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ABSTRACT VOLTERRA EQUATION WITH INITIAL POINT DATA

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We study the functional differential equation with abstract Volterra operator

$$\dot{x}(t) = (Vx)(t), \quad t \in [0, T), \ 0 < T \leq \infty, \tag{1}$$

and initial data

$$x(0) = x^0 \in R^n, \tag{2}$$

where V denotes a Volterra operator acting from a function space $E([0,T),R^n)$ into another function space $F([0,T),R^n)$. In case of measurable function spaces, (1) has to be understood as valid almost everywhere on $[0,T)$, and the operator V will be called of Volterra type (or causal).

We obtained four existence theorems under various assumptions. The method of the proof is based on contraction mapping principle or fixed point theorem in locally convex spaces.

Now assume that $x = x(t)$ is a solution of (1), satisfying condition (2), defined on an interval $I = [0,t_1] \subseteq [0,T)$. If V is integrable, then we can integrate both sides of (1) from 0 to $t \in [0,T)$, and take into account of (2), we obtain that

$$x(t) = x^0 + \int_0^t (Vx)(s)\, ds. \tag{3}$$

The right side of (3) is also a Volterra operator, since V is a Volterra operator. The equation (3) is similar to second kind Volterra integral equation of the special type with $f(t) = x^0$.

On the other hand, if $x = x(t)$ is a continuous solution of equation (3), defined on the interval I, and V is continuous, then we may take the derivatives on both sides in (3), which will lead to the equation (1). Condition (2) can be easily obtained by letting $t = 0$ in (3).

Hence any solution of (1) and (2) is a solution of equation (3), and vice versa, if continuous solutions are considered.

Let us first consider the Lebesgue space $L^p([0,T], R^n)$, where $0 < T < \infty$ and $1 \leq p < \infty$. These spaces are well investigated, and among various properties of them, the following two theorems are important in this paper. We mention them here without proof (see, for instance, Kantorovich and Akilov [1], Dunford and Schwartz [1], Kolmogorov and Fomin[1]).

Theorem 1 (Riesz compactness criterion)

Let $M \subset L^p([t_0, t_1], R^n)$, with $0 \leq t_0 \leq t_1 < \infty$ and $1 < p < \infty$. The necessary and sufficient conditions for the relative compactness of M in L^p are:

(i) M is bounded in L^p;

(ii) $\int_{t_0}^{t_1} | x(t+h) - x(t) |^p dt \to 0$, as $h \to 0$, uniformly with respect to $x \in M$.

Theorem 2 (A.N. Kolmogorov criterion)

Let $M \subset L^p([t_0,t_1], R^n)$, with $0 \leq t_0 \leq t_1 < \infty$ and $1 \leq p < \infty$. The necessary and sufficient conditions for the relative compactness of M in L^p are:

(i) M is bounded in L^p;

(ii) $x_h(t) = \frac{1}{h} \int_t^{t+h} x(u)\, du \to x(t)$, as $h \to 0$, uniformly with respect to $x \in M$.

Next theorem is Schauder-Tychonoff fixed point theorem. We list it here, again, without proof (the proof is rather lengthy, see, for example, R.E. Edwards [1], C. Corduneanu [1]).

Theorem 3

Let E be a locally convex (Fréchet) space, and assume T: $K \to E$ is a continuous map, with

K ⊂ E convex, and $TK \subset A \subset K$, where A is compact. Then there exists at least one fixed point for T, that is, $Tx = x$ for some $x \in K$.

Based on these theorems we will prove the following results.

Theorem 1.4 Assume that:

i) V is a Volterra type continuous operator on $L^p_{loc}([0, T), R^n)$, with $0 < T \leq \infty$ and $p > 1$.

ii) There exist functions A(t) and B(t), with A(t) a positive continuous solution of the inequality

$$\int_0^t K \cdot A(s)\, ds \leq A(t), \quad \text{with } K = 2^{p-1},$$

and B(t) a locally integrable function on [0,T), such that for any $x \in L^p_{loc}([0, T), R^n)$, and

$\int_0^t |x(s)|^p ds \leq A(t)$, $t \in [0,T)$, one has $|(Vx)(t)|^p \leq B(t)$, almost everywhere on [0,T).

