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ERROR ESTIMATES FOR FINITE ELEMENT APPROXIMATIONS OF SECOND ORDER ELLIPTIC PROBLEMS HAVING INHOMOGENEOUS ESSENTIAL BOUNDARY CONDITIONS

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Abstract. The analysis of a finite element method for problems with inhomogeneous essential boundary conditions is considered. For a model problem on a bounded polygonal domain Ω in $\mathbf{R}^2$ with Lipschitz boundary Γ, an error analysis is being done in the 'limit' case when the given function on Γ has 'minimum' of smoothness necessary for the solution to belong to a certain Sobolev space. Some numerical results are provided.

1. Introduction

Consider the inhomogeneous Dirichlet problem for second order elliptic partial differential equations:

(1.1) $\quad -\Delta u = f \quad \text{in} \quad \Omega$

(1.2) $\quad u = g \quad \text{on} \quad \Gamma$

where Ω is a bounded domain in $\mathbf{R}^2$ or $\mathbf{R}^3$ with boundary Γ.

One possible method to treat the inhomogeneity has been presented in [2],[4] and [6], where g is replaced by an approximation g^h. In [4], the analysis and implementation has been done for two cases:

1. g^h is an interpolant of g
2. g^h is a suitable projection of g.

Starting from the standard Galerkin weak formulation of (1.1)-(1.2), the associated approximated problem has been defined on the finite dimensional spaces $S^h \subset H^r(\Omega)$ (r is an integer, $r \geq 1$) and S^h_Γ is the restriction of S^h to the boundary, assuming that S^h possesses an appropriate approximation property and that Ω is a polygonal or polyhedral domain.

Optimal error estimates have been given [4] when the given functions f and g have the smoothness required for the solution u to be in $H^r(\Omega)$, but g is 'slightly' smoother than necessary: $g \in H^{r+\epsilon}(\Gamma)$, $\epsilon > 0$.

The purpose of this paper is to show that one can obtain the same error estimates even in the 'limit' case, when g possesses 'minimum' of smoothness necessary for u to belong to $H^r(\Omega)$.

2. Preliminaries

For the given problem, the standard Galerkin weak formulation is:
Given $g \in H^{1/2}(\Gamma)$ *and* $f \in H^{-1}(\Omega)$, *seek for a function* $u \in H^1(\Omega)$ *such that* $u|_\Gamma = g$ *and*

$$\int_\Omega \nabla u \nabla v \, d\Omega = \int_\Omega f v \, d\Omega \qquad \text{any } v \in H_0^1(\Omega) \tag{2.1}$$

To formulate the approximate problem, let $S^h \subset H^1(\Omega)$ be a finite dimensional vector space, parametrized by a parameter h; let S_Γ^h be the restriction of S^h to the boundary Γ and let $S_0^h = \{v^h \in S^h\colon v^h = 0 \text{ on } \Gamma\}$. For example, let S^h be a space of piecewise linear functions associated with a grid, and suppose Ω is a polygon or a polyhedron. Then, $S^h = span[\varphi_1^h, \ldots, \varphi_n^h, \varphi_{n+1}^h, \ldots, \varphi_{n+m}^h]$, where $\varphi_1^h, \ldots, \varphi_n^h$ are functions which are 1 at the interior nodes and 0 in rest, and $\varphi_{n+1}^h, \ldots, \varphi_{n+m}^h$ are functions which are 1 at the nodes on the boundary and 0 in rest, then $S_\Gamma^h = span[\psi_1^h, \ldots, \psi_m^h]$, where $\psi_j^h = \varphi_{n+j}^h|_\Gamma$ for $j = 1, \ldots, m$, and $S_0^h = span[\varphi_1^h, \ldots, \varphi_n^h]$. Note that $S_0^h \subset H_0^h(\Omega)$.

Let $g^h \in S_\Gamma^h$ be a function which approximates g. The approximated problem is then:

Given $f \in H^{-1}(\Omega)$, *seek for* $u^h \in S^h$ *such that*

$$u^h|_\Gamma = g^h \quad \text{and} \tag{2.2}$$

$$\int_\Omega \nabla u^h \nabla v^h \, d\Omega = \int_\Omega v^h f \, d\Omega \qquad \text{any } v^h \in S_0^h. \tag{2.3}$$

Throughout, C will denote a constant, taking different values in different instances; also, $P_h g$ will denote the $L^2(\Gamma)$-projection of g onto S_Γ^h.

