ON A THEOREM OF TARSKI*

Adrian Rezuş

In the mid-twenties Alfred Tarski raised the problem of axiomatizing classical propositional logic (TV, for short) by a single axiom and the rules of detachment (for material implication) and substitution as primitive rules of derivation. The problem was solved, by Tarski again, in 1925, for a large class of propositional logics, as eventually announced in [13], Theorem 8, but no proof of the result claimed there was ever published.

The method of Tarski for finding single axioms was frequently referred to in print (see, e.g., [9], [23], [11], [14], etc.) and though several persons - among which authors of textbooks of logic - have been certainly familiar to the principle involved in the original proof, none of them has ever documented the method for a larger audience.

We have discovered incidentally an analogue of Tarski's original method in the late 1979, relying on an extremely simple lambda-calculus argument (cf. [19]). This note reports some elaborate details of work contained, in essence, in [19], section 4, extending Tarski's result in various directions.

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1. Preliminaries

Throughout in the sequel, a *propositional language* is constructed as usual, from a denumerably infinite list of propositional variables p,q,r,s,t,u,v,w,... (possibly affected by numeric sub- and/or superscripts) and some unspecified propositional connectives. Whenever the latter are fixed we use Łukasiewicz's parentheses-free frontal notation in order to denote propositional formulae. In particular, C will be used as a binary connective and stands for any suitable notion of implication (material, intuitionistic, strict, multiple-valued, relevant, etc.).

A propositional language $\mathbf{L}$ is *implicative* if some notion of implication C is either primitive in $\mathbf{L}$ or can be defined in terms of the primitive notions in $\mathbf{L}$. A *purely implicative* language is a propositional language containing only C as a primitive (propositional) connective.

Lower-case light face Greek letters $\alpha,\beta,\gamma,\ldots$ (possibly affected by sub- or superscripts) will be used as meta-variables on propositional formulae in some implicative language, modulo uniform reletterings of the propositional variables they contain.

Similarly, lower-case bold face Roman letters will be used as constants for fixed propositional formulae modulo such reletterings. For convenience, we shall pick out a standard (e.g., lexical) ordering of the propositional variables and use formulae with their propositional variables occurring in this order as *ad hoc* representatives for specific bold face letters.

Examples (to be used later on):

$\mathbf{i}$:= Cpp or := Cp_0p_0
$\mathbf{k}$:= CpCqp
$\mathbf{k}'$:= CpCqq
$\mathbf{b}$:= CCpqCCrpCrq
$\mathbf{b}'$:= CCpqCCqrCpr
$\mathbf{c}$:= CCpCqrCqCpr
$\mathbf{c}'$:= CpCCqCprCqr
$\mathbf{c}_*$:= CpCCpqq

$$\underline{\underline{d}}_0 := \underline{\underline{d}} := CpCqCCpCqrr$$
$$\underline{\underline{d}}_n := \quad CpCqCCpCqrCs_1 \ldots Cs_n r$$
$$\underline{\underline{t}} := CpCqCrCCpCqCrss$$
$$\underline{\underline{s}} := CCpCqrCCpqCpr$$
$$\underline{\underline{s}}' := CCpqCCpCqrCpr$$
$$\underline{\underline{w}} := CCpCpqCpq.$$

As a shorthand, a constant formula (denoted by/represented by some bold face letter) β may occur as a subformula of a propositional formula α. In such cases the notational convention is that no propositional variable occurring in β should occur elsewhere in α. E.g., $\underline{\underline{k}}^+$ stands for $Cp\underline{\underline{k}}$, i.e., for $CpCqCrq$ or some lexical variant of it, but definitely not for $CpCpCrp, CpCqCpq$, etc. This will be sometimes made explicit in current auxiliary notation. Thus where

$$\underline{\underline{i}}_0 := \underline{\underline{i}} := Cp_0p_0, \qquad \underline{\underline{i}}_n := Cp_np_n \qquad (n \geq 1),$$

we will also set

$$\underline{\underline{k}}_0 := \underline{\underline{c}}_* := CpCCpqq, \quad \underline{\underline{k}}_n := CpCC\underline{\underline{i}}_1 \ldots C\underline{\underline{i}}_n CCpqq \qquad (n \geq 1),$$
$$\underline{\underline{k}}'_0 := \quad CCCppqq, \quad \underline{\underline{k}}'_n := CC\underline{\underline{i}}_1 \ldots C\underline{\underline{i}}_n C\underline{\underline{i}}_{n+1}qq \qquad (n \geq 1)$$

and

$$\underline{\underline{k}}^+_0 := CC\underline{\underline{k}}_0 rr := CCCpCCpqqrr, \quad \underline{\underline{k}}^+_n := CC\underline{\underline{i}}_1 \ldots C\underline{\underline{i}}_n C\underline{\underline{k}}_n rr \qquad (n \geq 1).$$

A (system of) propositional logic will be often confused with the set of its theorems, but whenever not otherwise specified, we shall understand by "propositional logic" a Hilbert-style presentation of some concept of logical derivation.

Where L is a propositional logic, the set of its (well formed) formulae will be denoted by Form_L.

A propositional logic $\underset{\wedge}{L}$ is implicative (relative to some specified notion of implication C) if (i) so is its underlying language and, moreover, (ii) the rule of detachment for C (modus ponens; (MP), for short)

$$C\alpha\beta, \alpha \Rightarrow \beta$$

is a derivable rule in $\underset{\wedge}{L}$ (say not only admissible in L; for the distinguo derivable/admissible in $\underset{\wedge}{L}$ see, e.g., [10]). A propositional logic is purely implicative if it is implicative and its underlying language is purely implicative. (This is mere technical jargon not intended to commit ourselves to some particular assumptions - philosophical or so - concerning what is to be meant by "implication" at all. But see [22], Chapter 1, for details.)

In particular, if some implicative logic is denoted by $\underset{\wedge}{L}$ then its pure (ly implicative) fragment (relative to the specified notion of implication) will be denoted by $\underset{\wedge}{L}_{\rightarrow}$.

In view of a remark of John von Neumann, it is immaterial if we choose to present a propositional logic with axioms and the rule of substitution (henceforth: (SB)) as a primitive rule of derivation or use axiom schemes and give up the rule (SB). So it will be convenient to forget any explicit reference to the applications of (SB), save in critical cases or in examples when we shall adopt the usual Polish school notation for substitutions (see, e.g., [12] or [18]).

While indicating proofs by (MP) (and (SB)) from particular sets of implicative formulae we shall make heavy use of (a slight refinement of) C.A. Meredith's condensed detachment operator (cdo, for short).

Initially thought of as a simple and convenient notational expedient (cf. [18]: Appendix II, [14], [8], [15], [16], [17], etc.), this abbreviative device has also some deeper motivation and applications as it will be seen later on.

Roughly speaking, where α,β are (purely) implicative formulae, $\underset{\wedge}{D}\alpha\beta$ stands for the most general result of the detachment of β or some substitution instance of it (as a minor premiss of (MP)) from α or some substitution instance of it (as a major premiss of (MP)). So $\underset{\wedge}{D}\alpha\beta$ makes sense for any two implicative formulae α,β such that there are substitution instances α',β' of α,β resp. with $\alpha' = C\beta'\gamma$ for some implicative formula γ. If this is the case we will say that $\underset{\wedge}{D}\alpha\beta$ is a proof of γ or that $\underset{\wedge}{D}\alpha\beta$ proves γ and it is easy to see that γ is uniquely determined up to uniform reletterings of its propositional variables. Here "the most general result" must be understood à la C. A. Meredith, in the sense that we should not make unnecessary identifications of propositional variables while performing the underlying (condensed) detachment.

Obviously, C. A. Meredith's $\underset{\wedge}{D}$-meta-notation for proofs by (MP) (and (SB)) allows a non-ambiguous restauring of the missing substitutions (= applications of (SB)), modulo uniform reletterings of propositional variables.

Examples: where $\underline{\underline{i}}, \underline{\underline{k}}, \underline{\underline{k}}', \underline{\underline{c}}$ and $\underline{\underline{c}}_*$ are as earlier we have that

$\underset{\wedge}{D}\underline{\underline{c}}\underline{\underline{k}}$ proves $\underline{\underline{k}}'$,

$\underset{\wedge}{D}\underline{\underline{k}}\underline{\underline{i}}$ proves $\underline{\underline{k}}'$

and

$\underset{\wedge}{D}\underline{\underline{ci}}$ proves $\underline{\underline{c}}_*$,

while, with applications of (SB) written up in full, the latter proof might have been displayed in the spirit of the Polish school as follows:

1 := $\underline{\underline{c}}$:= CCpCqrCqCpr

2 := $\underline{\underline{i}}$:= Cpp

1[p/Cpq,q/p,r/q] * C2[p/Cpq] - 3

3 := $\underline{\underline{c}}_*$:= CCpCCpqq.

The stipulation requiring the most general result of a detachment forbids taking say CpCpp for the result "proved by" $\underset{\wedge}{D}\underline{\underline{ck}}$ or $\underset{\wedge}{D}\underline{\underline{ki}}$. As a further notational convention we shall write $\underset{\wedge}{D}\alpha\beta = \underset{\wedge}{D}\alpha'\beta'$ if $\underset{\wedge}{D}\alpha\beta$ and $\underset{\wedge}{D}\alpha'\beta'$ prove the same formula γ (modulo the due reletterings). So one should have $\underset{\wedge}{D}\underline{\underline{ck}} = \underset{\wedge}{D}\underline{\underline{ki}}$.

More accurately, the cdo $\underset{\wedge}{D}$ may be thought of as being a partial binary operator on sets of implicative formulae whether pure or not. We shall give here a closer description of C. A. Meredith's $\underset{\wedge}{D}$ using the unification algorithm of J. A. Robinson (cf., e.g., [20], [21]). For convenience, let us restrict the frame of reference to purely implicative languages (the extension to arbitrary implicative languages is trivial).

Let $\underset{\wedge}{L}$ be a set of purely implicative formulae. Then the cdo is a partial mapping

$$\underset{\wedge}{D}: \underset{\wedge}{L}\mathrm{x}\underset{\wedge}{L} \rightsquigarrow \underset{\wedge}{L}$$

such that, for all α,β in $\underset{\wedge}{L}$, $\underset{\wedge}{D}\alpha\beta := \underset{\wedge}{D}(\alpha,\beta)$ is defined if

(i) $\alpha = C\gamma'\gamma''$ and

(ii) β and the antecedent γ' of α have a unifier in the sense of [20], else $\underset{\wedge}{D}\alpha\beta$ is undefined.

If $\underset{\wedge}{D}\alpha\beta$ is defined then β and the antecedent of α have a most general unifier the the sense of [20] (mgu, for short), β' say, and the due substitution instances α',β' of α,β resp. give $\alpha' = C\beta'\gamma$, for some γ, which is unique (up to uniform reletterings). Then $\underset{\wedge}{D}\alpha\beta := \gamma$.

It should be noted that the definition suggested above is constructive in the sense that the mgu of every two formulae (if any) can be found effectively by the so-called unification algorithm. (Actually, the algorithm allows to establish whether or not the unification is possible and if this is the case it finds out the due mgu. For details, see [20], [21] or consult [15] for an alternative equivalent account.)

2. Meredith D-Proofs.

For the purposes of this paper it will be convenient to formalize the metalanguage of D-proofs and to introduce some systematic abbreviations.

For any implicative logic L, let the set of D-proof expressions (pe's, for short) of L be the least set D_L such that

(i) any propositional meta-variable $(\alpha,\beta,\gamma,\ldots)$ is in D_L,

(ii) any propositional constant (i.e., a bold face letter denoting a theorem of L) is in D_L,

(iii) if x,y are in D_L then so is Dxy.

Hereafter, x,y,z,... (possibly with sub- or superscripts) will range on pe's of some arbitrary implicative logic L.

