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GALE'S FEASIBILITY THEOREM ON NETWORK FLOWS AND A BARGAINING SET FOR COOPERATIVE TU GAMES

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The bargaining set theory of cooperative TU games has been initiated by R. J. Aumann and M. Maschler in [2]; a good account of the state-of-the-art in the field has been given by M. Maschler in [8]. A new concept of bargaining set for TU games will be introduced in the present paper and a combinatorial characterization of the elements belonging to this set will be given, based upon a result due to D. Gale ([6]), applied to a network associated with the bargaining procedure. The case of three person games is fully discussed, to allow the comparison with Aumann/Maschler results given in [2].

1. The bargaining model.

Consider a cooperative TU game in coalitional form, that is a pair $G=(N,v)$, where N is a finite set of players $|N|=n$ and $v:2^N\rightarrow R$ is the so-called characteristic function, subject to $v(\phi)=0$. For any coalition S, $S\subseteq N$, $S\neq\phi$, the number $v(S)$ is the worth of S. A coalition structure is any partition of N; let

$$F_{\wp}=\{x:x\in R^n:\ x(S)=v(S),\ \forall S\in\wp\} \tag{1.1}$$

be the set of admissible payoffs for the coalition structure $\wp$, where $x(S)$ is the sum of x_i, $\forall i\in S$. Denote by F the set of all admissible payoffs which is the union of all $F_{\wp}$ for all coalition structures $\wp$. The core of G is defined by means of the excess function $e(S,x)=v(S)-x(S)$, $\forall S\subseteq N$, $x\in F$, as

$$C(G)=\{x\in F:\ e(S,x)\leq 0,\ \forall S\subseteq N\}, \tag{1.2}$$

(see Aumann/Dreze, [1]). For some coalition structure $\wp$, an $x\in F_{\wp}$ which belongs to the core is considered stable, because for any coalition S we have $x(S)\geq v(S)$, i.e., in x, the total win of players in S is at least equal to the worth of S. However, if $x \notin C(G)$ then it may happen that x is still stable depending on the meaning given to the word stable. We shall introduce a stability principle somehow similar to the one due to Aumann/Maschler, but having a strong combinatorial character.

A partial coalition structure of N is a set of pairwise disjoint and non-empty coalitions that may cover N or not.

A bargaining proposal (b.p.) against $x \in F_\wp$, $x \notin C(G)$, is a partial coalition structure $\mathfrak{T}=(T_1,..,T_q)$ subject to

$$e(T_h,x)>0,\ h=1,..,q. \tag{1.3}$$

Let $K_h \subseteq T_h$, $K_h \neq \phi$, $h=1,..,q$, be a family of coalitions having members in all blocks of $\mathfrak{T}$ and K be the union of all K_h. Obviously, we have $T_0 \cap K=\phi$, where $T_0=N-(T_1 \cup .. \cup T_q)$.

A bargaining distribution for $\mathfrak{T}$ initiated by K is any $y \in R^{n-t_0}$, $t_0=|T_0|$, such that

$$y(T_h)=v(T_h),\ h=1,..,q, \qquad y_i>x_i, \forall i \in K, \qquad y_i \geq x_i,\ \forall \in N-T_0. \tag{1.4}$$

Now, the meaning of the concept of bargaining proposal is shown by:

Lemma 1.1: For a partial coalition structure $\mathfrak{T}=(T_1,..,T_q)$, there exists a bargaining distribution initiated by some coalition K with $T_h \cap K \neq \phi$ $h=1,..,q$, if and only if $\mathfrak{T}$ is a bargaining proposal.

Proof: Obviously, (1.3) follows from (1.4), using the definition of the excess function. Conversely, if (1.3) hold, then we can define $y_i=x_i+e(T_h,\ x)/|K_h|, \forall i \in K_h$, $h=1,..,q$, and $y_i=x_i$ otherwise, and (1.4) will hold.

In words, a coalition K could find a better payoff than x if and only if there is a partial coalition structure where K has members in all blocks and the excesses of these blocks are positive. In this case, K may initiate a bargaining proposal, namely that partial coalition structure.