Furthermore, assume

$$\int_0^t B(s)ds \leq \left((A(t)-A(0)) \,/\, t^{\frac{p}{q}}\right), \quad 1/p + 1/q = 1, \text{ for } t \in (0,T).$$

Then, there exists a solution $x \in L^p_{loc}([0, T), R^n)$ of the equation (3), such that

$$\int_0^t |x(s)|^p ds \leq A(t),$$

provided $|x(0)|^p \leq A(0)$.

Before we prove Theorem 4 we need the following Lemma:

Lemma 5 There exists a positive continuous solution of

$$\int_0^t K \cdot A(s)\, ds \leq A(t), \text{ with } K > 0. \tag{4}$$

Proof.

Let us consider the function $A(t) = \varepsilon \cdot \exp(K \cdot t)$, for $\varepsilon>0$. Integrating both sides of (4) from 0 to $t \in (0,T)$, we obtain

$$\int_0^t K \cdot A(s)\, ds = \int_0^t K \cdot \varepsilon \cdot \exp(K \cdot s)\, ds = \varepsilon \int_0^t \exp(K \cdot s)\, d(Ks)$$

$$= \varepsilon(\exp(Kt)-1) \leq \varepsilon \cdot \exp(Kt) = A(t).$$

Thus, $A(t) = \varepsilon \cdot \exp(Kt)$ is a positive continuous solution of (4).

Now we are ready to prove Theorem 4.

Proof of Theorem 4. Consider the set

$$S = \Big\{ x;\ \int_0^t |x(s)|^p ds \leq A(t),\ t \in [0,T),\ x \in L^p_{loc}([0,\ T),\ R^n) \Big\}.$$

S is a closed, convex set belonging to $L^p_{loc}([0,\ T),\ R^n)$, as one can easily check. Defining

$$(V_1x)(t) = x^0 + \int_0^t (Vx)(s)ds,$$

the operator V_1 is also a Volterra type operator since V is a Volterra type operator. By Hölder's inequality, we obtain

$$|(V_1x)(t)| \leq |x^0| + \int_0^t |(Vx)(s)|ds \leq |x^0| + \Big(\int_0^t |(Vx)(s)|^p ds \Big)^{\frac{1}{p}} \cdot \Big(\int_0^t ds \Big)^{\frac{1}{q}}, \quad \text{where } \frac{1}{p}+\frac{1}{q} = 1.$$

Now since $x \in S$ implies $|(Vx)(t)|^p \leq B(t)$ a.e. on $[0,T)$, by using the inequality

$$(|a|+|b|)^p \leq 2^{p-1}(|a|^p+|b|^p), \quad \text{for } p \geq 1,$$

we have for $t > 0$:

$$|(V_1x)(t)|^p \leq \Big\{ |x^0| + \Big(\int_0^t |(Vx)(s)|^p ds \Big)^{\frac{1}{p}} \Big(\int_0^t ds \Big)^{\frac{1}{q}} \Big\}^p$$

$$\leq 2^{p-1} \Big\{ |x^0|^p + (t)^{\frac{p}{q}} \int_0^t B(s)\, ds \Big\}$$

$$\leq 2^{p-1}(\ A(0) + A(t) - A(0)\ ) = 2^{p-1}\ A(t). \tag{5}$$

Integrating both sides of (5) from 0 to t, we obtain

$$\int_0^t |(V_1x)(s)|^p ds \leq 2^{p-1} \int_0^t A(s)ds = K \int_0^t A(s)ds, \quad \text{where } K = 2^{p-1} > 0.$$

By Lemma 5, the existence of a solution of the inequality (4) insures that $V_1S \subseteq S$, for any $x \in S$.