A pe consisting of a single letter (propositional meta-variable or propositional constant) is *atomic*.

Further we write (xy) for Dxy and save parentheses by omitting the outermost pair and assuming association to the left.
E.g., d(kk)(dki) stands for DDdDkkDDdki.

We also adopt the following use of numeric superscripts $(n \geq 0)$: for any atomic pe x of some implicative logic L,

$$x^0 \text{ is the empty word,}$$
$$x^1 \equiv x \text{ and } x^{n+1} \equiv x^n x, \text{ for all } n \geq 0.$$

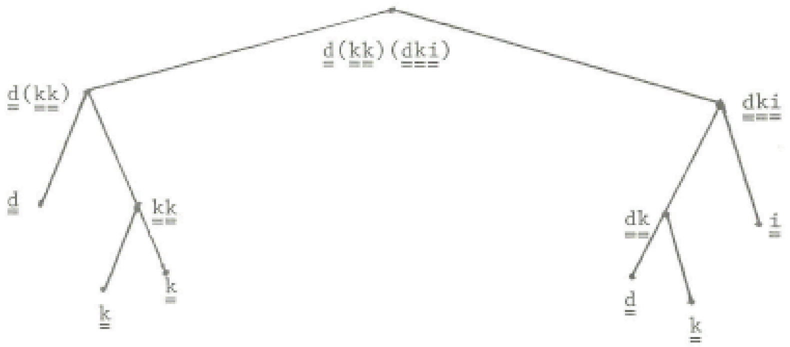
Similarly, we write C^n $(n \geq 0)$ for n consecutive occurrences of C in some implicative formula, $(CC)^n$ for 2xn consecutive occurrences of C, etc. But $p^i_j, q^i_j, \ldots (i,j \geq 0)$ are propositional variables and the superscript "i" has no iterative effect.

If a pe x is defined ("meaningful","denoting") we write $x\!\downarrow$ to indicate this, otherwise (if x is not defined) we write $x\!\uparrow$. It is reasonable to assume that a pe x is defined whenever it is atomic, so we won't write $\alpha\!\downarrow, \beta\!\downarrow$, etc. and $k\!\downarrow$, $i\!\downarrow$, etc. either.

Examples: $sii\!\uparrow$, $wi\!\uparrow$, but $ix\!\downarrow$, $kx\!\downarrow$, $k'x\!\downarrow$ whenever $x\!\downarrow$ (if x is atomic say).

In evaluating pe's we may adopt some obvious inside-out and left-to-right strategy which can be always represented in a tree-like manner.

E.g., with the example above: d(kk)(dki), we should first evaluate kk, dk, next d(kk), then dki and, finally, the entire pe.
Tree-like picture:



It should be clear how to define *components of* pe's (or *sub-D-proof expressions*; *sub-pe's*, for short). Further, a *proper* component (sub-pe) of a pe x is any component of x distinct from x itself. (Sub-pe's of a pe correspond, as expected, to subtrees, in the tree-representation of a pe.)

Now the evaluation of a pe depends on the evaluation of its components and it should respect in some immediate sense a variant of the *Frege principle of significance*. For, obviously, if some component x of a pe y is such that x↑ then the entire pe y is such that y↑, while if every proper component x of y is such that x↓ then one should also have y↓ (otherwise, x is a component of x, for any pe x).

So far we have introduced a class of (interpreted) formal languages suited for some slightly modified *typed combinatory logics* (cf. [4], [6] or [2]: Appendixes A,B). To see how they work we need not develop special theories of *reduction* for each implicative logic L but consider only the process of *evaluation of pe's*.

For any two pe's x,y define x = y if x↓, y↓ and x and y have the same value; clearly, = is an equivalence relation on pe's z with z↓. (So, to make a parallel with the classical case, the evaluation of a pe may be compared with a kind of "combinatory reduction", while = is supposed to be introduced by the familiar Church-Rosser property, where "x = y" means that x and y have a common "reduct". In fact, our pe's have the so-called "strong normalization property" whenever they are defined, *i.e.*, make sense as "typed terms", and hence our choice of "evaluation" - and not "reduction" - as a central concept in the "theory of $\underset{\wedge}{D}$-proofs". See also [4], 9E and [7].)

Now = can be weakened in the obvious way up to a partial equivalence relation $\simeq$ on pe's, and we will want to write "$x \simeq y$" for any two pe's x,y, whether they are defined ("meaningful") or not. (For later reference note that we have introduced $\simeq$ starting from the definition of = and not conversely.)

The relations = and $\simeq$ may be interpreted intuitively as equivalence (partial equivalence) of proofs (by (MP) and (SB)) in some implicative logic $\underset{\wedge}{L}$.

As earlier, if x,y are pe's of some $\underset{\wedge}{L}$ and y is atomic then $y\downarrow$ and we say that x proves y whenever $x = y$. Also note that, by construction, definiteness of pe's $(\ldots\downarrow)$ is preserved by =. So for all pe's x,y if x proves y then also $x\downarrow$ (for y is atomic, hence $y\downarrow$, while = preserves definiteness).

It is easy to check the following combinatory-like "equations":

Lemma 1.

For all pe's $x,y,x,z_1,\ldots,z_n$ $(n \geq 1)$:

(1)	$\underline{\underline{i}}x \simeq x$	(2)	$\underline{\underline{k}}xy \simeq x$
(3)	$\underline{\underline{k}}'xy \simeq y$	(4)	$\underline{\underline{k}}^{+}xyz \simeq y$
(5)	$\underline{\underline{b}}xyz \simeq x(yz)$	(6)	$\underline{\underline{b}}'xyz \simeq y(xz)$
(7)	$\underline{\underline{c}}xyz \simeq xzy$	(8)	$\underline{\underline{c}}'xyz \simeq yzx$
(9)	$\underline{\underline{c}}_{*}xy \simeq yx$	(10)	$\underline{\underline{d}}xyz \simeq zxy$
(11)	$\underline{\underline{d}}_n xyzz_1\ldots z_n \simeq zxy$		
(12)	$\underline{\underline{t}}xyzu \simeq uxyz$	(13)	$\underline{\underline{w}}xy \simeq xyy$
(14)	$\underline{\underline{s}}xyz \simeq xz(yz)$	(15)	$\underline{\underline{s}}'xyz \simeq yz(xz)$.

Proof. Straightforward consequences from the definition of $\underset{\wedge}{D}$.

Now = and $\simeq$ are congruences w.r.t. $\underset{\wedge}{D}$. That is:

Lemma 2.

For all pe's x,y,z,

(1) if $x = y$ and $xz\downarrow$ (or $yz\downarrow$) then $xz = yz$,

(2) if $x = y$ and $zx\downarrow$ (or $zy\downarrow$) then $zx = zy$,

and finally,

(3) if $x \simeq y$ then $xz \simeq yz$ and $zx \simeq zy$.

Proof. Obvious.

Remark 3.

For all $n \geq 0$,

(1) if $x\downarrow$ and $y\downarrow$ then $\underline{\underline{d}}_n x\downarrow$ and $\underline{\underline{d}}_n xy\downarrow$,

(2) if $x\downarrow$, $y\downarrow$ and $z\downarrow$ then $\underline{\underline{t}}x\downarrow$, $\underline{\underline{t}}xy\downarrow$ and $\underline{\underline{t}}xyz\downarrow$.

Some of the following consequences from definitions will be useful later on.

Lemma 4.

For all implicative formulae α,β in some $\hat{L}$ and all $n \geq 0$,

(1) $\underline{\underline{i}}\alpha = \alpha$	(2) $\underline{\underline{k}}\alpha\beta = \alpha$
(3) $\underline{\underline{k}}'\alpha\beta = \beta$	(4) $\underline{\underline{k}}\underline{\underline{i}} = \underline{\underline{c}}\underline{\underline{k}} = \underline{\underline{k}}'$
(5) $\underline{\underline{k}}\underline{\underline{i}}\alpha = \underline{\underline{c}}\underline{\underline{k}}\alpha = \underline{\underline{k}}'\alpha = \underline{\underline{i}}$	(6) $\underline{\underline{k}}\underline{\underline{k}} = \underline{\underline{k}}^+$
(7) $\underline{\underline{k}}_n\alpha\beta = \beta\underline{\underline{i}}^n\alpha$	(8) $\underline{\underline{k}}'_n\alpha = \alpha\underline{\underline{i}}^{n+1}$
(9) $\underline{\underline{k}}^+_n\alpha = \alpha\underline{\underline{i}}^n\underline{\underline{k}}_n$	

Proof. Straightforward.

Lemma 5.

(1) $\underline{\underline{c}}\underline{\underline{b}} = \underline{\underline{b}}'$	(2) $\underline{\underline{c}}\underline{\underline{b}}' = \underline{\underline{b}}'\underline{\underline{c}}_*(\underline{\underline{b}}'\underline{\underline{b}}') = \underline{\underline{b}}$
(3) $\underline{\underline{b}}'\underline{\underline{b}}'(\underline{\underline{b}}'\underline{\underline{c}}_*) = \underline{\underline{b}}(\underline{\underline{c}}_*\underline{\underline{c}}')\underline{\underline{c}}' = \underline{\underline{c}}'\underline{\underline{c}}'\underline{\underline{c}}' = \underline{\underline{c}}$	
(4) $\underline{\underline{b}}\underline{\underline{b}}\underline{\underline{c}}_* = \underline{\underline{c}}\underline{\underline{c}} = \underline{\underline{c}}'$	(5) $\underline{\underline{c}}\underline{\underline{i}} = \underline{\underline{c}}_*$
(6) $\underline{\underline{c}}\underline{\underline{s}} = \underline{\underline{s}}'$	(7) $\underline{\underline{c}}\underline{\underline{s}}' = \underline{\underline{s}}$.

Proof. Easy.

Lemma 6.

For all $n \geq 0$, $m \geq 2$,

(1) $\underline{\underline{k}}_n\underline{\underline{i}} = \underline{\underline{k}}'_n$	(2) $\underline{\underline{k}}_n\underline{\underline{k}} = \underline{\underline{k}}^+_n$
(3) $\underline{\underline{k}}_n\underline{\underline{i}}\underline{\underline{i}} = \underline{\underline{i}}$	
(4) $\underline{\underline{k}}_n\underline{\underline{i}}^m = \underline{\underline{i}}$.	

Proof. Easy.

Where x,y are pe's, let $x[y]$ stand for "y is a component of x". Then we have a

Corollary 7.

If $x[y] \simeq z$ and $y \simeq y_0$ then $x[y_0] \simeq z$, for all pe's x,y,y_0,z.

Proof. Use Lemma 2.

The evaluation strategy indicated earlier is guaranteed by the following consequence of Corollary 7.

Corollary 8.

For all pe's x, y, y_0, z of some $\underset{\wedge}{L}$, if $x[y]$ proves z and y proves y_0 then $x[y_0]$ proves (also) z.

Proof. Indeed, if $x[y]$ proves z then z is atomic and one has $z\downarrow$, by definition. So $x[y]\downarrow$ and also $y\downarrow$, by the "Frege principle" noted earlier; hence y has a "value". But y proves y_0 so this "value" must be y_0. So $x[y]$ and $x[y_0]$ must have the same "value", by Corollary 7.

Let $\underset{\wedge}{L}$ be some implicative logic. If β is a theorem of $\underset{\wedge}{L}$ and

$$\beta \underline{\underline{i}}^m := \beta \underline{\underline{i}}_1 \ldots \underline{\underline{i}}_m$$

for some $m \geq 0$, we say that β is m-solvable. Similarly, a set $\{\beta_j : j \in I\}$ $(I \subseteq \mathbb{N})$ of theorems of $\underset{\wedge}{L}$ is m-solvable if β_j is m-solvable for all j in $I \subseteq \mathbb{N}$. (In the above, $\underline{\underline{i}}_1, \ldots, \underline{\underline{i}}_m$ are pairwise distinct lexical variants of $\underline{\underline{i}}$, as in Section 1.)