Note that the pair $(y;\mathfrak{T})$ is similar to an Aumann/Maschler objection; the differences are: no rationality conditions are imposed to x in $F_\wp$, then K may not be a subset of a block in $\mathcal{P}$, and $(y;\mathfrak{S})$ is not an objection against another group of players but it is an objection against $(x;\mathcal{P})$.

Note also that for a bargaining proposal $\mathfrak{S}$, there are a lot of bargaining distributions initiated by a given K, namely they are solutions of the linear system (1.4) or equivalently

$$\alpha(T_h)=e(T_h,x),\ h=1,..,q,\qquad \alpha_i>0,\ \forall i\in K,\qquad \alpha_i\geq 0,\forall i\in N-T_0 \tag{1.5}$$

connected to (1.4) by $\alpha_i=y_i-x_i,\forall i\in N-T_0$.

Bargaining proposals and bargaining distributions have been considered in earlier papers of the author ([4], [5]); these terms have been used for avoiding any confusion with the Aumann/Maschler objections.

Now, if $x\in F_{\mathcal{P}}$ does not belong to the core, then obviously there are bargaining proposals because at least one excess is positive. We introduce a new countering procedure which will allow us to define a new stability principle, even though there are coalitions able to improve their payoffs relative to x. The new assumption of our bargaining model is that before announcing their bargaining proposal the players in $N-T_0$ agree upon the following commitment: a group of players $\bar{K}\subseteq N-T_0$ will be able to leave $\mathfrak{S}$ and join coalitions of another partial coalition structure only if $\bar{K}$ contains players of all blocks of $\mathfrak{S}$, and for each T_h the players in $\bar{K}\cap T_h$ will compensate T_h with $e(T_h,x)$ from their extra wins in their new coalitions. If a partial coalition structure would offer to $\bar{K}$ such extra wins, perhaps more, then we shall consider that can counter $\mathfrak{S}$.

Consider a partial coalition structure $=(U_1,..,U_p)$, different of $\mathfrak{S}$, such that

$$(N-U_0)\cap T_h\neq\phi,\ h=1,..,q,\qquad U_0=N-(U_1\cup..\cup U_p) \tag{1.6}$$

and

$$e(U_j,x)>0,\ j=1,..,p. \tag{1.7}$$

Note that " different of $\mathfrak{S}$" means that no block in is a block in $\mathfrak{S}$. Obviously, a family of coalitions $\bar{K}_h\subseteq(N-U_0)\cap T_h$, $\bar{K}_h\neq\phi$, $h=1,..,q$, whose union is $\bar{K}$, may be willing to move to , if each group $\tilde{K}_h$ is able to satisfy the commitment of compensating T_h with $e(T_h,x)$. The bargaining model becomes clear by the following definitions:

A compensatory bargaining counter distribution (c.b.c.d.) for a partial coalition structure , subject to (1.6) and (1.7), against $\mathfrak{S}$, is any $z \in R^{n-u_0}$, $u_0 = |U_0|$, satisfying

$$\beta(U_j) = e(U_j,x),\ j=1,\ldots,p, \qquad \beta_i \geq 0, \forall i \in N-U_0, \tag{1.8}$$

where $\beta_i = z_i - x_i$, $\forall i \in N-U_0$, and

$$\sum \beta_i \geq e(T_h,x),\ h=1,\ldots,q, \tag{1.9}$$

where the sum is extended to all $i \in (N-U_0) \cap T_h$.

Clearly, (1.8) are conditions imposed to a payoff and (1.9) are saying that the former members of T_h could compensate all members of T_h with $e(T_h,x)$ from their extra wins in .

Therefore, if there is such a $z \in R^{n-u_0}$, then can counter $\mathfrak{S}$.

A compensatory bargaining counter proposal (c.b.c.p.) is any partial coalition structure subject to (1.6) and (1.7) for which there exists a compensatory bargaining counter distribution against $\mathfrak{S}$.

