{F}$ is a constant. Namely, with the above notations we shall have $\overline{f}(W_{a_1}, \dots, W_{a_n}) = W_a$.

Let Contr_m be the family of all contractive algebras having only m-ary operations. If $\mathcal{A} \varepsilon \mathrm{Contr}_m$ then we can cover A with a cascade $B \omega C$ such that B has constant operations, according to Lemma 4 and Theorem 3.

The base set of the Yoeli algebra attached to $\mathcal{B}$ is $K = \{(a, W_b) | a \neq b\}$; the base set for $\mathcal{C}$ is a quotient set K/τ, where τ is an equivalence such that $\theta_{\mathcal{D}} \cap \tau = \iota_K$. This means that if $a \neq b$ and $a' \neq b$ (i.e. if $a \in W_b$ and $a' \in W_b$) then $((a, W_b), (a', W_b)) \in \tau$ implies $a = a'$. Moreover, it is impossible to choose τ such that K/τ contains $|A| - 1$ elements.

If f is an m-ary operation from $\overline{F}$ we have the operation $< [(a_1, W_{b_1})]_{\theta_{\mathcal{D}}}, \dots, [(a_m, W_{b_m})]_{\theta_{\mathcal{D}}}, f >$, for every collection $\{[(a_1, W_{b_1})]_{\theta_{\mathcal{D}}}, \dots, [(a_m, W_{b_m})]_{\theta_{\mathcal{D}}}\}$ of $\theta_{\mathcal{D}}$-classes. Assuming that $[(a_j, W_{b_j})]_{\theta_{\mathcal{D}}} \cap [(a'_j, W_{b'_j})]_\tau = (a'_m, W_{b''_j})$ for $1 \leq j \leq m$, we shall have

$$< [(a_1, W_{b_1})]_{\theta_{\mathcal{D}}}, \dots, [(a_m, W_{b_m})]_{\theta_{\mathcal{D}}}, f > ([(a'_1, W_{b'_1})]_\tau, \dots, [a'_m, W_{b'_m}]_\tau) =$$

$$= [\overline{f}((a''_1, W_{b''_1}) \qquad (a''_m, W''_{b_m}))_\tau = [(f(a''_1, \dots, a''_m), \overline{f}(W_{b''_1}, \dots, W_{b''_m}))]_\tau .$$

Using the definition of $\theta_{\mathcal{D}}$ it follows that $b''_j = b_j$ for $1 \leq j \leq m$. Therefore, if $\overline{f}(W_{b_1}, \dots, W_{b_m}) = W_b$ then we shall obtain as a result of the above operation the τ-class $[f(a''_1, \dots, a''_m), W_b)]_\tau$. If any of the sets $[(a_j, W_{b_j})]_{\theta_{\mathcal{D}}} \cap [(a'_j, W_{b'_j})]_\tau$ is void then we shall adopt as the value of the above operation $[(f(a_1',\ldots,a_m'),W_b)]_\tau$. We infer that we can not exhaust all the elements of the quotient set K/τ by considering the set $\{[f(a_1'',\ldots,a_m''),W_b]_\tau \mid a_j'' \in A$ for $1 \leq j \leq m\}$; this points that C is also a contractive algebra. Since the base set of C contains $|A| - 1$ elements we can repeat the previous contruction and obtain a contractive algebra with $|A| - 2$ elements, etc. We have thus proved

Theorem 4. *Let* $A = (A,F)$ *be a finite algebra from* $Contr_m$. *There exists a cascade of some algebras from* $Contr_m$, *having constant operations, which covers* A.

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