Then one has immediately

Lemma 9.

For all α, β in some implicative logic $\underset{\wedge}{L}$,

(1) if β is m-solvable $(m > 0)$ then $\underline{\underline{k}}_m \alpha\beta = \alpha$,

(2) $\underline{\underline{k}}_m$ is m-solvable, for $m \geq 2$,

(3) $\underline{\underline{k}}_m \alpha \underline{\underline{k}}_m = \alpha$, for all $m \geq 2$.

Proof. (1) Use Lemma 4: (6) and (1) with Corollary 8.

(2) Note that $\underline{\underline{k}}_m \underline{\underline{i}}\underline{\underline{i}} = \underline{\underline{i}}\underline{\underline{i}}^m \underline{\underline{i}} = \underline{\underline{i}}$, for all $m \geq 0$.

(3) By (1) and (2).

Let $\underset{\wedge}{L}$ be an arbitrary implicative logic. Define, for each $m, n \geq 1$, mappings

$$\underset{\sim}{d}_{m,n} : \mathrm{Form}^n_{\underset{\wedge}{L}} \to \mathrm{Form}_{\underset{\wedge}{L}},$$

by: for all $\beta_1, \ldots, \beta_n$ in $\mathrm{Form}_{\underset{\wedge}{L}}$, not containing $p^i_j, q^i_j (1 \leq i \leq n-1,\ 1 \leq j \leq m)$,

$$\underset{\sim}{d}_{m,n}(\beta_1, \ldots, \beta_n) := \begin{cases} (CC)^{n-1}\beta_1 C\beta_2 p_1 Cq^1_1 \ldots Cq^1_m p_1 \ldots C\beta_n p_{n-1} Cq^{n-1}_1 \ldots Cq^{n-1}_m p_{n-1} & \text{if } n > 1, \\ \beta_1, & \text{if } n = 1. \end{cases}$$

In particular, set for $m = 0, \underset{\sim}{d}_n := \underset{\sim}{d}_{0,n}$, with, for all $\beta_1,\ldots,\beta_n$ in $\mathrm{Form}_{\underset{\wedge}{L}}$, not containing $p_i (1 \leq i \leq n - 1)$,

$$\underset{\sim}{d}_n(\beta_1,\ldots,\beta_n) := \begin{cases} (CC)^{n-1}\beta_1 C\beta_2 p_1 p_1 \ldots C\beta_n p_{n-1} p_{n-1}, & \text{if } n > 1 \\ \\ \beta_1, & \text{if } n = 1. \end{cases}$$

Lemma 10.

(1) For all $\alpha_1,\ldots,\alpha_n$ in some $\underset{\wedge}{L}$ $(n \geq 1)$, not containing $p_1,\ldots p_{n-1}$, one has

$$\underset{=}{d}^{n-1}\alpha_1\ldots\alpha_n \text{ proves } \underset{\sim}{d}_n(\alpha_1,\ldots,\alpha_n).$$

(2) For all $\alpha_1,\ldots,\alpha_n$ in some $\underset{\wedge}{L}$ and all $m > 0$, $n \geq 1$, where the α_k's do not contain p^i_j, $q^i_j (1 \leq i \leq n - 1, 1 \leq j \leq m)$, and where we have set $\underset{=}{d}_m^{n-1} := (\underset{=}{d}_m)^{n-1}$,

$$\underset{=}{d}_m^{n-1}\alpha_1\ldots\alpha_n \text{ proves } \underset{\sim}{d}_{m,n}(\alpha_1,\ldots,\alpha_n).$$

Proof. Tedious but trivial.

Let also $\underset{\wedge}{L}$ be as earlier and $\underset{\sim}{t}$ be a mapping

$$\underset{\sim}{t} : \mathrm{Form}^3_{\underset{\wedge}{L}} \to \mathrm{Form}_{\underset{\wedge}{L}}$$

such that, for all β_1,β_2,β_3 in $\mathrm{Form}_{\underset{\wedge}{L}}$ not containing p,

$$\underset{\sim}{t}(\beta_1,\beta_2,\beta_3) := CC\beta_1 C\beta_2 C\beta_3 pp.$$

Then one can see easily that

Lemma 11.

For all $\alpha_1,\alpha_2,\alpha_3$ in some implicative logic $\underset{\wedge}{L}$,

$$\underset{=}{t}\alpha_1\alpha_2\alpha_3 \text{ proves } \underset{\sim}{t}(\alpha_1,\alpha_2,\alpha_3).$$

Proof. Clear.

3. A Generalization of Tarski's Theorem.

Let $\underset{\wedge}{L}$ be a propositional logic. Where $(R_1),\ldots,(R_r)$, $(r \geq 1)$ are derivable rules in $\underset{\wedge}{L}$ we say that $\underset{\wedge}{L}$ is finitely axiomatizable in the set of rules $\{(R_1),\ldots,(R_r)\}$ if $\underset{\wedge}{L}$ has a Hilbert-style formulation with

(i) a finite number of axioms $\alpha_1,\ldots,\alpha_n$, $(n \geq 1)$, and

(ii) the rules $(R_1),\ldots,(R_r),(SB)$ as primitive rules of derivation.

Let now $\underset{\wedge}{L}$ be an implicative logic with $(MP),(R_1),\ldots,(R_r),(r \geq 0)$, derivable rules in $\underset{\wedge}{L}$. We say that $\underset{\wedge}{L}$ is <u>Tarski axiomatizable in</u> the set of rules $\{(R_1),\ldots,(R_r),(MP)\}$ if $\underset{\wedge}{L}$ has a Hilbert-style formulation with

(i) a single axiom and

(ii) (MP), for the specified notion of implication, together with

(iii) $(R_1),\ldots,(R_r),(SB)$ as primitive rules of derivation.

(Alternatively, one can take a single/a finite number of <u>axiom scheme/s</u> and leave out the rule (SB), as indicated earlier.)

Clearly, for $\underset{\wedge}{L}$ implicative, if $\underset{\wedge}{L}$ is Tarski axiomatizable in some set of rules then it is also finitely axiomatizable in the same set. We establish sufficient conditions, generalizing Theorem 8 in [13], and allowing to prove the converse of the above.

<u>Theorem 12</u>.

Let $\underset{\wedge}{L}$ be an implicative logic (relative to some specified notion C of implication). If

(i) $\underset{\wedge}{L}$ is finitely axiomatizable in the set of rules

$$\{(R_1),\ldots,(R_r),(MP)\} \qquad (r \geq 0)$$

and

(ii) for some $m \geq 0$, $\underline{d}_{=m}$ is a theorem of $\underset{\wedge}{L}$,

(iii) $\underline{\underline{k}}$ is a theorem of $\underset{\wedge}{L}$

then $\underset{\wedge}{L}$ is Tarski axiomatizable in the same set of rules.

<u>Proof</u>. Let $\underset{\wedge}{L}_n$ be the (Hilbert-style) formulation of $\underset{\wedge}{L}$ with axioms

$$\alpha_1,\ldots,\alpha_n \qquad (n \geq 1)$$

and rules

$$(R_1),\ldots,(R_r),(MP),(SB) \qquad (r \geq 0).$$

(If $n = 1$, there is nothing to prove but we include this as a limit case.)

Define, for $m \geq 0$ and $n \geq 1$, (m,n <u>fixed</u>),

$$\underline{\underline{h}}_{m,n} := \underset{\sim}{d}_{m,n}(\alpha_1,\ldots,\alpha_n).$$

Then, by Lemma 10, one finds that

$$\underline{\underline{d}}_m^{n-1}\alpha_1\ldots\alpha_n \quad \text{proves} \quad \underline{\underline{h}}_{m,n}$$

and since $\underline{\underline{d}}_m$ is a theorem of $\underset{\wedge}{L}_n$, $\underline{\underline{h}}_{m,n}$ is also a theorem of $\underset{\wedge}{L}_n$ (for $\underline{\underline{h}}_{m,n}$ can be proved by (MP) and (SB) from the axioms of $\underset{\wedge}{L}_n$; hence the result holds for $r = 0$, too).

Now set

$$\underline{\underline{g}}_{m,n} := d_{m,3}(\underline{\underline{k}},\underline{\underline{k}}^+,\underline{\underline{h}}_{m,n}),$$

with $\underline{\underline{k}},\underline{\underline{k}}^+$ and $\underline{\underline{h}}_{m,n}$ as above.

Our claim is that $\underline{\underline{g}}_{m,n}$ is the needed single axiom. Firstly, $\underline{\underline{g}}_{m,n}$ is a theorem of $\underset{\wedge}{L}_n$. Indeed, by Lemma 4: (6), one finds that

$$\underline{\underline{k}}\underline{\underline{k}} \quad \text{proves} \quad \underline{\underline{k}}^+$$

and, by Lemma 10,

$$\underline{\underline{d}}_m\underline{\underline{d}}_m\underline{\underline{k}}\underline{\underline{k}}^+\underline{\underline{h}}_{m,n} \quad \text{proves} \quad \underline{\underline{g}}_{m,n},$$

hence the result follows by Corollary 8 (for $\underline{\underline{k}},\underline{\underline{d}}_m$ are theorems of $\underset{\wedge}{L}_n$, by hypothesis).

On the other hand, let $\underset{\wedge}{L}^*$ be the (Hilbert-style) formulation of L with $\underline{\underline{g}}_{m,n}$ as single axiom and the same rules as earlier. (Explicitely, $\underline{\underline{g}}_{m,n}$ is

$$\underline{\underline{g}}_{m,n} := CCCC\underline{\underline{k}}C\underline{\underline{k}}^+r_1Cs_1\ldots Cs_m r_1 C\underline{\underline{h}}_{m,n}r_2Ct_1\ldots Ct_m r_2.)$$

Using $\underset{\wedge}{D}$-proofs one can show quickly that

$$\underline{\underline{g}}_{m,n}\underline{\underline{g}}_{m,n}(\underline{\underline{g}}_{m,n})^m \quad \text{proves} \quad \underline{\underline{k}},$$

and, in particular, for $m = 0$ and with $\underline{\underline{g}}_n := \underline{\underline{g}}_{0,n}$, for convenience, one finds that

$$\underline{\underline{g}}_n\underline{\underline{g}}_n \quad \text{proves} \quad \underline{\underline{k}}.$$

It is a tedious affair to find explicitely the needed substitutions, but the matter is completely trivial and we have only to take some care in applying correctly J. A. Robinson's unification algorithm. For instance, displaying substitutions <u>à la</u> Lukasiewicz [12], one has, with $\underline{\underline{K}} := Cp_1Cq_1p_1$ and $\underline{\underline{h}}_n := \underline{\underline{h}}_{0,n}$,

$$1 := \underline{\underline{g}}_n := CCCCpCqpCCrCsCtsuuC\underline{\underline{h}}_n vv$$

$$2 := \underline{\underline{g}}'_n := 1[p/CC\underline{\underline{h}}_n\underline{\underline{k}}C\underline{\underline{h}}_n\underline{\underline{k}}, q/CrCsCts, r/\underline{\underline{h}}_n, s/p_1, t/q_1, u/C\underline{\underline{h}}_n C\underline{\underline{h}}_n\underline{\underline{k}}, v/\underline{\underline{k}}]$$

$$3 := \underline{\underline{g}}''_n := 1[p/C\underline{\underline{h}}_n\underline{\underline{k}}, q/q, r/r, s/s, t/t, u/CC\underline{\underline{h}}_n\underline{\underline{k}}C\underline{\underline{h}}_n C\underline{\underline{h}}_n\underline{\underline{k}}, v/C\underline{\underline{h}}_n\underline{\underline{k}}]$$

$$2 * 3 - 4$$

$$4 := \underline{\underline{k}} := Cp_1Cq_1p_1,$$

i.e., $\underline{\underline{g}}'_n := C\underline{\underline{g}}''_n\underline{\underline{k}}$.)