We shall be able in the second section to characterize the compensatory bargaining counter proposals, i.e., those subject to (1.6), (1.7) for which (1.8), (1.9) is consistent. This will be possible by using a feasibility theorem for network flows due to D. Gale. Now, we can introduce the stability principle:

An admissible payoff x for a coalition structure $\mathfrak{S}$ is compensatory stable (c-stable) if either $x \in C(G)$, or $x \notin C(G)$ and for each b.p. there is a c.b.c.p. against the b.p.

The compensatory bargaining set M_c of G is the set of c-stable payoffs.

2. Gale's feasibility theorem and the compensatory bargaining counter proposals.

Consider a bipartite graph $\Gamma = (U,T,E)$, where $U \cup T$ is the set of vertices, $U \cap T = \phi$, and E is the set of edges; if $U = (u_1,\ldots,u_p)$ and $T = (t_1,\ldots,t_q)$, then E contains only edges (u_j,t_h), $u_j \in U$, $t_h \in T$. Now, build a network $\mathfrak{R}(U,T)$ as follows: to Γ add all edges (u_j, t_h)

which were not in E, define a capacity function by $c(u_j, t_h)=\infty$ if $(u_j, t_h)\in E$ and $c(u_j, t_h)=0$ otherwise; a supply function is defined on the set of vertices, namely $\sigma(u_j)>0$ is the supply available at u_j, $j=1,..,p$, and $\sigma(t_h)=-\delta(t_h)<0$ is showing the demand $\delta(t_h)$ *at* t_h, $h=1,..,q$.

A flow $f=(f_{jh})$ in $\mathfrak{R}(U,T)$ is a function defined on the set of edges in the network and satisfying

$$\sum_h f_{jh}\le\sigma(u_j),\ j=1,..,p, \qquad \sum_j f_{jh}\ge\delta(t_h),\ h=1,..,q, \tag{2.1}$$

$$0\le f_{jh}\le c(u_j, t_h),\ j=1,..,p,\ h=1,..,q,$$

where f_{jh} is the flow on edge (u_j, t_h).

It is used to say that a network with demands and supplies is feasible if there is a flow, i.e., the linear system (2.1) is consistent (see [6]). A feasibility theorem for such networks has been given by D. Gale in [7], for more general networks than our above network. If Gale's theorem is applied to our network $\mathfrak{R}(U,T)$, we get:

Theorem 2.1: The network $\mathfrak{R}(U,T)$ is feasible if and only if for each subset $X\subset U$ (including the empty set), we have

$$\sum_{u_j\notin X}\sigma(u_j)\ge\sum_{t_h\notin I(X)}\delta(t_h) \tag{2.2}$$

where $I(X)$ is the set of vertices in T adjacent to vertices in X in the original graph Γ.

Note that the network considered by Gale has no edges with infinite capacities, so that the above statement may not be considered as a valid application of Gale's theorem (see [6], Th. 1, pg. 38). In this case, a proof similar to Gale's proof could be given, namely: we should impose the condition that in the extended network the capacity of every cut which contains no edge of infinite capacity is at least equal to the capacity of the cut comprising all exit edges; by examining all cuts and eliminating the redundant conditions we end up with (2.2). Note also that algorithmically we should not check (2.2) to determine the feasibility, but we should rather find a maximum flow in the extended network and see whether all exit edges are saturated.

Let $\mathfrak{T}=(T_1,..,T_q)$ be a bargaining proposal against a payoff $x\in F_{\wp}$ which does not belong to the core; let $=(U_1,.., U_p)$ be a partial coalition structure subject to

$$(N-U_0)\cap T_h\neq\phi,\ h=1,\dots,q, \qquad (N-T_0)\cap U_j\neq\phi,\ j=1,\dots,p, \tag{2.3}$$

and

$$e(U_j,x)>0,\ j=1,\dots,p. \tag{2.4}$$

Note that beside (1.6) and (1.7) we have imposed to   the second group of conditions (2.3); we shall show later that we can confine ourselves to such partial coalition structures without any loss of generality.