Obviously, V_1S is uniformly bounded in $L^p_{loc}([0, T), R^n)$, $p>1$. In order to prove that V_1S is relatively compact in $L^p_{loc}([0, T), R^n)$, we need to check that for any $x(t) \in S$,

$$\int_0^t |(V_1x)(s+h) - (V_1x)(s)|^p ds \leq \int_0^t \left| \int_0^{s+h} (Vx)(u)du - \int_0^s (Vx)(u)du \right|^p ds$$

$$= \int_0^t \left| \int_s^{s+h} (Vx)(u)du \right|^p ds$$

$$\leq \int_0^t \left| \left(\int_s^{s+h} |(Vx)(u)|^p du \right)^{\frac{1}{p}} \cdot \left(\int_s^{s+h} du \right)^{\frac{1}{q}} \right|^p ds$$

$$\leq (h)^{\frac{p}{q}} \cdot \int_0^t \left(\int_s^{s+h} B(u)\ du\ \right) ds$$

$$\leq (h)^{\frac{p}{q}} \cdot \int_0^{t_1} \left(\int_0^{t_1} B(u)\ du\ \right) ds = (h)^{\frac{p}{q}} \cdot N, \quad \text{for } t \in [0,t_1],$$

where $N= \int_0^{t_1}\int_0^{t_1} B(u)\ du\ ds < \infty$, and $0 < t_1 < T$, such that the t_1 could be chosen as close to T as possible. Hence

$$\int_0^t |(V_1x)(s+h) - (V_1x)(s)|^p ds \to 0, \text{ as } h \to 0, \text{ uniformly with respect to } x \in S.$$

By the sufficiency part of Theorem 1, (Riesz compactness criterion), the set V_1S is relatively compact in $L^p_{loc}([0, T), R^n)$.

Applying Schauder–Tychonoff fixed point theorem, the existence of a solution of (3) is proven. (For the original idea of the proof, see C. Corduneanu [2] and [3]).

In case $p=1$, we apply Theorem 2 to prove the relative compactness.

Theorem 6 Assume that:

i) V is a Volterra type continuous operator on $L^1_{loc}([0, T), R^n)$ with $0 < T \leq \infty$,

ii) there exist functions A(t) and B(t), with A(t) a positive continuous solution of $\int_0^t A(s)\,ds \leq A(t)$,

and B(t) a locally integrable function on [0,T), such that for any $x \in L^1_{loc}([0, T), R^n)$ with $\int_0^t |x(s)|ds \leq A(t)$ one has $|(Vx)(t)| \leq B(t)$ a.e. on [0,T), and furthermore assume

$$\int_0^t B(s)\,ds \leq A(t)-A(0), \text{ for } t \in [0,T).$$

Then there exists a solution $x \in L^1_{loc}([0, T), R^n)$ of (3), such that

$$\int_0^t |x(s)|\,ds \leq A(t),$$

provided $|x(0)| \leq A(0)$.

Proof. Consider the set

$$S = \left\{ x\,;\ \int_0^t |x(s)|\,ds \leq A(t),\ \ t \in [0,T),\ x \in L^1_{loc}([0, T), R^n) \right\}.$$

Defining

$$(V_1x)(t) = x^0 + \int_0^t (Vx)(s)\,ds,$$

we easily obtain from ii):

$$|(V_1x)(t)| \leq |x^0| + \int_0^t |(Vx)(s)|\,ds \leq |x^0| + \int_0^t B(s)\,ds, \text{ a.e.}$$

$$\leq A(0) + A(t)-A(0) = A(t). \tag{6}$$

Integrating both sides of (6) from 0 to t and applying Lemma 5, we have

$$\int_0^t |(V_1x)(s)|ds \le \int_0^t A(s)ds \le A(t) \text{ , for } t \in [0,T).$$

Thus, $V_1S \subset S$ and V_1S is bounded in $L^1_{loc}([0, T), R^n)$.