Now, $\underline{\underline{h}}_{m,n}$ is a theorem of $\underset{\wedge}{L}^*$ for one can establish that

$$\underline{\underline{g}}_{m,n}(\underline{\underline{kk}})\underline{\underline{k}}^{m+1} \quad \text{proves} \quad \underline{\underline{h}}_{m,n},$$

while the axioms $\alpha_1,\ldots,\alpha_n$ of $\underset{\wedge}{L}_n$ can be extracted from $\underline{\underline{h}}_{m,n}$ as follows.

With, for $1 \leq j \leq n-1$, set

$$\underline{\underline{h}}_{m,n-j} := \underline{\underline{h}}_{m,n}\underline{\underline{k}}^{(m+1)\times j}$$

(that is:

$$\underline{\underline{h}}_{m,n-1} := \underline{\underline{h}}_{m,n}\underline{\underline{kk}}^{m} = \underline{\underline{h}}_{m,n}\underline{\underline{k}}^{m+1}$$

$$\underline{\underline{h}}_{m,n-2} := \underline{\underline{h}}_{m,n-1}\underline{\underline{kk}}^{m} = \underline{\underline{h}}_{m,n}\underline{\underline{k}}^{(m+1)\times 2}$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\underline{\underline{h}}_{m,1} := \underline{\underline{h}}_{m,2}\underline{\underline{kk}}^{m} = \underline{\underline{h}}_{m,n}\underline{\underline{k}}^{(m+1)\times(n-1)}).$$

It is easy to see that, for $1 \leq j \leq n$,

$$\underline{\underline{h}}_{m,j}(\underline{\underline{kk}})\underline{\underline{k}}^{m+1} \quad \text{proves} \quad \alpha_j,$$

so the axioms of $\underset{\wedge}{L}_n$ can be proved from $\underline{\underline{g}}_{m,n}$ by (MP) and (SB) only (i.e., for $r = 0$, too).

(These axioms imply also that $\underline{\underline{d}}_m$ is a theorem of $\underset{\wedge}{L}^*$, by hypothesis.)

So $\underset{\wedge}{L}_n$ and $\underset{\wedge}{L}^*$ are equivalent.

<u>Remark 13.</u>

Tarski's Theorem 8 in [13] is a particular case of our Theorem 12 with $m = 0$ and $m = 1$.

<u>Remark 14.</u>

Our method of proving Theorem 12 does not provide <u>organic</u> axioms, in the sense of M. Wajsberg (for an axiom system $\underset{\wedge}{L}$, an axiom of $\underset{\wedge}{L}$ is <u>organic</u> if it has no subformulae, except itself, that are theorems in $\underset{\wedge}{L}$) and this was also the case with Tarski's original method of proof, as reported in [11]. (The import of an organic axiomatization is explained in [24].)

Another practical inconvenient of both methods (see [23] and [11] for Tarski's examples) is in the fact the single axioms obtained thereby are very long.

In the end, Theorem 12 is of some <u>theoretical</u> interest since there are systems of propositional logic that are finitely axiomatizable in (MP) and still not Tarski axiomatizable in (MP). E.g., the purely implicative fragment $\underset{\wedge\to}{T}$ of the logic of "Ticket Entailment" of A. R. Anderson (cf. [1]) cannot be axiomatized with a single axiom, (MP) and (SB) only. (This result is due to Z. Parks; see [1], 8.5.2., for details.)

<u>Remark 15.</u>

If we had considered the additional condition

(iv) $\mathbf{t}$ is a theorem of $\underset{\wedge}{L}$

among the hypotheses of Theorem 12, the construction of the single axiom $\mathbf{g}_{m,n}$ might have been somewhat different. Indeed, with $\mathbf{h}_{m,n}$ as earlier, set

$$\mathbf{g}^*_{m,n} := \underset{\sim}{t}(\mathbf{k}, \mathbf{h}_{m,n}, \mathbf{k}) := CC\mathbf{k}C\mathbf{h}_{m,n}C\mathbf{k}rr$$

for the new single axiom.

Now as

$$\mathbf{t}\mathbf{k}\mathbf{h}_{m,n}\mathbf{k} \quad \text{proves} \quad \mathbf{g}^*_{m,n}$$

by Lemma 11, $\mathbf{g}^*_{m,n}$ is also a theorem of (the new) $\underset{\wedge}{L}$.

Let $\underset{\wedge}{L}^{\&}$ be the new formulation of $\underset{\wedge}{L}$ with $\mathbf{g}^*_{m,n}$ as single axiom and primitive rules as earlier.

It is easy to see that

$$\mathbf{g}^*_{m,n}\mathbf{g}^*_{m,n}\mathbf{g}^*_{m,n} \quad \text{proves} \quad \mathbf{k}$$

and

$$\mathbf{g}^*_{m,n}(\mathbf{g}^*_{m,n}\mathbf{g}^*_{m,n}) \quad \text{proves} \quad \mathbf{h}_{m,n}$$

(for this note that

$$\mathbf{g}^*_{m,n}\mathbf{g}^*_{m,n} = \mathbf{k}\mathbf{k} = \mathbf{k}^{+}),$$

while the α_j's $(1 \leq j \leq n)$ can be obtained from $\mathbf{h}_{m,n}$, with $\mathbf{k}$ readily available, as in the proof of Theorem 12.

(The combinatory argument behind the construction of $g^*_{m,n}$ is due, in essence, to J. B. Rosser [unpublished]; cf. [19], 1.4. and Theorem 19 below.)

Let $BCK_\to$ be the Meredith (purely) implicative logic (cf. [18], Appendix I, [14], [17], etc.), formulated with (MP), (SB) as primitive rules and axioms b, c, k (for alternative axiomatizations see [14] and [19]). As pointed out by H. B. Curry, K. Iséki, R. Routely, R. K. Meyer, D. Meredith et al., there is some interest in studying $BCK_\to$ and its extensions from both a combinatory and an algebraic point of view (cf. the references in [19]). But it is also a logical landmark in axiomatization problems. Indeed, one has the following consequence of Theorem 12 above.

Corollary 16.

Any finitely axiomatizable extension of $BCK_\to$ in some set of rules $\{(R_1),\dots,(R_r),(MP)\}$, $(r \geq 0)$, is Tarski axiomatizable in the same set of rules.

Proof. Note that

ckk proves i, Lemma 4:(5)

ci proves c_*, Lemma 5:(5)

bcc_* proves $d_0 := d$,

then apply Theorem 12, with $m = 0$.

Alternatively, one has also that

$b(bc)(bcc_*)$ proves t,

so t is a theorem of $BCK_\to$ and one can apply the argument of Remark 15 above.

Now $BCK_\to$ is known to be a subsystem of many familiar (propositional) logics. The following list is far of being complete.

Corollary 17.

The following logics are Tarski axiomatizable in (MP):

(i) the classical logic TV and its purely implicative fragment $TV_\to$ (cf. [13], [11], [18], etc.);

(ii) the intuitionistic logic H and its purely implicative fragment $H_\to$ (cf. [5]);

(iii) Hilbert's positive logic, Johansson's Minimalkalkül and any (finitely axiomatizable) intermediate logic (in (MP); see [5]),

(iv) Łukasiewicz's many-valued logics $Ł_n$ $(n \geq 3)$ and $\aleph_0$ (cf. [13]), etc.

Proof. Trivial derivations, using Corollary 16.

Remark 18.

Arguments similar to those used earlier apply, mutatis mutandis, to quantificational extensions of the logics named above. (Do the exercises of [19], section 4.)

4. Refinements for Relevant Logics.

A. N. Prior noticed (cf. [14], page 181) that the original methods of Tarski for obtaining single axioms do not work in the absence of the "paradoxical" Law of Simplification $\underline{\underline{k}}$ (:= CpCqp). In particular, this comment applies to several interesting (implicative) logics among which the relevant logics R, E and some of their neighbours or rivals (see [1], [22], [28], etc.).

It will be clear from what follows that Prior's statement no longer holds for our methods.

Actually, the problem of axiomatizing Church's weak implication (in [3], i.e., the system $R_{\rightarrow}$ of [1]) was raised incidentally in [1], 8.5.1. in [19] we claimed that $R_{\rightarrow}$ and the Anderson-Belnap Pure Entailment system $E_{\rightarrow}$ of [1] are Tarski axiomatizable in (MP) but the effective example of single axiom suggested there, for $R_{\rightarrow}$, contained an oversight.

In this section we shall state explicitely - this was not the case in [19] - some alternative lists of conditions guaranteeing the Tarski axiomatizability of a large class of (purely implicative) relevant logics (in the sense of [1],[22]), among which $R_{\rightarrow}$, $E_{\rightarrow}$ etc.

We shall first introduce some convenient terminology.

Let L be a (purely) implicative logic. A theorem of L is solvable if it is m-solvable for some $m \geq 0$, otherwise it is unsolvable. Sets of theorems of L will be referred to similarly.

Examples: $\underline{\underline{b}}, \underline{\underline{b}}', \underline{\underline{c}}, \underline{\underline{c}}_*, \underline{\underline{i}}, \underline{\underline{k}}, \underline{\underline{k}}_m$ $(m \geq 0)$ are solvable, while $\underline{\underline{w}}, \underline{\underline{s}}, \underline{\underline{s}}'$ are unsolvable.

Clearly, unsolvable sets may contain solvable elements, but we needn't distinguish among such subtleties.

Let L (implicative) be now finitely axiomatizable in some set of rules $\{(R_1), \ldots, (R_r), (MP)\}$ - (MP) for the specified notion of implication - with $r \geq 0$, and all its axioms in the set

$$\underset{\sim}{B}(\underset{\wedge}{L}) = \{\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q\}$$

$(p,q \geq 0, p + q \geq 1)$ such that the α_i's $(1 \leq i \leq p)$ are unsolvable and the β_j's $(1 \leq f \leq q)$ are solvable. Usually, $\underset{\sim}{B}(\underset{\wedge}{L})$ is called <u>a basis for/of</u> $\underset{\wedge}{L}$. Hereafter, "$\underset{\sim}{B}(\underset{\wedge}{L})$" will refer to this description with p,q varying as indicated $(n := p + q$ positive, with possibly either $p = 0$ or $q = 0$; so no basis is empty, but it may contain no unsolvable, resp. no solvable elements).

Consider now the functions $\underset{\sim}{d}_n := \underset{\sim}{d}_{0,n}$ of section 2. Let $m \geq 0$.

A basis $\underset{\sim}{B}(\underset{\wedge}{L})$ for some implicative logic $\underset{\wedge}{L}$ is <u>sequentially</u> m-<u>quasisolvable</u> if its elements can be arranged in a sequence

$$\alpha_1,\ldots,\alpha_p,\beta_1,\ldots,\beta_q \qquad (p+q \geq 1;\ p,q \geq 0),$$

with the α_j's unsolvable, and there are theorems

$$\alpha_1,\ldots,\alpha_p$$

of $\underset{\wedge}{L}$ such that the following formulae are m-solvable:

(i) $\gamma_i,\beta_j,$ $\qquad (1 \leq i \leq p,\ 1 \leq j \leq q)$

(ii) $\underset{=}{v}_i := \underset{\sim}{d}_2(\alpha_i,\gamma_i)$ $\qquad (1 \leq i \leq p)$

(iii) $\underset{=}{f}_i := \underset{\sim}{d}_i(\underset{=}{v}_1,\ldots,\underset{=}{v}_i)$ $\qquad (1 \leq i \leq p)$

(iv) $\underset{=}{h}_j := \underset{\sim}{d}_{j+1}(\underset{=}{f}_p,\beta_1,\ldots,\beta_j)(1 \leq j \leq q).$

(Note that $\underset{=}{f}_1 := \underset{=}{v}_1 := \underset{\sim}{d}_2(\alpha_1,\gamma_1)$ and $\underset{=}{h}_1 := \underset{\sim}{d}_{p+1}(\underset{=}{v}_1,\ldots,\underset{=}{v}_p,\beta_1)$, by the definition of $\underset{\sim}{d}_1,\underset{\sim}{d}_2,\ldots,\underset{\sim}{d}_{p+1}$.)