For our pair of partial coalition structures (, $\mathfrak{T}$) we can associate a graph $\Gamma=(U,T,E)$ as follows: each block $U_j\in$ will be represented by a vertex $u_j\in U$ and each block $T_h\in\mathfrak{T}$ will be represented by a vertex $t_h\in T$; we take an edge $(u_j,\ t_h)$ in E if $U_j\cap T_h\neq\phi$. Note that the first conditions (2.3) are saying that each vertex t_h is connected to at least one vertex in   and the second group of conditions (2.3) are saying that each vertex u_j is connected to at least one vertex in T. Now, we build a network $\mathfrak{R}(U,T)$, by adding first edges as shown above and defining similarly their capacities and by taking: $\sigma(u_j)=e(U_j,x)$, $j=1,\dots,p$, and $\delta(t_h)=e(T_h,x)$, $h=1,\dots,q$. Now, in this network we have the result:

Theorem 2.2: Let $\mathfrak{T}$ be a b.p. and   be a partial coalition structure subject to (2.3) and (2.4); then   is a c.b.c.p. against $\mathfrak{T}$ if and only if the network $\mathfrak{R}(U,T)$ associated with (,$\mathfrak{T}$) is feasible.

Proof: Suppose that   is a c.b.c.p. against $\mathfrak{T}$, that is (1.8), (1.9) is consistent; let β_i, $\forall i\in N-U_0$, be a solution of this linear system. Define $f_{jh}=0$, if $U_j\cap T_h=\phi$; define $f_{jh}=\sum\beta_i$, where the sum extends to all $i\in U_j\cap T_h$, if $U_j\cap T_h\neq\phi$. From this definition and (1.8), (1.9) we get

$$\sum_h f_{jh}=\sum_{i\in(N-T_0)\cap U_j}\beta_i\leq\sum_{i\in U_j}\beta_i=e(U_j,x),\ j=1,\dots,p,$$

$$\sum_j f_{jh}=\sum_{i\in(N-U_0)\cap T_h}\beta_i\geq e(T_h,x),\ h=1,\dots,q,$$

and if $c_{jh}=0$, i.e., $U_j\cap T_h=\phi$, we have $f_{jh}=c_{jh}=0$, where $c_{jh}=c(u_j,t_h)$. Hence f defined above satisfies (2.1), i.e., the network is feasible. Conversely, suppose that the network is feasible, i.e., (2.1) is consistent; let f_{jh}, $j=1,..,p$ and $h=1,..,q$, be a solution of this linear system. Define β_i, $i\in N-U_0$ as follows:

$$\beta_i=0,\quad if\quad i\in N-U_0,\ i\notin N-T_0, \tag{2.5}$$

$$\beta_i=f_{jh}/|U_j\cap T_h|+d_j/|(N-T_0)\cap U_j|\quad if\quad i\in U_j\cap T_h\neq\phi,$$

where

$$d_j=e(U_j,x)-\sum_h f_{jh},\ j=1,..,p. \tag{2.6}$$

As all flows and all differences d_j, $j=1,..,p$, are nonnegative by (2.1) with $\sigma(u_j)=e(U_j,x)$, $j=1,..,p$, and $\delta(t_h)=e(T_h,x)$, $h=1,..,q$, we have $\beta_i\geq 0$, $\forall i\in N-U_0$. For $U_j\cap T_h\neq\phi$ we get from (2.5) and (2.6) that for every U_j we have:

$$\sum_{i\in U_j}\beta_i=\sum_{h/U_j\cap T_h\neq\phi}\ \sum_{i\in U_j\cap T_h}\beta_i=\sum_h f_{jh}+d_j=e(U_j,x),\ j=1,..,p. \tag{2.7}$$

On the other hand, we get from (2.5):

$$\sum_{i\in U_j\cap T_h}\beta_i\geq f_{jh}; \tag{2.8}$$

as $\beta_i=0$ if $i\in N-U_0$, $i\notin N-T_0$, from (2.1) and (2.8) we get for each T_h:

$$\sum_{i\in(N-U_0)\cap T_h}\beta_i\geq\sum_{j/U_j\cap T_h\neq\phi}f_{jh}=\sum_j f_{jh}\geq e(T_h,x),\ h=1,..,q, \tag{2.9}$$

because $f_{jh}=0$ when $U_j\cap T_h=\phi$. Now, (2.7) and (2.9) show that the numbers β_i, $\forall i\in N-U_0$, defined by (2.5), satisfy (1.8) and (1.9), hence is a c.b.c.p. against $\mathfrak{S}$.