Now let us consider for any $x \in S$:

$$\left| \frac{1}{h}\int_t^{t+h} (V_1x)(u)du - (V_1x)(t) \right| = \left| \frac{1}{h}\int_t^{t+h} (V_1x)(u)\, du - \frac{1}{h}\int_t^{t+h} (V_1x)(t)\, du \right|$$

$$\le \frac{1}{h}\int_t^{t+h} |(V_1x)(u) - (V_1x)(t)|\, du$$

$$= \frac{1}{h}\int_t^{t+h} \left| x^0 + \int_0^u (Vx)(s)\, ds - x^0 - \int_0^t (Vx)(t)\, ds \right| du$$

$$\le \frac{1}{h}\int_t^{t+h} \left| \int_t^u |(Vx)(s)|\, ds \right| du$$

$$\le \frac{1}{h}\int_t^{t+h} \left| \int_t^u B(s)\, ds \right| du.$$

Since $B(t)$ is locally integrable, then for any $\varepsilon > 0$, there exists $\delta > 0$, such that $\left| \int_t^u B(s)\, ds \right| < \varepsilon$, provided $|u-t| < \delta$, $0 \le u, t \le t_1 < T$. Now let $|u-t| \le h < \delta$, then

$$\frac{1}{h}\int_t^{t+h} \left| \int_t^u B(s)\, ds \right| du < \frac{1}{h}\int_t^{t+h} \varepsilon\, du = \varepsilon.$$

Thus, $\frac{1}{h}\int_t^{t+h} (V_1x)(s)\, ds \to (V_1x)(t)$, as $h \to 0$, uniformly with respect to $x \in S$.

By A.N.Kolmogorov criterion, the set $V_1S \subseteq S \subseteq L^1_{loc}([0, T), R^n)$ is relatively compact in $L^1_{loc}([0,T), R^n)$. The Schauder–Tychonoff fixed point theorem insures the existence of a solution $x(t)$ of equation (3).

As we mentioned before, the spaces E and F should not necessarily be the same. The following theorem deals with the operator V acting from the absolutely continuous function space $AC_{loc}([0,T), R^n)$ to a space of measurable (locally integrable) functions.

The space $AC_{loc}([0,T),R^n)$ is a subspace of $C([0,T), R^n)$; each element $a \in AC_{loc}([0,T), R^n)$ has a derivative almost everywhere which is locally integrable. The indefinite integral of a locally integrable function is an absolutely continuous function (for more details, see, for example, H.L. Royden [1], A.E. Taylor[1]). A convenient family of seminorm on $AC_{loc}([0,T), R^n)$ could be defined as:

$$|x(t)|^m_{AC} = \left(|x^0| + \int_0^{t_m} |\dot{x}(s)|\, ds \right), \text{ where } m \text{ is an arbitrary positive integer and } t_m \uparrow T.$$

Theorem 7 Assume that

i) V is a Volterra type continuous operator from $AC_{loc}([0,T), R^n)$ into $L^1_{loc}([0,T), R^n)$.

ii) V is compact.

iii) there exist functions A(t) and B(t), such that A(t) is positive and continuous, B(t) is locally integrable, nonnegative, with the property that $x \in AC_{loc}([0,T), R^n)$ and

$$|x^0| + \int_0^t |\dot{x}(s)|\, ds \leq A(t)$$

implies $|(Vx)(t)| \leq B(t)$ a.e. on $[0,T)$, where $\int_0^t B(s)\, ds \leq A(t) - A(0)$, for $t \in [0,T)$.

Then equation (3) has a solution $x(t) \in AC_{loc}([0,T), R^n)$ such that

$$|x^0| + \int_0^t |(Vx)(s)|\, ds \leq A(t),\ t \in [0,T)$$

holds true, as soon as $|x^0| \leq A(0)$.

Proof. Define a convex, closed subset in $AC_{loc}([0,T), R^n)$ as:

$$S = \left\{ x;\ \ x \in AC_{loc}(\ [0,T),\ R^n),\ |\, x^0\, |\ +\ \int_0^t |\, \dot{x}(s)\, |\, ds \le A(t),\ \ t \in [0,T) \right\}.$$

The closedness of S follows easily from the definition of convergence in $AC_{loc}(\ [0,T),\ R^n)$.