In particular, $\underset{\sim}{B}(\underset{\wedge}{L})$ is <u>sequentially</u> m-<u>solvable</u> if $p = 0$; that is, $\underset{\sim}{B}(\underset{\wedge}{L}) = \{\beta_1,\ldots,\beta_q\}$ contains only m-solvable axioms and for some permutation π of the set $\{1,\ldots,q\}$, the formulae

$$\underset{=}{h}_j^{\pi} := \underset{\sim}{d}_j(\beta_{\pi(1)},\ldots,\beta_{\pi(j)}) \qquad (1 \leq j \leq q)$$

are also m-solvable (viz., $\underset{=}{h}_j^{\pi}\underset{=}{i}^m$ proves $\underset{=}{i}$, for all j, $1 \leq j \leq q$).

For simplicity, we shall consider first (purely) implicative logics possessing at least one (sequentially) m-solvable basis, for some $m \geq 2$. Next we shall extend our discussion to implicative logics possessing arbitrary bases.

Theorem 19.

Let $\underset{\wedge}{L}$ be an implicative logic (for some specified notion C of implication) such that

(i) $\underset{\wedge}{L}$ is finitely axiomatizable in the set of rules

$$\{(R_1),\ldots,(R_r),(MP)\} \qquad (r \geq 0),$$

If, for some $m \geq 2$,

(ii) $\underset{\wedge}{L}$ has a sequentially m-solvable basis

(iii) $\underline{\underline{k}}_m$ is a theorem of $\underset{\wedge}{L}$,

and

(iv) $\underline{\underline{d}},\underline{\underline{t}}$ are theorems of $\underset{\wedge}{L}$

then $\underset{\wedge}{L}$ is Tarski axiomatizable in the same set.

Proof. Let $\alpha_1,\ldots,\alpha_n$ $(n \geq 1)$ be the axioms of $\underset{\wedge}{L}$ (taking $n = 1$, trivially, as a limit case of the Theorem) and construct, for an appropriate π, formulae

$$\underline{\underline{h}}_j^{\pi} := \underline{\underline{d}}_j(\alpha_{\pi(1)},\ldots,\alpha_{\pi(j)}) \qquad (1 \leq j \leq n).$$

Sequential m-solvability means that, for some π, one has

$$\alpha_j\underline{\underline{i}}^m \text{ proves } \underline{\underline{i}} \qquad (1 \leq j \leq n),$$

$$\underline{\underline{h}}_j^{\pi}\underline{\underline{i}}^m \text{ proves } \underline{\underline{i}} \qquad (1 \leq j \leq n).$$

Now recall that, by Lemma 9:(2) we have

$$\underline{\underline{k}}_m\underline{\underline{i}}^m \text{ proves } \underline{\underline{i}}, \qquad \text{for all } m \geq 2.$$

We claim the needed single axiom is

$$\underline{\underline{g}}_{(m,n)}^{\pi} := \underline{\underline{t}}(\underline{\underline{k}}_m,\underline{\underline{h}}_n^{\pi},\underline{\underline{k}}_m) := CC\underline{\underline{k}}_m C\underline{\underline{h}}_n^{\pi} C\underline{\underline{k}}_m pp$$

(observing the relettering convention of section 1 above).

Indeed, $\underline{\underline{k}}_m,\underline{\underline{d}}$ are theorems of $\underset{\wedge}{L}$ (the hypothesis of the Theorem), so $\underline{\underline{h}}_n^{\pi}$ is a theorem of $\underset{\wedge}{L}$, for, by Lemma 10,

$$\underline{\underline{d}}^{n-1}\alpha_1\ldots\alpha_n \text{ proves } \underline{\underline{h}}_n^{\pi}, \qquad \text{for any } \pi.$$

As $\underline{\underline{t}}$ is a theorem of $\underset{\wedge}{L}$ (by hypothesis, again), one has also that, for an appropriate π, $\underline{\underline{g}}_{(m,n)}^{\pi}$ is a theorem of $\underset{\wedge}{L}$, since

$$\underline{\underline{t}}\underline{\underline{k}}_m\underline{\underline{h}}_n^{\pi}\underline{\underline{k}}_m \text{ proves } \underline{\underline{g}}_{(m,n)}^{\pi},$$

by Lemma 11.

Conversely, let L_* be the formulation of L with $g_* := g^{\pi}_{(m,n)}$, for convenience (m,n fixed), and primitive rules as earlier.

One works in L_*, deriving first k_m from g_*, next h^{π}_n and, finally, the α_j's ($1 \leq j \leq n$). This can be done as follows:

$$g_*g_* \text{ proves } k_m^+,$$

for h^{π}_n is m-solvable, by the hypothesis of the Theorem.

Recall also that, by Lemma 6:(2),

$$k_m k_m \text{ proves } k_m^+, \qquad (m \geq 0),$$

so

$$g_*g_* = k_m k_m = k_m^+, \qquad (m \geq 2)$$

and hence

$$g_*g_*k_m^+ = k_m^+ k_m^+ = k_m \qquad (m > 2),$$

given the m-solvability of k_m for $m \geq 2$, (Lemma 9:(2) above), while

$$g_*g_*k_m^+ \text{ proves } i \qquad \text{for } m = 2,$$

and therefore

$$g_*g_*i = k_m k_m i = ii^m k_m \qquad (m = 2),$$

by Lemma 4:(7) and Corollary 7, etc.

So far we have shown that k_m is a theorem of L_*, for all $m \geq 2$.

Now

$$g_*k_m^+ = g_*(k_m k_m) = g_n^{\pi},$$

for k_m is m-solvable ($m \geq 2$), by Lemma 9:(2).

That is, collecting the facts,

$$g_*(g_*g_*) \text{ proves } h^{\pi}_n \qquad (\text{for } m \leq 2$$

so, anyway, k_m, k_m and h^{π}_n can be proved from g_* by (MP) and (SB) only and the result holds for $r = 0$, too.

Next, since $\alpha_j i^m$ proves i, for $1 \leq j \leq m$, we have

$$h^{\pi}_n k_m = h^{\pi}_{n-1}$$

$$h^{\pi}_{n-1} k_m = h^{\pi}_n k_m^2 = h^{\pi}_{n-2}$$

$$\ldots \quad \ldots \quad \ldots$$

$$\underline{\underline{h}}_2^\pi \underline{\underline{k}}_m = \underline{\underline{h}}_n^\pi \underline{\underline{k}}_m^{n-1} = \underline{\underline{h}}_1^\pi = \alpha_1 .$$

Finally, for all $j, 1 \leq j \leq n$,

$$\underline{\underline{h}}_j^\pi \underline{\underline{k}}_m^+ \underline{\underline{k}}_m \text{ proves } \alpha_j \qquad (m \geq 2)$$

since

$$\underline{\underline{h}}_j^\pi \underline{\underline{i}}^m \text{ proves } \underline{\underline{i}} \qquad (1 \leq j \leq n)$$

and

$$\underline{\underline{k}}_m \underline{\underline{i}}^m \text{ proves } \underline{\underline{i}} \qquad (m \geq 2).$$

Therefore, each α_j $(1 \leq j \leq n)$ is provable from $\underline{\underline{g}}_* := \underline{\underline{g}}^\pi_{(m,n)}$ by (MP) and (SB) only, and this completes the proof of the Theorem with $r \geq 0$.

Unlike in Theorem 12, the hypotheses of Theorem 19 allow also applications to (purely implicative) relevant logics (in the sense of [1], [22], [25], [26], [28]). But, as earlier, in section 3, where we paused on C. A. Meredith's $\widehat{BCK}_\rightarrow$, we prefer to reach that point _via_ some intermediary landmark. The motivation behind this _détour_ will appear soon.

Let $\widehat{BCI}_\rightarrow$ be _the Jaśkowski-Meredith purely implicative logic_ (cf. [18], Appendix I, [14], section 7, or even [17], whose "Postscript" gives the reason we had to use the name above). This, by the way, a relevant logic in the sense of [1], [22]. Specifically, it coincides with the purely implicative fragment of what the defenders of relevance use to call "Relevance without Contraction" "$\widehat{R}$-W", for short, where both "W" and "Contraction" denote our formula $\underline{\underline{w}}$, that is: "the Hilbert formula" of the post-war Dublin residents, whether Polish or not) and has been studied - on different grounds - by various persons among which S. Jaśkowski, C. A. Meredith (as principal proponents; see references given earlier), A. Church, N. D. Belnap Jr., A. Urquhart (cf. [25], [27]), R. Routley, R. K. Meyer (see [22] and the references given there), and the author ([19]).

By definition, $\widehat{BCI}_\rightarrow$ if finitely axiomatizable in (MP) with, as axioms, $\underline{\underline{b}}, \underline{\underline{c}}$ and $\underline{\underline{i}}$. C. A. Meredith has also established its Tarski axiomatizability in (MP) (cf. [18], Appendix I, [14], section 7; for alternative axiomatizations see also [19]).

We will be interested in extensions of $\widehat{BCI}_\rightarrow$ still possessing this property.

One has the following straightforward consequence of Theorem 19.

Corollary 20.

Any finitely axiomatizable extension L of $BCI_{\rightarrow}$ in some set of rules $\{(R_1),\ldots,(R_r),(MP)\}$, $(r \geq 0)$, such that L has a sequentially m-solvable basis, for some $m \geq 2$, is Tarski axiomatizable in the same set of rules.

Proof. Note that

$$\underline{\underline{ci}} \text{ proves } \underline{\underline{k}}_0 := \underline{\underline{c}}_*,$$

by Lemma 5:(5), and, as in Remark 15,

$$\underline{\underline{bcc}}_* \text{ proves } \underline{\underline{d}} := \underline{\underline{d}}_0$$

and

$$\underline{\underline{b}}(\underline{\underline{bc}})\underline{\underline{d}} \text{ proves } \underline{\underline{t}}.$$

Also, for all $n \geq 0$,

$$\underline{\underline{c}}(\underline{\underline{k}}_n\underline{\underline{i}}) \text{ proves } \underline{\underline{k}}_{n+1},$$

so $\underline{\underline{d}},\underline{\underline{t}}$ and the $\underline{\underline{k}}_m$'s $(m \geq 0)$ are all theorems of $BCI_{\rightarrow}$.

Finally, apply Theorem 19 to the case in point.

It is not very difficult to see that Corollary 20 is not completely pointless (viz. there are implicative logics satisfying its hypotheses - indeed, somewhat involved -; cf. Remark 23 below).

However, this result (as well as Theorem 19 above) does not seem to be very useful in getting the Tarski axiomatizability of Church's weak implication - i.e., the system $R_{\rightarrow}$ of [1], [3] - or of any "intermediate" implicative logic between $BCI_{\rightarrow}$ and $R_{\rightarrow}$.

To put the finger on the nature of the difficulty let us examine several axiomatizations of $R_{\rightarrow}$ (with (MP) and (SB)).

Recall first that, in [3], $R_{\rightarrow}$ was finitely axiomatizable in (MP) with, as axioms, $\underline{\underline{w}},\underline{\underline{b}},\underline{\underline{c}}$ and $\underline{\underline{i}}$. Henceforth, $R_{\rightarrow}$ will denote this formulation of Church's system. (But note that the basis $\{\underline{\underline{w}},\underline{\underline{b}}',\underline{\underline{c}},\underline{\underline{i}}\}$ has the same effect as Church's in view of Lemma 5:(1) and (2), and similarly, with $\underline{\underline{c}}$ replaced by either $\underline{\underline{c}}_*$ or $\underline{\underline{c}}'$ and/or $\underline{\underline{w}}$ replaced by $\underline{\underline{s}}$ or $\underline{\underline{s}}'$, etc.; see [19] for details.)