From Theorems 2.1 and 2.2 follows:

Theorem 2.3: Let $\mathfrak{S}$ be a b.p. against $x\in F_{\mathcal{P}}$, $x\notin C(G)$,, and be a partial coalition structure subject to (2.3) and (2.4); then is a c.b.c.p. against $\mathfrak{S}$, if and only if for each proper subset X of blocks in (including $X=\phi$), we have

$$\sum_{U_j \notin X} e(U_j, x) \geq \sum_{T_h \notin I(X)} e(T_h, x) \tag{2.10}$$

where $I(X)$ is the set of blocks in $\mathfrak{T}$ having common players with the blocks in X.

Note that Theorem 2.3 is a combinatorial characterization of those c.b.c.p. which are subject also to the second group of conditions (2.3). The following Lemma shows that we can confine ourselves to such partial coalition structures when we try to discover whether $\mathfrak{T}$ could be countered.

Lemma 2.4: For a b.p. $\mathfrak{T}$, there is a c.b.c.p. if and only if there is a c.b.c.p. satisfying $(N-T_0)\cap U_j \neq 0$, $j=1,..,p$.

Proof: For a partial coalition structure *, which does not satisfy this condition for all $j=1,..,p$, even though * satisfies the other conditions (2.3) and (2.4), the linear system (1.8) and (1.9), where U_j are replaced by U_j^*, has the property: if $(N-T_0)\cap U_j^* = \phi$ for some j, then the subsystem (1.9) does not contain unknowns β_i with $i \in U_j^*$, while that part of (1.8) which contains these unknowns is consistent and does not contain other variables. Therefore, those blocks of * which have nonempty intersection with $N-T_0$ form a c.b.c.p. whenever * forms a c.b.c.p.

Note that in checking whether a partial coalition structure subject to (2.3) and (2.4) is a c.b.c.p. against a given b.p. $\mathfrak{T}$, we may not check whether (2.10) holds; instead we may prefer to find a maximum flow in the extended network $\mathfrak{R}(U,T)$ and check whether this flow saturates the exit arcs. The proof of Theorem 2.1 shows that these are two alternative methods. Anyway, in general, we shall meet huge problems; however, better alternatives are available for important particular cases, as we shall show in the next section, based upon the above results.

3. A characterization for the compensatory bargaining set.

Recall from the first section that for an admissible payoff $x \in F_{\wp}$, where $\wp$ is a coalition structure, we may have a c-stable x if $x \in C(G)$, or if $x \notin C(G)$ but for every b.p. against x there is a c.b.c.p. against the b.p.; all c-stable payoffs form the compensatory bargaining set M_c. While definition (2.1) is characterizing nicely the elements of M_c belonging to the core, a characterization of the c-stable payoffs which do not belong to the core is now possible due to Theorem 2.3 of the second section. We get as a corollary:

Theorem 3.1: An admissible payoff vector $x \in F_\wp$, $x \notin C(G)$, is an element of the bargaining set M_c, if and only if for each partial coalition structure $\mathfrak{T}=(T_1,..,T_q)$ subject to $e(T_h,x)>0$, $h=1,..,q$, there is a partial coalition structure $=(U_1,..,U_p)$ subject to $T_h \cap (\cup U_j) \neq \phi$, $h=1,..,q$, and $U_j \cap (\cup T_h) \neq \phi$, $j=1,..,p$ and $e(U_j,x)>0$, $j=1,..,p$, such that for each proper subset X of blocks in , (including $X=\phi$), we have

$$\sum_{U_j \notin X} e(U_j,x) \geq \sum_{T_h \notin I(X)} e(T_h,x),$$

where $I(X)$ is the subset of blocks in $\mathfrak{T}$ having common players with the blocks in X.

