

COMPACT SELF ADJOINT OPERATORS AND
EXISTENCE OF SOLUTIONS TO SINGULAR
NONLINEAR BOUNDARY VALUE PROBLEMS
IN A REAL HILBERT SPACE

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Abstract

Properties of Sturm Liouville problems will be used to establish existence of solutions to various singular two point boundary value problems in a real Hilbert space.

1 Introduction and Preliminaries.

This paper presents existence results for the second order differential equation

$$\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t), p(t)y'(t)), \quad 0 < t < 1$$

with $y(t)$ satisfying either the boundary condition

(i). $y(0) = ry'(1) - ky(1) = 0$; $r^2 + k^2 > 0$, $r \geq 0$ and $r \neq kp(1) \int_0^1 \frac{ds}{p(s)}$

or

(ii). $\alpha y'(0) + \beta y(0) = ry'(1) - ky(1) = 0$; $r^2 + k^2 > 0$, $\alpha^2 + \beta^2 > 0$, $\alpha > 0$ and $r > 0$.

Here our solution will lie in a real Banach space B . In addition $f : [0, 1] \times B^2 \rightarrow B$ will be continuous, $q \in C(0, 1)$ with $q > 0$ on $(0, 1)$ and $p \in C[0, 1] \cap C^1(0, 1]$ will satisfy $p > 0$ on $(0, 1]$ if (i) is satisfied whereas we assume $p > 0$ on $[0, 1]$ if (ii) is satisfied.

Differential equations in Banach spaces have been widely studied over the last twenty years, see [1,8,9,10] and their references. Much of the interest in second order boundary value problems has been focused on $B = \mathfrak{R}$ and $p \equiv 1$, see [6,13,14]. However recently many authors [2,3,5,11,15,16] have considered $B = \mathfrak{R}$ and a more general p , namely $p \in C[0, 1] \cap C^1(0, 1)$ together with $p > 0$ on $(0, 1)$, usually with Dirichlet or mixed boundary data i.e. $y(0) = y(1) = 0$ or $y(0) = y'(1) = 0$. In this paper existence results will be obtained for more general boundary conditions and in a more abstract setting. In addition new existence results will be obtained even for the case $B = \mathfrak{R}$ and $p \equiv 1$.

We remark that the technique used in this paper together with the ideas in [13] could be used to examine higher order singular boundary value problems; however we choose to omit the details. It should also be noted that many other boundary conditions, for example,

(iii). $y(1) = ry'(0) + ky(0) = 0$; $r^2 + k^2 > 0$, $r \geq 0$ and $r \neq kp(0) \int_0^1 \frac{ds}{p(s)}$

(iv). $y(0) = \lim_{t \rightarrow 1^-} p(t)y'(t) = 0$

or

$$(v). y(1) = \lim_{t \rightarrow 0^+} p(t)y'(t) = 0$$

could be considered; however since the analysis is verbatim no discussion is necessary. Moreover the results reported here are for homogeneous boundary data, however the nonhomogeneous case could be considered by a change of variables [14].

The following notation will be used throughout this paper. For appropriate functions u define $|u|_0 = \sup_{[0,1]} |u(t)|$, $|u|_1 = \max\{|u|_0, |u'|_0\}$, $|u|_p = \max\{|u|_0, \sup_{[0,1]} |p(t)u'(t)|\}$,

$\|u\|_1 = \int_0^1 p|u| dt$ and $\|u\| = \left\{ \int_0^1 p|u|^2 dt \right\}^{\frac{1}{2}}$. Let $L_p^1[0, 1]$ be the space of measurable functions $u : [0, 1] \rightarrow \mathfrak{R}$ such that $\|u\|_1 < \infty$ and $L_p^2[0, 1]$ be the space of measurable functions $u : [0, 1] \rightarrow \mathfrak{R}$ such that $\|u\| < \infty$.

For any real Banach space B with norm $|\cdot|$, $C([0, 1], B)$ is the Banach space of continuous functions u with norm $|u|_0$ and $C^1([0, 1], B)$ the Banach space of functions u , such that u' is continuous, with norm $|u|_1$. Also let $K^1 = \{u \in C[0, 1] \cap C(0, 1] : |u|_p < \infty\}$ which is a Banach space and $K_B^1 = \{u \in K^1 : u \text{ satisfies the stated boundary condition}\}$. Denote by $L^1([0, 1], B)$ the Banach space of measurable functions $u : [0, 1] \rightarrow B$ which are Bochner integrable (with respect to Lebesgue measure on $[0, 1]$); see [4,19] for properties of the Bochner integral. In particular we state without proof [19]:

Theorem 1.1 *Let $u : [0, 1] \rightarrow B$ be absolutely continuous and assume $u'(t)$ exists almost everywhere and is Bochner integrable. Then*

$$u(t) - u(0) = \int_0^t u'(s) ds.$$

Theorem 1.2 *Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Assume $u, v : [0, 1] \rightarrow B$ are absolutely continuous and that u', v' exist almost everywhere and are integrable. Then*

$$\int_0^1 \langle u, v' \rangle dt = \langle u, v \rangle|_0^1 - \int_0^1 \langle u', v \rangle dt.$$

We also state the following version of the Arzela-Ascoli theorem [10].