Clearly, Church's basis is unsolvable, due to the presence of $\underline{\underline{w}}$ (for $\underline{\underline{wi}}$ is already undefined; and similarly for the remaining four-element bases suggested earlier). So even if $BCI_{\rightarrow}$ is trivially a subsystem of $R_{\rightarrow}$, we have no means to apply our Corollary 20 (or, equivalently, Theorem 19) to Church's system unless we can find a solvable basis for it that is sequentially so, too. Is this, after all, possible? We incline, at present writing, to a negative answer of this question and will try to briefly explain why next.

We shall do first of all some more axiom chopping (around $R_{\to}$).

Define two new boldface types, namely:

$$\mathbf{a} := \text{CCpCqrCCspCqCsr},$$
$$\mathbf{w}' := \text{CpCCpCpqq}.$$

The combinatory argument behind half of the following Lemma relies on a similar construction which could have been traced back to the work of a pioneer in combinatory logic, <u>viz</u>. to F. B. Fitch's Yale dissertation (1934).

<u>Lemma 21</u>.

$R_{\to}$ is finitely axiomatizable in (MP) with, as bases,

(i)	$\{\mathbf{w},\mathbf{i},\mathbf{a}\}$	(ii)	$\{\mathbf{w}',\mathbf{i},\mathbf{a}\}$
(iii)	$\{\mathbf{w}',\mathbf{b},\mathbf{c},\mathbf{i}\}$	(iv)	$\{\mathbf{w}',\mathbf{b}',\mathbf{c},\mathbf{i}\}$
(v)	$\{\mathbf{w}',\mathbf{b},\mathbf{c}_*,\mathbf{i}\}$	(vi)	$\{\mathbf{w}',\mathbf{b}',\mathbf{c}_*,\mathbf{i}\}$
(vii)	$\{\mathbf{w}',\mathbf{b},\mathbf{c}',\mathbf{i}\}$	(viii)	$\{\mathbf{w}',\mathbf{b}',\mathbf{c}',\mathbf{i}\}$,

etc.

<u>Proof</u>. Let, for convenience, $B_i(R_{\to})$, with $1 \leq i \leq 8$, be the corresponding bases and $B_0(R_{\to})$ be Church's basis $\{\mathbf{w},\mathbf{b},\mathbf{c},\mathbf{i}\}$

(i) :$B_1(R_{\to})$ produces $B_0(R_{\to})$ for one has that

$\mathbf{ai}$ proves $\mathbf{c}$,
$\mathbf{ci}$ proves $\mathbf{c}_*$, Lemma 5:(5)
$\mathbf{ac}_*$ proves $\mathbf{b}'$,

and

$\mathbf{cb}'$ proves $\mathbf{b}$, Lemma 5:(1).

Conversely,

$\mathbf{b}(\mathbf{bc})\mathbf{b}$ proves $\mathbf{a}$,

so $B_0(R_{\to})$ produces $B_1(R_{\to})$.

(ii) :$B_1(R_{\to})$ and $B_2(R_{\to})$ are equivalent, for

$$\mathbf{cw} \text{ proves } \mathbf{w}'$$

and

$$\mathbf{cw}' \text{ proves } \mathbf{w}.$$

(iii)-(viii): Use (ii) and Lemma 5.

Lemma 22.

For all $\alpha, \beta_1, \beta_2, \beta_3, \beta_4$ in any implicative logic $\underset{\wedge}{L}$,

(1) $\mathbf{w}'\alpha = \mathbf{d}\alpha\alpha = \underset{\sim}{d}_2(\alpha,\alpha)$;

(2) $\mathbf{w}'\mathbf{ii}$ is undefined (so, finally, $\mathbf{w}'$ is unsolvable);

(3) $\mathbf{w}\ \mathbf{bi} = \mathbf{bii} = \mathbf{i}$; (4) $\mathbf{w}\ \mathbf{b}'\mathbf{i} = \mathbf{b}'\mathbf{ii} = \mathbf{i}$;

(so $\mathbf{b}$ and $\mathbf{b}'$ are 2-solvable and therefore m-solvable for $m \geq 2$).

(5) $\mathbf{wai} = \mathbf{aii} = \mathbf{c}_*$; (6) $\mathbf{c}_*\mathbf{ii} = \mathbf{i}$ ($\mathbf{c}_*$ is 2-solvable);

(7) $\mathbf{a}\beta_1\beta_2\beta_3\beta_4 \simeq \beta_1(\beta_2\beta_4)\beta_3$ and $\mathbf{a}$ is m-solvable with $m \geq 4$.

(8) $\mathbf{c}, \mathbf{c}'$ are 3-solvable (and hence m-solvable with $m \geq 3$).

Proof. Trivial.

Remark 23.

Corollary 20 applies easily to $\underset{\wedge\wedge\wedge\to}{BCK}$. Take, e.g., the standard basis $\{\mathbf{k},\mathbf{b},\mathbf{c}\}$ (with (MP)). This basis is, clearly, 3-solvable (hence m-solvable, for $m \geq 3$) while, with $\mathbf{h}_1 := \mathbf{k}, \mathbf{h}_2 := \underset{\sim}{d}_2(\mathbf{k},\mathbf{b})$ and $\mathbf{h}_3 := \underset{\sim}{d}_3(\mathbf{k},\mathbf{b},\mathbf{c})$, one finds immediately that $\mathbf{k}$ is 2-solvable, $\mathbf{h}_2$ is 4-solvable and $\mathbf{h}_3$ is 3-solvable. So the $\underset{\wedge\wedge\wedge\to}{BCK}$-basis above is sequentially 4-solvable and $\underset{\wedge\wedge\wedge\to}{BCK}$ is, obviously, a proper extension of $\underset{\wedge\wedge\wedge\to}{BCI}$.

Remark 24.

Let $\mathbf{i}_n$ ($n \geq 0$) be lexical variants of $\mathbf{i} := \mathbf{i}_0$, as in section 1. Then one has clearly, for $m \neq n$,

$$\mathbf{di}_m\mathbf{i}_n = \underset{\sim}{d}_2(\mathbf{i}_m,\mathbf{i}_n) := CC\mathbf{i}_m C\mathbf{i}_n rr,$$

so $\mathbf{d}$ is 3-solvable.

However, for any $m \geq 0$,

$$\mathbf{w}'\mathbf{i}_m = \mathbf{di}_m\mathbf{i}_m = \underset{\sim}{d}_2(\mathbf{i}_m,\mathbf{i}_m),$$

and $\mathbf{w}'\mathbf{i}_m$ proves only $CC\mathbf{i}_m C\mathbf{i}_m rr :=$ CCCppCCpprr, *viz.* a (proper) substitution instance of $CC\mathbf{i}_m C\mathbf{i}_n rr$. Now it is not difficult to see that $\mathbf{w}'\mathbf{i}_m\mathbf{i}_n$ is undefined for any $m,n \geq 0$. Thus $\mathbf{w}'$ is <u>unsolvable</u> and, as it was the case with the $\mathbf{R}_{\to}$-bases considered earlier, we cannot use Lemma 22:(2) in order to apply Corollary 20 (or Theorem 19) to the case of $\mathbf{R}_{\to}$.

We will try now to explain why is Corollary 20 inappropriate for our present purposes (the Tarski axiomatizability of $\mathbf{R}_{\to}$).

A (pure) implicative formula α is a <u>prima facie</u> theorem of $\mathbf{R}_{\to}$ if it is a theorem of $\mathbf{R}_{\to}$ without thereby being a (proper) substitution instance of some theorem of $\mathbf{R}_{\to}$ (that is: it does not arise from a theorem of $\mathbf{R}_{\to}$ by identifying two or more propositional variables it contains). E.g., $\mathbf{w} := CCpCpqCpq$ is <u>prima facie</u>, but $CCpCCppCpp$ is not so. Further, let a <u>prima facie</u> theorem of $\mathbf{R}_{\to}$ be <u>non-linear</u> if it is not already a theorem of $\mathbf{BCI}_{\to}$.

<u>Conjecture 25</u>.

The non-linear theorems of $\mathbf{R}_{\to}$ are unsolvable.

If our conjecture is true then it is already clear that no axiomatization of $\mathbf{R}_{\to}$ can satisfy the hypotheses of Corollary 20 (resp. Theorem 19).

Fortunately, we can prove the following generalization of Theorem 19.

<u>Theorem 26</u>.

Theorem 19 holds with "sequentially m-solvable" replaced by "sequentially m-quasi-solvable".

<u>Proof</u>. Let $\mathbf{L}$ be finitely axiomatizable in some set of rules containing (MP), for the specified notion of implication, with axioms

$$\alpha_1,\ldots,\alpha_p,\ \beta_1,\ldots,\beta_q \qquad (p,q \geq 0,\ p+q \geq 1),$$

where the α_i's are unsolvable and the β_j's are m-solvable, for some $m \geq 2$. Construct now, for $\gamma_1,\ldots,\gamma_p$ m-solvable theorems of $\mathbf{L}$, formulae

$$\begin{aligned} \mathbf{v}_i &:= \mathbf{d}_2(\alpha_i,\gamma_i) && (1 \leq i \leq p),\\ \mathbf{f}_i &:= \mathbf{d}_i(\mathbf{v}_1,\ldots,\mathbf{v}_i) && (1 \leq i \leq p),\\ \mathbf{h}_j &:= \mathbf{d}_{j+1}(\mathbf{f}_p,\beta_1,\ldots,\beta_j) && (1 \leq j \leq q). \end{aligned}$$

By the hypothesis of the Theorem these formulae must be m-solvable, too. The needed single axiom for $\mathbf{L}$ is now

$$\mathbf{g} := \mathbf{g}_{(p,q,m)} := \mathbf{t}(\mathbf{k}_m,\mathbf{h}_q,\mathbf{k}_m) := CC\mathbf{k}_m C\mathbf{h}_q C\mathbf{k}_m ss$$

and one has, as earlier in the proof of Theorem 19, that

$$\underline{\underline{gg}}(\underline{\underline{gg}}) \text{ proves } \underline{\underline{k}}_m,$$

for $\underline{\underline{h}}_q$ is m-solvable (and so is $\underline{\underline{k}}_m$) and

$$\underline{\underline{g}}(\underline{\underline{gg}}) \text{ proves } \underline{\underline{h}}_q.$$

"Sequentially" points out, as in the case of sequential solvability, to the way of obtaining the α_i's and the β_j's $(1 \leq i \leq p,\ 1 \leq j \leq q)$. Specifically, the $\underline{\underline{h}}_j$'s $(1 \leq j < q)$ and $\underline{\underline{f}}_p$ can be obtained from $\underline{\underline{h}}_q$, i.e.:

$$\underline{\underline{h}}_j\underline{\underline{k}}_m \text{ proves } \underline{\underline{h}}_{j-1}, \underline{\underline{h}}_1\underline{\underline{k}}_m \text{ proves } \underline{\underline{f}}_p \qquad (1 \leq j < q),$$

for the β_j's $(1 \leq j \leq q)$ are m-solvable.
Then the $\underline{\underline{f}}_i$'s $(1 \leq i < p)$ can be obtained from $\underline{\underline{f}}_p$ as follows:

$$\underline{\underline{f}}_i\underline{\underline{k}}_m \text{ proves } \underline{\underline{f}}_{i-1} \qquad (1 < i \leq p),$$

for the $\underline{\underline{v}}_i$'s $(1 < i \leq p)$ are m-solvable. (Note that $\underline{\underline{f}}_1 := \underline{\underline{v}}_1$.)
Now the remaining $\underline{\underline{v}}_i$'s $(1 < i \leq p)$ can be obtained from the $\underline{\underline{f}}_i$'s, i.e.:

$$\underline{\underline{f}}_i(\underline{\underline{k}}_m\underline{\underline{k}}_m)\underline{\underline{k}}_m \text{ proves } \underline{\underline{v}}_i \qquad (1 < i \leq p),$$

for the $\underline{\underline{f}}_i$'s $(1 \leq i < p)$ are m-solvable.