Note that from this result we can derive some other results helpful in checking whether a given payoff belongs to M_c. The case of superadditive games will be considered. Recall that $G=(N,v)$ is superadditive, if for any pair of disjoint coalitions S_1 and S_2 we have $v(S_1)+v(S_2) \leq v(S_1 \cup S_2)$. An important subclass of the set of superadditive games is that of convex games; recall that $G=(N,v)$ is convex, if for any pair of coalitions T_1 and T_2 we have $v(T_1)+v(T_2) \leq v(T_1 \cup T_2)+v(T_1 \cap T_2)$. We need the simple result:

Lemma 3.2: If G is a superadditive game, then for any pair of disjoint coalitions S_1 and S_2, we have $e(S_1,x)+e(S_2,x) \leq e(S_1 \cup S_2,x)$, for any $x \in F$; the inequality holds for a finite set of pairwise disjoint coalitions, too.

Proof: The inequality follows from the definition of the excess function and the superadditivity of the game; an induction is needed for more than two coalitions.

From Lemma 3.2 we shall be able to derive a result saying that for superadditive games threats by means of multicoalitional b.p.'s and countering by multicoalitional c.b.c.p.'s are meaningless.

Lemma 3.3: In a superadditive game G, any b.p. $\Im=(T_1,..,T_q)$ with $q>1$ against a payoff vector $x\in F$, $x\notin C(G)$, is countered by $=(U_1)$, where U_1 is the union of all T_h, $h=1,..,q$.

Proof: Obviously, U_1 satisfies (2.3); from Lemma 3.2 we get the inequality $e(U_1,x)\geq\sum_h e(T_h,x)$, in which the right-hand side is positive by (1.3), so that (2.4) holds either. The same inequality shows that (2.10) holds ($X=\phi$ is the only X to consider), hence by Theorem 2.3 our $=(U_1)$ is a c.b.c.p. against $\Im$.

Lemma 3.4: In a superadditive game G, any b.p. $\Im=(T_1)$ which is countered by a c.b.c.p. $=(U_1,..,U_p)$ with $p>1$, is also countered by $^*=(U^*)$, where U^* is the union of all U_j, $j=1,..,p$.

Proof: Obviously, * satisfies (2.3); from Lemma 3.2 we get the inequality $e(U^*,x)\geq\sum_j e(U_j,x)$, in which the right-hand side is positive from (2.4), because U^* satisfies also (2.4). The same inequality together with (2.10) for $X=\phi$, where the right-hand side is $e(T_1,x)$, is giving $e(U^*,x)\geq e(T_1,x)$; therefore, by Theorem 2.3 our * is a c.b.c.p. against $\Im$.

Now, from Lemmas 3.3 and 3.4 we prove easily the following:

Theorem 3.5: In a superadditive game G, an admissible payoff $x\in F$, $x\notin C(G)$, belongs to the compensatory bargaining set $M_c(G)$, if and only if for each coalition T^* with $e(T^*,x)>0$ there is a coalition U^* with $U^*\cap T^*\neq\phi$ and $e(U^*,x)>0$ such that $e(U^*,x)\geq e(T^*,x)$.

Proof: If $x\in M_c(G)$ and T^* is a coalition with $e(T^*,x)>0$, then to the b.p. $\Im^*=(T^*)$ corresponds a c.b.c.p. countering $\Im^*$; then, by Lemma 3.4, $\Im^*$ is also countered by $^*=(U^*)$, where U^* is the union of coalitions in . Conversely, suppose that for some $x\in F$, $x\notin C(G)$, any b.p. made of one coalition is countered by a c.b.c.p. made also of one coalition; consider a b.p. $\Im=(T_1,..,T_q)$. If $q=1$, then $\Im$ can be countered by a c.b.c.p. made of one coalition; if $q>1$, then by Lemma 3.3, $\mathfrak{S}$ can also be countered by $=(U_1)$, where is the union of all T_h, $h=1,..,q$. Hence any b.p. can be countered and x belongs to $M_c(G)$.