Assume that there exists a $k \geq 1$ such that S^h possesses the following approximation property:

$$\inf_{w^h \in S^h} \| w - w^h \|_s \leq C h^{r-s} \| w \|_r \qquad \text{for } s = 0, 1 \tag{2.4}$$

where r is an integer, $1 \leq r \leq k+1$. For example, S_h consists of piecewise polynomials of degree $\leq k$. Also, assume that the space S_Γ^h possesses the approximation property:

$$\inf_{v^h \in S_\Gamma^h} \| v - v^h \|_{0,\Gamma} \leq C h^{r-1/2} \| v \|_{r-1/2,\Gamma} \tag{2.5}$$

The following results concerning error estimates hold:

Proposition 2.1 ([4], Lemma 2). *Given $g \in H^{r+\epsilon}(\Gamma)$ for any $\epsilon > 0$ and $f \in H^{r-2}(\Omega)$, so that the solution u of (1.1)-(1.2) belongs to $H^r(\Omega)$, assume that $g^h = P_h g$. Then*

$$(2.6) \qquad \inf_{\substack{\hat{u}^h \in S^h \\ \hat{u}^h|_\Gamma = P_h g}} \| u - \hat{u}^h \|_1 \le C h^{r-1} \left\{ \| u \|_r + \| g \|_{r+\epsilon,\Gamma} \right\}$$

As a consequence of Proposition 2.1:

Proposition 2.2 ([4], Corollary 3). *In the same hypotheses as Proposition 2.1,*

$$(2.7) \qquad \| u - u^h \|_s \le C h^{r-s} \left\{ \| u \|_r + \| g \|_{r+\epsilon,\Gamma} \right\} \qquad \text{for } s = 0, 1$$

where u^h is a solution of the approximated problem (2.2)-(2.3).

Some remarks about the $L^2(\Gamma)$- projection of g are needed for the results in the next section:

Proposition 2.3. *If $g, \tilde{g} \in H^{1/2}(\Gamma)$, then*

$$(2.8) \qquad \| P_h g - P_h \tilde{g} \|_{1/2,\Gamma} \le K_h \| g - \tilde{g} \|_{1/2,\Gamma}$$

where K_h is a constant which depends on h.

Proof: Assume that $P_h g = \sum_{j=1}^{m} \lambda_j \psi_j^h$. Let u^h be a solution of (2.2)-(2.3), so u_h must be of the form $u^h = \sum_{j=1}^{n+m} c_j \varphi^h{}_j$, with $\lambda_j = c_{n+j}$ for $j = 1, \ldots, m$. Let $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)^T$. Then $\underline{\lambda}$ is a solution of the equation $A\underline{\lambda} = \underline{b}$, where $A = [a_{ij}]_{1 \le i,j \le m}$, $a_{ij} = \int_\Gamma \psi_i \psi_j \, d\Gamma$ and $\underline{b} = [b_i]_{1 \le i \le m}$, $b_i = \int_\Gamma g \psi_i \, d\Gamma$.

Analogously, if $P_h \tilde{g} = \sum_{j=1}^{m} \tilde{\lambda}_j \psi^h{}_j$, then $\tilde{\underline{\lambda}} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_m)^T$ is a solution of the equation $A\tilde{\underline{\lambda}} = \tilde{\underline{b}}$, where $\tilde{\underline{b}} = [\tilde{b}_i]_{1 \le i \le m}$, $\tilde{b}_i = \int_\Gamma \tilde{g} \psi_i \, d\Gamma$, so $\underline{\lambda} - \tilde{\underline{\lambda}}$ satisfies: $A\left(\underline{\lambda} - \tilde{\underline{\lambda}}\right) = \underline{b} - \tilde{\underline{b}}$.