Finally, the α_i's can be obtained from the $\underline{\underline{v}}_i$'s $(1 \leq i \leq p)$, since the γ_i's are supposed to be m-solvable and hence

$$\underline{\underline{v}}_i\underline{\underline{k}}_m \text{ proves } \alpha_i \qquad (1 \leq i \leq p),$$

while the β_j's $(1 \leq j \leq q)$ can be obtained from the corresponding $\underline{\underline{h}}_j$'s, as expected, i.e., by

$$\underline{\underline{h}}_j(\underline{\underline{k}}_m\underline{\underline{k}}_m)\underline{\underline{k}}_m \text{ proves } \beta_j \qquad (1 \leq j \leq q),$$

for $\underline{\underline{h}}_1, \ldots, \underline{\underline{h}}_{q-1}$ are m-solvable.

On the other hand, $\underline{\underline{g}} := \underline{\underline{g}}_{(p,q,m)}$ is a theorem of $\underset{\wedge}{L}$, for so is $\underline{\underline{h}}_q$ (the γ_i's, $1 \leq i \leq p$, were supposed to be theorems of $\underset{\wedge}{L}$, while $\underline{\underline{d}}$ is a theorem of $\underset{\wedge}{L}$, by the hypothesis of the Theorem), $\underline{\underline{k}}_m$ and $\underline{\underline{t}}$. This completes the proof (Lemmas 10 and 11 were used tacitly).

Remark 27.

An analogue of Corollary 20 holds with "sequentially m-solvable" replaced by "sequentially m-quasi-solvable".

Corollary 28.

$\underset{\sim}{R}_{\rightarrow}$ is Tarski axiomatizable in (MP).

Proof. This is a straightforward application of Theorem 26 (Corollary 27), Consider Church's axiomatization of $R_{\rightarrow}$ with

$$\alpha_1 := \underline{\underline{w}}, \beta_1 := \underline{\underline{i}}, \beta_2 := \underline{\underline{c}}, \ \beta_3 := \underline{\underline{b}}.$$

Set $\gamma_1 := \underline{\underline{b}}'$ (recall that $\underline{\underline{cb}}$ proves $\underline{\underline{g}}'$.

Clearly, $\underline{\underline{v}}_1 := \underline{\underline{f}}_1 := \underset{\sim}{d}_2(\underline{\underline{w}},\underline{\underline{b}}')$. Now, with the conventions of the proof of Theorem 26, one has

$$\underline{\underline{h}}_1 := \underset{\sim}{d}_2(\underline{\underline{f}}_1,\underline{\underline{i}}) := \underset{\sim}{d}_3(\underline{\underline{w}},\underline{\underline{b}}',\underline{\underline{i}}),$$

$$\underline{\underline{h}}_2 := \underset{\sim}{d}_3(\underline{\underline{f}}_1,\underline{\underline{i}},\underline{\underline{c}}) := \underset{\sim}{d}_4(\underline{\underline{w}},\underline{\underline{b}}',\underline{\underline{i}},\underline{\underline{c}}),$$

and

$$\underline{\underline{h}}_3 := \underset{\sim}{d}_4(\underline{\underline{f}}_1,\underline{\underline{i}},\underline{\underline{c}},\underline{\underline{b}}) := \underset{\sim}{d}_5(\underline{\underline{w}},\underline{\underline{b}}',\underline{\underline{i}},\underline{\underline{c}},\underline{\underline{b}}).$$

Using now Lemma 1, Remark 3, Corollary 8 and Lemma 22:(4) one can show that $\underline{\underline{f}}_1, \underline{\underline{h}}_1$ and $\underline{\underline{h}}_2$ are 2-solvable, while $\underline{\underline{h}}_3$ is 4-solvable. As the set of β_j's $(j = 1,2,3)$ is 3-solvable and $\gamma_1 := \underline{\underline{b}}'$ is 2-solvable (cf. Lemma 22:(3), (8), etc.), we have already established that Church's basis is sequentially 4-quas-solvable. On the other hand, we know that $\underline{\underline{d}}, \underline{\underline{t}}$, and $\underline{\underline{k}}_4$ are $\underset{\sim}{R}_{\rightarrow}$-theorems (cf. the proof of Corollary 20), so Church's system is Tarski axiomatizable in (MP) with, as single axiom,

$$\underline{\underline{r}} := \underset{\sim}{t}(\underline{\underline{k}}_4,\underline{\underline{h}}_3,\underline{\underline{h}}_4) := CC\underline{\underline{k}}_4 C\underline{\underline{h}}_3 C\underline{\underline{k}}_4 pp,$$

where $\underline{\underline{h}}_3$ is as above.

Remark 29.

It is possible to shorten the latter single axiom $\underline{\underline{r}}$, obtained in Corollary 28, using the fact that $\underline{\underline{aii}}$ proves $\underline{\underline{c}}_*$ (cf. Lemma 22:(5)). Indeed, set, with the conventions of the proof of Theorem 26,

$$\alpha_1 := \underline{\underline{w}}, \beta_1 := \underline{\underline{i}}, \beta_2 := \underline{\underline{b}} \ \text{ and } \ \gamma_1 := \underline{\underline{a}}.$$

(We have seen in Lemma 21 that $\underline{\underline{a}}$ is a theorem of $\underset{\wedge\wedge\wedge}{BCI}_{\rightarrow}$ and hence of $\underset{\sim}{R}_{\rightarrow}$.)

Now construct formulae $\underline{\underline{v}}_1 := \underset{\sim}{d}_2(\underline{\underline{w}},\underline{\underline{a}})$, $\underline{\underline{f}}_1 := \underline{\underline{v}}_1, \underline{\underline{h}}_1 := \underset{\sim}{d}_2(\underline{\underline{f}}_1,\underline{\underline{i}}) ;= \underset{\sim}{d}_3(\underline{\underline{w}},\underline{\underline{a}},\underline{\underline{i}}_3)$ and $\underline{\underline{h}}_2 := \underset{\sim}{d}_3(\underline{\underline{f}}_1,\underline{\underline{i}},\underline{\underline{b}}) := \underset{\sim}{d}_4(\underline{\underline{w}},\underline{\underline{a}},\underline{\underline{i}},\underline{\underline{b}})$. One finds easily that $\underline{\underline{h}}_2\underline{\underline{i}}^3$ proves $\underline{\underline{c}}_*$

(for $\mathbf{h}_2\mathbf{i}^3 = \mathbf{h}_1\mathbf{bii} = \mathbf{bf}_1\mathbf{iii} = \mathbf{f}_1(\mathbf{ii})\mathbf{i} = \mathbf{f}_1\mathbf{ii} = \mathbf{wai} = \mathbf{aii} = \mathbf{c}_*$); so one has also that $\mathbf{h}_2\mathbf{i}^5 = \mathbf{i}$, for $\mathbf{c}_*$ is 2-solvable. Next $\mathbf{h}_1$ is 4-solvable for $\mathbf{h}_1\mathbf{i}^4 = \mathbf{f}_1\mathbf{i}^4 = \mathbf{wai}^3 = \mathbf{ai}^4 = \mathbf{i}$ and, finally, $\mathbf{v}_1 := \mathbf{f}_1$ is 4-solvable, by the same token ($\mathbf{f}_1\mathbf{i}^4 = \mathbf{wai}^3$, etc.). So the set $\{\mathbf{w},\mathbf{i},\mathbf{b}\}$ is sequentially 5-quasi-solvable and we may set as single axiom (for $\mathsf{R}_\to$) $\mathbf{r}^\& := CC\mathbf{k}_5C\mathbf{h}_2C\mathbf{k}_5pp$, with $\mathbf{h}_2$ as above. The additional trick (beyond the pattern of proof of Theorem 26) consists of getting $\mathbf{c}_*$ from $\mathbf{h}_2$ and $\mathbf{i}$. But the set $\{\mathbf{w},\mathbf{i},\mathbf{b},\mathbf{c}_*\}$ is a basis for $\mathsf{R}_\to$ (by Lemma 5).

Theorem 19 admits now of a generalization in some other direction, viz. by generalizing the concept of solvability.

Let L be some (purely) implicative logic and $\vec{\beta} := \beta_1,\ldots,\beta_m$ be a sequence of theorems of L (possibly with repetitions and possibly empty). For $\vec{\beta}$ fixed, set

$$\mathbf{k}_{\vec{\beta}} := CpCC\beta_1\ldots C\beta_m CCpqq.$$

(One can easily see that for $\vec{\beta}$ empty one has $\mathbf{k}_{\vec{\beta}} := \mathbf{k}_o := \mathbf{c}_*$.)

A set $\{\alpha_1,\ldots,\alpha_n\}$ of theorems of L is $\vec{\beta}$-solvable (for $\vec{\beta}$ fixed) if $\alpha_j\vec{\beta} := \alpha_j\beta_1\ldots\beta_m$ proves $\mathbf{i}$ for all j, $1 \le j \le n$.

Construct now for some sequence $\alpha_1,\ldots,\alpha_n$ (of theorems of L) formulae $\mathbf{h}_j := \underset{\sim}{d}_j(\alpha_1,\ldots,\alpha_j)$ with $1 \le j \le n$.

A set $\{\alpha_1,\ldots,\alpha_n\}$ of theorems of L is sequentially $\vec{\beta}$-solvable (for $\vec{\beta}$ fixed) if its elements can be arranged (without repetitions) in a sequence $\alpha_1,\ldots,\alpha_n$ say such that

(i) each α_j is $\vec{\beta}$-solvable $(1 \le j \le n)$,

(ii) each $\mathbf{h}_j$ (constructed as above) is $\vec{\beta}$-solvable $(1 \le j \le n)$ and finally,

(iii) $\mathbf{k}_{\vec{\beta}}$ is $\vec{\beta}$-solvable.

Clearly, if each β_i in $\vec{\beta}$ $(1 \le i \le m)$ is the formula $\mathbf{i}$, $\vec{\beta}$-solvability for this $\vec{\beta}$ amounts to m-solvability and sequential $\vec{\beta}$-solvability coincides with sequential m-solvability for $m \ge 2$.

One can prove by a straightforward extension of the methods used earlier that the following generalization of Theorem 19 holds.

Theorem 30.

Theorem 19 holds with "sequential m-solvability" replaced by "sequential $\vec{\beta}$-solvability" for some fixed sequence $\vec{\beta} := \beta_1,\ldots,\beta_m$ of theorems of $\underset{\wedge}{L}$ and with "$\underline{\underline{k}}_m$ is a theorem of $\underset{\wedge}{L}$" replaced by "$\underline{\underline{k}}_{\vec{\beta}}$ is a theorem of $\underset{\wedge}{L}$".

Proof. Mutatis mutandis, as for Theorem 19.

Remark 31.

Note also that Theorem 19 becomes a particular case of Theorem 30 with $\vec{\beta} := \underline{\underline{i}}_1,\ldots,\underline{\underline{i}}_m$ (the $\underline{\underline{i}}_j$'s are lexical variants of $\underline{\underline{i}}$, as in section 1 above).

Theorems 19, 26 and 30 do not apply as such to the purely implicative fragment $\underset{\wedge\to}{E}$ of the Anderson-Belnap Entailment system (cf. [1]) due to the fact $\underline{\underline{d}},\underline{\underline{t}}$ and the $\underline{\underline{k}}_m$'s ($m \geq 0$) are not theorems of $\underset{\wedge\to}{E}$. But the pattern of proof used earlier still works for some slight modification of the corresponding hypotheses.