Let us consider now the operator

$$(V_1x)(t) = x^0 + \int_0^t (Vx)(s)ds,$$

which is acting on $AC_{loc}(\ [0,T),\ R^n)$. Then consider the set

$$V_1S = \left\{ y(t):\ y(t) = (V_1x)(t),\ \text{for } x \in S \right\},$$

obviously

$$y(0) = x^0 \text{ and } \dot{y}(t) = (Vx)(t).$$

Hence for any $x \in S$, taking into account iii) , we have the following estimate:

$$|\, y(0)\, | + \int_0^t |\, \dot{y}(s)\, |\, ds = |\, x^0 | + \int_0^t |(Vx)(s)|\ ds\ \le\ A(0) + \int_0^t B(s)\ ds,\ t \in [0,T),$$

$$\le A(0) + A(t) - A(0)\ = A(t), \tag{7}$$

provided $|x^0| \le A(0)$.

Thus, $V_1S \subseteq S \subseteq AC_{loc}(\ [0,T),\ R^n)$.

We need to prove now that the set $V_1S \subset AC_{loc}(\ [0,T),\ R^n)$ is relatively compact (thus, its closure $\overline{V_1S} \subset S$ is compact in $AC_{loc}(\ [0,T),\ R^n)$). This means that, for any $\bar{t}$, $0 < \bar{t} < T$, the family of functions in $AC_{loc}(\ [0,\ \bar{t}\ ],\ R^n)$, consisting of restrictions to $[0,\ \bar{t}\ ]$ of all functions in V_1S, must be relatively compact. This is easily seen because the set of their derivatives is relatively compact in $L^1([0,\ \bar{t}\ ],\ R^n)$, according to the property ii) in the statement of Theorem 7.

Finally, Schauder–Tychonoff fixed point theorem applied to S and V_1 gives the desired solution of equation (3), that is the solution $x(t) \in AC_{loc}(\ [0,T),\ R^n)$ such that

$$|\, y(0)\, | + \int_0^t |\, \dot{y}(s)\, |\, ds = |\, x^0\, |\ +\ \int_0^t |\, (Vx)(s)\, |\, ds \le A(t),\ t \in [0,T)$$

holds true, as soon as $|x^0| \leq A(0)$.

This completes the proof of Theorem 7.

In order to prove the uniqueness of the solution we need to make more assumptions on the operator V. Probably, the most natural way is to assume that V satisfies a Lipschitz condition.

We use a weighted norm, Lipschitz condition and contraction mapping to obtain a unique solution of equation (3) in next theorem.

Let $C_g([0,T),R^n)$ be the Banach space of all continuous mapping from [0,T) in to R^n, such that

$$|x|_g = \sup_{0 \leq t < \infty} \left(|x(t)| \Big/ g(t) \right) < \infty,$$

where

$g : [0,T) \to (0,\infty)$ is a continuous map.

(More details about the space $C_g([0,T),R^n)$ can be found in C. Corduneanu [1] and [2].)

Theorem 8 Assume that

i) V is an abstract Volterra type operator on $C([0,T),R^n)$;

ii) V satisfies the Lipschitz condition of the form

$$|(Vx)(t)-(Vy)(t)| \leq L(t) \cdot \sup_{0 \leq s \leq t} |x(s)-y(s)| , \; t \in [0,T),$$

for any x, $y \in C([0,T),R^n)$, where L(t) is a nonnegative locally integrable function;

iii) $(V\theta)(t) \in L^1_{loc}([0, T), R^n)$.

Let

$$g(t) = \exp\left(\frac{1}{\alpha} \int_0^t L(s) \cdot ds \right), \text{ with } 0 < \alpha < 1.$$

Then there exists a unique solution $x \in C_g([0,T),R^n)$ of the equation (3).

Proof.

First we want to point out that the condition ii) implies the continuity of the operator V.