Note that A is invertible, so $\underline{\lambda}$ and $\tilde{\underline{\lambda}}$ are uniquely determined. Also, if $|.|_1$ is the l^1_m-norm, then:

$$(2.9) \qquad |\underline{\lambda} - \tilde{\underline{\lambda}}|_1 \le K_{1,h} \, |\underline{b} - \tilde{\underline{b}}|_1$$

where $K_{1,h}$ is a constant which depends on m, so therefore on h. Also,

$$(2.10) \qquad |\underline{b} - \tilde{\underline{b}}|_1 = \sum_{i=1}^{m} \left| \int_\Gamma (g - \tilde{g}) \psi_i \, d\Gamma \right| \le K_{2,h} \| g - \tilde{g} \|_{1/2,\Gamma} \qquad \text{and}$$

$$(2.11) \qquad \| P_h g - P_h \tilde{g} \|_{1/2,\Gamma} \le K_{3,h} \; |\underline{\lambda} - \tilde{\underline{\lambda}}|_1$$

where $K_{2,h}$ and $K_{3,h}$ are constants which depend on h.

Combining (2.9), (2.10) and (2.11) yields (2.8).

□

3. Error estimates for the 'limit' case $g \in H^{r-1/2}(\Gamma)$

Theorem 3.1. *Assume that* $g \in H^{r-1/2}(\Gamma)$, $f \in H^{r-2}(\Omega)$ *and* Ω *is a bounded domain with Lipschitz boundary* Γ. *Then*

$$(3.1) \qquad \inf_{\substack{\hat{u}^h \in S^h \\ \hat{u}^h|_\Gamma = P_h g}} \| u - \hat{u}^h \|_1 = O(h^{r-1})$$

Proof: In the above hypotheses, the solution u belongs to $H^r(\Omega)$. According to a density property [1], there exists a sequence of functions $\{u_k\}_{k \ge 1}$, $u_k \in H^r(\Omega) \cap C^\infty(\bar{\Omega})$ for any k, such that $u_k \to u$ in $H^r(\Omega)$, as $k \to \infty$. Then

$$(3.2) \qquad \| u_k \|_r \le M \qquad \text{any } k,$$

where M is a constant which does not depend on k, but on $\| u \|_r$.

Also note that due to an imbedding theorem for bounded domains with Lipschitz boundary ([1]),

$$(3.3) \qquad \| u_k \|_{r+1} \le C_1 \| u_k \|_r$$

For each k, denote $u_k|_\Gamma$ by g_k, $-\Delta u_k$ by f_k, and by $P_h g_k$ the $L^2(\Gamma)$-projection of g_k onto $S^h{}_\Gamma$. Then $g_k \in C^\infty(\Gamma)$ and

$$(3.4) \qquad \| g_k \|_{r+1/2,\Gamma} \le C_2 \| u_k \|_{r+1}$$

It follows from (3.2), (3.3) and (3.4) that

$$(3.5) \qquad \| g_k \|_{r+1/2,\Gamma} + \| u_k \|_r \le C$$

where $C = C_1 C_2 M + 1$ is a constant which does not depend on h and k.

Now note that u_k is a solution of the problem

$$(3.6) \qquad -\Delta u_k = f_k \qquad \text{in } \Omega$$

$$(3.7) \qquad u_k = g_k \qquad \text{on } \Gamma$$

Let k be sufficiently large, such that $\| u - u_k \|_r < C h^{r-1}$, so that

$$(3.8) \qquad \| u - u_k \|_1 < h^{r-1} \qquad \text{and}$$

$(3.9) \qquad K_h \| g - g_k \|_{1/2,\Gamma} < h^{r-1}$ hold, where K_h is the constant that appears in

Proposition 2.3.

According to Proposition 2.1,

$$(3.10)\qquad \inf_{\substack{\hat{u}^h_k \in S^h \\ \hat{u}^h_k|_\Gamma = P_h g_k}} \| u_k - \hat{u}^h{}_k \|_1 \leq C h^{r-1} \Big\{ \| u_k \|_r + \| g_k \|_{r+\epsilon,\Gamma} \Big\} \qquad \text{any } \epsilon > 0.$$

To fix the ideas, let $\epsilon = \frac{1}{2}$. Let $u^h_k \in S^h$ be such that

$$(3.11)\qquad \| u_k - u^h_k \|_1 = \inf_{\substack{\hat{u}^h_k \in S^h \\ \hat{u}^h_k|_\Gamma = P_h g_k}} \| u_k - \hat{u}^h_k \|_1$$

Then, from (3.4), (3.5), (3.10) and (3.11) it follows that:

$$(3.12)\qquad \| u_k - u^h_k \|_1 \leq C h^{r-1}$$

where C is a constant which does not depend on k and h, but on u and r.