Define first, with $\hat{p} := Cp'p''$, $\hat{q} := Cq'q''$ and $\hat{r} := Cr'r''$, the following formulae:

$$\underline{\underline{\hat{d}}} := C\hat{p}C\hat{q}CC\hat{p}C\hat{q}rr,$$

$$\underline{\underline{\hat{t}}} := C\hat{p}C\hat{q}C\hat{r}CC\hat{p}C\hat{q}C\hat{r}ss,$$

$$\underline{\underline{\hat{k}}}_0 := \underline{\underline{\hat{c}}}_* := C\hat{p}CC\hat{p}qq,$$

$$\underline{\underline{\hat{k}}}_n := C\hat{p}CC\underline{\underline{i}}_1\ldots C\underline{\underline{i}}_n CC\hat{p}qq, \qquad (n \geq 1)$$

$$\underline{\underline{\hat{c}}} := CCpC\hat{q}rC\hat{q}Cpr.$$

(All these formulae are theorems of $\underset{\wedge\to}{E}$, as one can easily check using the Fitch style formulation of $\underset{\wedge\to}{E}$ in [1]. But see Corollary 33 below for the corresponding Hilbert style derivations, with condensed detachment.)

We can now establish the following (stronger) form of Theorem 19.

Theorem 32.

Theorem 19 holds with $\underline{\underline{k}}_m$ ($m \geq 2$), $\underline{\underline{d}},\underline{\underline{t}}$ replaced by $\underline{\underline{\hat{k}}}_m,\underline{\underline{\hat{d}}},\underline{\underline{\hat{t}}}$ resp.

Proof. As earlier, for Theorem 19.

Let now $B\hat{C}I_{\to}$ be the (purely) implicative logic axiomatized by $\underline{b}, \hat{\underline{c}}$ and $\underline{i}$ with (MP) and (SB) as primitive rules of derivation. It is known that $B\hat{C}I_{\to}$ can be axiomatized also with (MP), (SB) and, as axioms, $\underline{b}'$ and $\underline{0} := \underline{k}_0'$, *i.e.*,

$$\underline{0} := CCCppqq,$$

(cf., *e.g.*, [14] or [1], 8.5.1.; the result is due to M. Wajsberg, C. A. Meredith and N. Belnap Jr., independently). C. A. Meredith has also found that $B\hat{C}I_{\to}$ is Tarski axiomatizable in (MP) (cf. [14], section 10 or [1], 8.5.1.).

It can be easily seen that the following (stronger) form of Corollary 20 obtains.

Corollary 33.

Corollary 20 holds with $BCI_{\to}$ replaced by $B\hat{C}I_{\to}$.

Proof. Note that $\underline{b}\hat{\underline{c}}\hat{\underline{c}}_*$ proves $\hat{\underline{d}}, \underline{b}(\underline{b}\hat{\underline{c}})\hat{\underline{d}}$ proves $\hat{\underline{t}}, \hat{\underline{k}}_0 := \hat{\underline{c}}_*$ and $\hat{\underline{c}}\underline{i}$ proves $\hat{\underline{c}}_*$, while, for all $m \geq 0$, $\hat{\underline{c}}(\hat{\underline{k}}_m \underline{i})$ proves $\hat{\underline{k}}_{m+1}$. Then apply Theorem 32.

One has also the following (stronger) analogues of Theorem 26 and Remark 27.

Theorem 34.

Theorem 19 holds with $\underline{d}, \underline{t}, \underline{k}_m$ replaced by $\hat{\underline{d}}, \hat{\underline{t}}, \hat{\underline{k}}_m$ resp. $(m \geq 2)$ and "sequentially m-solvable" replaced by "sequentially m-quasi-solvable."

Proof. *Mutatis mutandis*, as for Theorem 26.

Corollary 35.

Corollary 20 holds with $BCI_{\to}$ replaced by $B\hat{C}I_{\to}$ and "sequentially m-solvable" replaced by "sequentially m-quasi-solvable."

Proof. Note that $\hat{\underline{d}}, \hat{\underline{t}}$, and the $\hat{\underline{k}}_m$'s $(m \geq 0)$ are theorems of $B\hat{C}I_{\to}$, then apply Theorem 34.

Recall now (from [1], say) that the following sets of formulae axiomatize the Pure Entailment system $E_{\to}$ with (MP) and (SB) as primitive rules of derivation:

$$B_0(E_{\to}) = \{\underline{w}, \underline{b}, \hat{\underline{c}}, \underline{i}\}, \quad B_1(E_{\to}) = \{\underline{w}, \underline{b}', \underline{0}\}$$

(the latter one is Belnap's preferred basis for $E_{\to}$). Clearly, $E_{\to}$ is a (proper) extension of $B\hat{C}I_{\to}$ (though not of $BCI_{\to}$) and we may readily apply Corollary 35 to

Belnap's basis $B_1(E_{\to})$ say, getting the expected result, *viz.*

Corollary 36.

$E_{\to}$ is Tarski axiomatizable in (MP).

Proof. (Nearly completed above. Still, for the sake of effectiveness we can afford the following considerations.)
Set $\alpha_1 := \mathbf{w}, \beta_1 := \mathbf{0}, \beta_2 := \mathbf{b}'$ and $\gamma_1 := \mathbf{b}'$. Further, with conventions as in the proof of Theorem 26 (resp. Theorem 34), set $\mathbf{v}_1 := \mathbf{f}_1 := d_2(\mathbf{w},\mathbf{b}')$, $\mathbf{h}_1 := d_2(\mathbf{f}_1,\mathbf{0}) := d_3(\mathbf{w},\mathbf{b}',\mathbf{0})$ and $\mathbf{h}_2 := d_3(\mathbf{f}_1,\mathbf{0},\mathbf{b}') := d_4(\mathbf{w},\mathbf{b}',\mathbf{0},\mathbf{b}')$. Now $\mathbf{h}_2$ is 2-solvable for $\mathbf{h}_2\mathbf{ii} = \mathbf{h}_1\mathbf{b}'\mathbf{i} = \mathbf{b}'\mathbf{f}_1\mathbf{0i} = \mathbf{0}(\mathbf{f}_1\mathbf{i}) = \mathbf{0}(\mathbf{wb}') = \mathbf{wb}'\mathbf{i} = \mathbf{b}'\mathbf{ii} = \mathbf{i}$; $\mathbf{h}_1$ is 1-solvable for $\mathbf{h}_1\mathbf{i} = \mathbf{f}_1\mathbf{0} = \mathbf{0}(\mathbf{wb}') = \mathbf{wb}'\mathbf{i}$, etc. and $\mathbf{v}_1 := \mathbf{f}_1$ is 2-solvable ($\mathbf{f}_1\mathbf{ii} = \mathbf{wb}'\mathbf{i}$, etc.). Finally, $\mathbf{b}'$ is 2-solvable and $\mathbf{0}$ is 1-solvable. Thus, Belnap's basis is sequentially 2-quasi-solvable and the single axiom for $E_{\to}$ might be now, with $\mathbf{h}_2$ as above,

$$\mathbf{e} := t(\mathbf{t}_2,\mathbf{h}_2,\mathbf{k}_2) := CC\mathbf{k}_2C\mathbf{h}_2C\mathbf{k}_2pp,$$

on the usual pattern employed earlier for $R_{\to}$.

Remark 37.

As expected, a (stronger) variant of Theorem 30 holds with $\hat{\mathbf{k}}_{\vec{\beta}} := CCp'p''CC\beta_1\ldots C\beta_mCCCp'p''qq$ instead of $\mathbf{k}_{\vec{\beta}}$, for some fixed sequence $\vec{\beta} := \beta_1,\ldots,\beta_m$ ($m \geq 2$ say) of formulae of L. The corresponding statement is an obvious generalization of Theorem 32.

Remark 38.

Unlike the methods of proof available in the presence of $\mathbf{k}$ (in section 3 above) the methods of finding single axioms used in this section do not apparently apply to implicative logics containing conjunction and/or disjunction (these ingredients would obviously block the application of the unification algorithm while evaluating pe's). In particular, the full Relevance logic R (bf. [1], [22], [25]), the Entailment system E of Anderson-Belnap (cf. [1], [25]) and the Prawitz-Urquhart system S (cf. [28], say) as well as many other (propositional) relevant logics reviewed in [1] and [22] are cases in point. Specifically, it is an open problem whether the relevant logics R, E, S, etc. are Tarski axiomatizable in (MP) and the Adjunction rule (ADJ).

Added in proof (September 6, 1982). David Meredith noticed, in correspondence, that the formula $\underline{\underline{j}}$:= CCpCqqCpCqCpq falsifies our Conjecture 25. This suggest that $\underline{R}_{\rightarrow}$ may possess axiomatizations satisfying the hypotheses of Theorem 19 and/or Corollary 20, but does _not_ alter the content of our discussion in 4.

APPENDIX

On a Singleton Basis for the Set of Closed Lambda-terms.

As reported in [14, page 180], C. A. Meredith maintained once that the following lambda-term

$$\underline{G} =: \lambda xyz.y(\lambda u.z)(xz)$$

is a basis for the set of closed lambda-(K-) terms. Still he did never supply a proof of this claim in print and several attempts to a reconstruction of the missing derivations (see, e.g., [17, page 283] or [19, pp. 10-11]), even with the help of a computer (programmed in 16K LISP by Professor W. L. van der Poel), have been unsuccessful so far.

Actually, the needed argument is relatively simple.

Recall first some current notation in [19]:

$\underline{I}$ =: $\lambda x.x$, $\underline{K}$ =: $\lambda xy.x$, $\underline{K}'$ = : $\lambda xy.y$, $\underline{B}$ =: $\lambda xyz.x(yz)$,

$\underline{B}'$ =: $\lambda xyz.y(xz)$, $\underline{C}$ =: $\lambda xyz.xzy$, $\underline{C}_*$ =: $\lambda xy.yx$, $\underline{W}$ =: $\lambda xy.xyy$,

$\underline{W}_*$ =: $\lambda x.xx$, $\underline{S}$ =: $\lambda xyz.xz(yz)$, $\underline{S}'$ =; $\lambda xyz.yz(xz)$.

Further, set for any closed lambda-term X, $X_1 =: X$, $X_{n+1} =: X_n X$ (n a positive integer). = will denote beta-convertibility.

Note first that $\underline{G}_3 = \lambda xy.y(xx)$. One has immediately that

$\underline{C}_* = \underline{GG_3G}$, $\underline{I} = \underline{GC_*C_*}$, $\underline{K}' = \underline{GIC_*I}$ and $\underline{K} = \underline{GK'C_*}$.

Now one can obtain $\underline{B}'$, $\underline{B}$ and $\underline{C}$. Indeed, set $\underline{E} =: \underline{G}(\underline{KC_*})\underline{G}$. (Note that $\underline{E} = \lambda xy.x(\underline{K}y) = \underline{B}'\underline{K}$.).

Then $\underline{B}' = \underline{G}_2(\underline{KE}) = \underline{EG_2E}$, $\underline{B} = \underline{B}'\underline{C}_*(\underline{B}'\underline{B}')$ and $\underline{C} = \underline{B}'\underline{B}'(\underline{B}'\underline{C}_*)$.

Finally, one needs any one of $\underline{W}_*, \underline{W}, \underline{S}'$ or $\underline{S}$. The former two are easy to get: take, e.g., $\underline{W}_* = \underline{GI}(\underline{G_3I}) = \underline{GI}(\underline{C_*I})$ or $\underline{W} = \underline{G}_2(\underline{K}(\underline{C_*G_3}))$.

For $\underline{S}'$ and $\underline{S}$ one may proceed as in [19], viz. by realizing that

$$\underline{S}' = \underline{B}'\underline{G}(\underline{B}'(\underline{B}'(\underline{C_*G}))) \text{ and, finally, } \underline{S} = \underline{CS}'.$$

Question: is there any basis shorter than $\underline{G}$?

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