Now let us define a new Volterra operator:

$$(V_1x)(t) = x^0 + \int_0^t (Vx)(s)\, ds.$$

Consider ii) and iii), for any $x(t) \in C_g([0,T),R^n)$, we obtain that

$$| (V_1x)(t) | \leq | x^0 | + \int_0^t | (Vx)(s) | \, ds$$

$$\leq | x^0 | + \int_0^t | (Vx)(s) - (V\theta)(s) | \, ds + \int_0^t | (V\theta)(s) | \, ds$$

$$\leq | x^0 | + \int_0^t L(s) \cdot \sup_{0 \leq u \leq s} |x(u) - \theta| \, ds + M, \quad \text{where } M = \int_0^\infty | (V\theta)(s) | \, ds$$

$$\leq N + \int_0^t L(s) \cdot \sup_{0 \leq u \leq s} |x(u) \cdot \exp(-\tfrac{1}{\alpha} \int_0^u L(v)\, dv) \cdot \exp(\tfrac{1}{\alpha} \int_0^u L(v)\, dv)| \, ds, \quad \text{where } N = |x^o| + M,$$

$$\leq N + \int_0^t L(s) \cdot \sup_{0 \leq u \leq s} |x(u) \cdot \exp(-\tfrac{1}{\alpha} \int_0^u L(v)\, dv)| \cdot \sup_{0 \leq u \leq s} | \exp(\tfrac{1}{\alpha} \int_0^u L(v)\, dv)| \, ds$$

$$\leq N + |x(t)|_g \int_0^t L(s) \cdot \exp(\tfrac{1}{\alpha} \int_0^s L(v)\, dv)\, ds$$

$$= N + (|x(t)|_g \cdot \alpha \Big(\exp(\tfrac{1}{\alpha} \int_0^t L(v)\, dv) - 1 \Big)$$

$$\leq N + |x(t)|_g \cdot \alpha \Big(\exp(\tfrac{1}{\alpha} \int_0^t L(v)\, dv \Big).$$

Thus, $| (V_1x)(t) | \cdot \exp(-\frac{1}{\alpha} \int_0^t L(v)\, dv) \leq N \cdot \exp(-\frac{1}{\alpha} \int_0^t L(v)\, dv) + \alpha \, |x(t)|_g \leq N + \alpha \, |x(t)|_g < \infty,$

which means that $(V_1x)(t) \in C_g([0,T),R^n)$, for any $x(t) \in C_g([0,T),R^n)$.

If x(t), y(t) $\in C_g([0,T),R^n)$, then

$$| (V_1x)(t) - (V_1y)(t) | \leq \int_0^t | (Vx)(s)-(Vy)(s) | \, ds$$

$$\leq \int_0^t L(s) \cdot \sup_{0 \leq u \leq s} |x(u)-y(u)| \, ds$$

$$\leq \int_0^t L(s) \cdot \sup_{0 \leq u \leq s} | \, |x(u)-y(u)| \cdot \exp(-\tfrac{1}{\alpha} \int_0^u L(v) \, dv)| \cdot \sup_{0 \leq u \leq s} | \exp(\tfrac{1}{\alpha} \int_0^u L(v) \, dv)| \, ds$$

$$\leq |x(t)-y(t)|_g \cdot \int_0^t L(s) \cdot \exp(\tfrac{1}{\alpha} \int_0^s L(v) \, dv) \, ds$$

$$= |x(t)-y(t)|_g \cdot \alpha \left(\exp(\tfrac{1}{\alpha} \int_0^t L(v) \, dv) - 1 \right)$$

$$\leq |x(t)-y(t)|_g \cdot \alpha \left(\exp(\tfrac{1}{\alpha} \int_0^t L(v) \, dv \right). \tag{8}$$

From (8), we obtain immediately that

$$| (V_1x)(t) - (V_1y)(t) |_g \leq \alpha \cdot |x(t)-y(t)|_g \text{ with } 0 < \alpha < 1.$$

Thus, V_1 is a contraction mapping in the Banach space $C_g([0,T),R^n)$. Hence there exists a unique solution in this space, such that

$$(V_1x)(t) = x(t) = x^0 + \int_0^t (Vx)(s) \, ds.$$

Therefore the proof is complete.

Note: During the proof, we use the fact that $\sup | a \cdot b | \leq \sup | a | \cdot \sup | b |$.

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