Consider now the 3-person case, because in [2] this is a fully discussed case and the reader can compare easier the results given by the two theories; for this reason, we take the game (G):$v(1)=v(2)=v(3)=0$, $v(12)=a$ $v(23)=b$, $v(13)=c$, and $v(123)=d$, where a, b, c, and d are nonnegative numbers, the same game as in [2]. We intend to determine the coalitionally rational noncore elements of the compensatory bargaining set, because the coalitional rationality was a condition imposed in [2]. Recall that for a coalition structure $\wp$, an admissible payoff vector x is coalitionally rational, if we have $x(S)\geq v(S)$ for every $S\subseteq S_k\in\wp$. We shall be based upon the auxiliary result:

<u>Theorem 3.6</u>: For the 3-person game *(G)*, an admissible payoff vector x, $x\notin C(G)$, which is coalitionally rational, belongs to the compensatory bargaining set $M_c(G)$, if and only if $H=\max_S e(S,x)>0$ is reached for at least two coalitions.

<u>Proof</u>: If $x\in F$ is coalitionally rational we should have $x_i\geq 0$, $i=1,2,3$, so that $e(i,x)=-x_i\leq 0$, $i=1,2,3$, hence no singleton can be a coalition in a b.p. or a c.b.c.p. It follows that each b.p. and each c.b.c.p. is made of one coalition only, each one of cardinality at least two. Two coalitions of this size will always have at least one player in common so that to check whether x is c-stable, for each coalition T with $e(T,x)>0$, $|T|\geq 2$, (three such coalitions may exist at most), we should check whether there is a coalition U with $e(U,x)>0$, $|U|\geq 2$, (two such coalitions may exist at most), such that $e(U,x)\geq e(T,x)$. If yes, U forms a c.b.c.p. against the b.p. formed by T. From these remarks, the result trivially follows.

If Theorem 3.6 is used for all possible coalition structures $\wp$, we get:

<u>Theorem 3.7</u>: For the 3-person game $v(1)=v(2)=v(3)=0$, $v(12)=a$, $v(23)=b$, $v(13)=c$, and $v(123)=d$, with a, b, c and d nonnegative, we have:

(A) if $\wp=(1,2,3)$, $x=(0,0,0)$, then $x\in M_c-C(G)$ if and only if $H=\max(a,b,c,d)>0$ and at least two of the four numbers equal H;

(B) if ρ consists of one coalition of cardinality two and one singleton and *(G)* satisfies $a+b+c \geq 2d$ and the triangle inequalities $a \leq b+c$, $b \leq a+c$, $c \leq a+b$, then in M_c-$C(G)$ we have:

$(0,\ 1/2(a+b-c),\ 1/2(-a+b+c))$ *for* $\rho=(23,1)$, *if* $b<a+c$,
$(1/2(a-b+c),\ 0,\ 1/2(-a+b+c))$ *for* $\rho=(13,2)$, *if* $c<a+b$,
$(1/2(a-b+c),\ 1/2(a+b-c),\ 0)$ *for* $\rho=(12,3)$, *if* $a<b+c$;

(C) if ρ consists of one coalition of cardinality two and one singleton and *(G)* satisfies $a+b+c<2d$ and the inequalities $a \leq d$, $b \leq d$, $c \leq d$, then in M_c-$C(G)$ we have:

$(o,d-c,b+c-d)$ and/or $(o,a+b-d,d-a)$ for $\rho=(23,1)$ if $b<d$ and $d \leq b+c$, or $d \leq a+b$
$(a+c-d,o,d-a)$ and/or $(d-b,o,b+c-d)$ for $\rho=(13,2)$ if $c<d$ and $d \leq a+c$, or $d \leq b+c$
$(d-b,a+b-d,0)$ and/or $(a+c-d,d-c,0)$ for $\rho=(12,3)$ if $a<d$ and $d \leq a+b$,, or $d \leq a+c$

(D) if $\rho=(123)$, then M_c-$C(G)=\phi$.

Note that the 3-person game *(G)* has c-stable admissible payoffs outside the core in some cases, hence the concept of compensatory bargaining set is meaningful. On the other hand, as Theorem 3.7 shows, there are coalition structures, like $\rho=(1,2,3)$ for the 3-person game, for which no coalitionally rational payoff belongs to M_c; indeed, this is the case if one of the numbers *a, b, c, d* is greater than the other three. The existence problem seems to be very difficult.

For details concerning the 3-person case, see also [3].

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