Let $\hat{u}^h \in S^h$ be such that $\hat{u}^h|_\Gamma = P_h g$. Then

$$(3.13)\qquad \| u^h_k - \hat{u}^h \|_1 \leq C \| P_h g_k - P_h g \|_{1/2,\Gamma}$$

Applying Proposition 2.3, it follows that:

$$(3.14)\qquad \| u^h_k - \hat{u}^h \|_1 \leq C \| g_k - g \|_{1/2,\Gamma}$$

and since (3.9) is valid,

$$(3.15)\qquad \| u^h_k - \hat{u}^h \|_1 \leq C h^{r-1}$$

holds.

Combining the triangle inequality:

$$(3.16)\qquad \| u - \hat{u}^h \|_1 \leq \| u - u_k \|_1 + \| u_k - u^h_k \|_1 + \| u^h_k - \hat{u}^h \|_1$$

with (3.8), (3.12) and (3.15) and taking the infimum over $\hat{u}^h \in S^h$, $\hat{u}^h|_\Gamma = P_h g$ yield (3.1).

□

Corollary 3.2. *In the same hypotheses as Theorem 3.1,*

$$(3.17)\qquad \| u - u^h \|_s = O(h^{r-s}) \qquad \textit{for } s = 0, 1.$$

The proof is similar to the proof of Corollary 3 [4].

4. Numerical Results

In Table 1 we provide some sample computational results for the following case: $g^h = P_h g$ (the $L^2(\Gamma)$-projection of g onto S^h_Γ); Ω is the unit square in $\mathbf{R}^2$; Ω is divided into squares of side h and each square is further subdivided into two triangles; S^h is the space of piecewise linear polynomials determined within each triangle by their values at the vertices; the exact solution is:

(4.1) $u = r^\alpha sin(\alpha\theta)$ where $r = (x^2 + y^2)^{1/2}$ and $tan\ \theta = \frac{y}{x}$.

The smoothness of u is controlled by varying α. The computational results agree with the estimates (3.17).

Other computations, using bilinear finite elements, provide analogous results.

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Table 1. Computational results for the exact solution (4.1), using linear finite elements

α	No. of intervals	L^2-error	L^2-rate	H^1-error	H^1-rate
2.0	4	.1804(-1)		.2357	
$(2^{2.0} = 4.0)$			4.0000		2.0008
$(2^{1.0} = 2.0)$	8	.4510(-2)		.1178	
			4.0017		1.9993
	16	.1127(-2)		.5892(-1)	
	6	.8018(-2)		.1571	
			4.0009		1.9997
	12	.2004(-2)		.7856(-1)	
.7	4	.3244(-1)		.3013	
$(2^{1.7} = 3.2490)$			3.2440		1.6225
$(2^{0.7} = 1.6245)$	8	.1000(-1)		.1857	
			3.2456		1.6232
	16	.3081(-2)		.1144	
	6	.1630(-1)		.2271	
			3.2450		1.6233
	12	.5023(-2)		.1399	
.5	4	.3312(-1)		.2991	
$(2^{1.5} = 2.8284)$			2.8235		1.4115
$(2^{0.5} = 1.4142)$	8	.1173(-1)		.2119	
			2.8244		1.4136
	16	.4153(-2)		.1499	
	6	.1805(-1)		.2445	
			2.8234		1.4124
	12	.6393(-2)		.1731	
.2	4	.2149(-1)		.1885	
$(2^{1.2} = 2.2973)$			2.2925		1.1471
$(2^{0.2} = 1.1486)$	8	.9374(-2)		.1644	
			2.2964		1.1480
	16	.4082(-2)		.1432	
	6	.1323(-1)		.1740	
			2.2948		1.1477
	12	.5765(-2)		.1516	