Theorem 1.3 *A subset S of $C([0, 1], B)$ is relatively compact iff S is equicontinuous and the set $\{u(t) : u \in S\}$ is relatively compact in B for each $t \in [0, 1]$.*

2 Existence Principles and Sturm Liouville Problems.

This section establishes existence principles for second order equations of the form

$$\frac{1}{p(t)}(p(t)y'(t))' = q(t)f(t, y(t), p(t)y'(t)), \quad 0 < t < 1$$

subject to various boundary data; here solutions will lie in a real Banach space $(B, |\cdot|)$. We begin with the problem

$$(2.1) \quad \begin{cases} \frac{1}{p}(py')' = qf(t, y, py'), & 0 < t < 1 \\ y(0) = ry'(1) - ky(1) = 0; & r^2 + k^2 > 0, \quad r \geq 0 \text{ and } r \neq kp(1) \int_0^1 \frac{ds}{p(s)} \end{cases}$$

with the following conditions satisfied:

$$(2.2) \quad \begin{cases} f : [0, 1] \times B^2 \rightarrow B \text{ is continuous, } q \in C(0, 1) \text{ with } q > 0 \text{ on } (0, 1). \\ \text{Also for each } r > 0 \text{ there exists } \eta_r \in C[0, 1] \text{ such that } |z|, |w| \leq r \\ \text{imply } |f(t, z, w)| \leq \eta_r(t) \text{ for } t \in [0, 1] \end{cases}$$

and

$$(2.3) \quad p \in C[0, 1] \cap C^1(0, 1] \text{ together with } p > 0 \text{ on } (0, 1] \text{ and } \int_0^1 \frac{ds}{p(s)} < \infty.$$

By a solution to (2.1) we mean a function $y \in C([0, 1], B) \cap C^1((0, 1], B) \cap C^2((0, 1), B)$ with $py' \in C([0, 1], B)$ which satisfies the differential equation on $(0, 1)$ and the stated boundary condition in (2.1). Associated with (2.1) we have a family of problems

$$(2.4)_\lambda \quad \begin{cases} \frac{1}{p}(py')' = \lambda qf(t, y, py'), 0 < t < 1 \\ y(0) = ry'(1) - ky(1) = 0 \end{cases}$$

where $0 < \lambda < 1$. A *nonlinear alternative of Leray-Schauder* type [6] will now be used to establish the following existence principle.

Theorem 2.1 *Suppose (2.2) and (2.3) are satisfied. In addition assume*

$$(2.5) \quad \int_0^1 p(t)q(t) dt < \infty$$

and

$$(2.6) \quad \begin{cases} \text{for each bounded set } \Omega \subseteq K_B^1 \text{ and for each } t \in [0, 1], \text{ the set} \\ \{ \int_t^1 p(s)q(s)f(s, y(s), p(s)y'(s)) ds : y \in \Omega \} \\ \text{is relatively compact.} \end{cases}$$

Also suppose there is a constant M , independent of λ , such that $\|y\|_p \leq M$ for each solution y to $(2.4)_\lambda$ for each $\lambda \in (0, 1)$. Then (2.1) has at least one solution in K^1 .

PROOF: Solving $(2.4)_\lambda$ is equivalent to finding a $y \in C([0, 1], B) \cap C^1((0, 1], B)$ with $\sup_{(0,1]} |p(t)y'(t)| < \infty$ which satisfies

$$(2.7) \quad p(t)y'(t) - p(1)y'(1) = -\lambda \int_t^1 p(s)q(s)f(s, y(s), p(s)y'(s)) ds.$$

Define mappings $T : K_B^1 \rightarrow C_1([0, 1], B)$ and $F : K_B^1 \rightarrow C_1([0, 1], B)$ by setting

$$(Ty)(t) = p(t)y'(t) - p(1)y'(1)$$

and

$$(Fy)(t) = - \int_t^1 p(s)q(s)f(s, y(s), p(s)y'(s)) ds.$$

Here $C_1([0, 1], B) = \{u \in C([0, 1], B) : u(1) = 0\}$. Now F is continuous and completely continuous by the *Arzela-Ascoli theorem*. To see this let $\Omega \subseteq K_B^1$ be bounded. The bound-

edness of $F\Omega$ is trivial and the equicontinuity on $[0, 1]$ follows from assumption (2.5) and the following inequality (here $y \in \Omega$ and $0 \leq s \leq t \leq 1$),

$$|(Fy)(t) - (Fy)(s)| \leq \int_s^t p(z)q(z)|f(z, y(z), p(z)y'(z))| dz.$$

The above together with (2.6) and the *Arzela-Ascoli theorem* (Theorem 1.3) implies that F is completely continuous. In addition T^{-1} is continuous. To see this let $g \in C_1([0, 1], B)$ so the equation $(Ty)(t) = g(t)$ has the general solution

$$y(t) = k \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1} \int_0^1 \frac{g(s)}{p(s)} ds \int_0^t \frac{ds}{p(s)} + \int_0^t \frac{g(s)}{p(s)} ds$$

so we conclude that T is both one-to-one and onto and hence invertible. The continuity of T^{-1} is immediate.

Thus $(2.4)_\lambda$ (i.e. (2.7)) is equivalent to the fixed point problem $y = \lambda Ny$ where $N = T^{-1}F$. Notice $N : K_B^1 \rightarrow K_B^1$ is continuous and completely continuous. Set $U = \{u \in K_B^1 : |u|_p < M + 1\}$, $K = K_B^1$ and apply the *Nonlinear Alternative of Leray-Schauder* [6, Theorem 2.3] to deduce that N has a fixed point i.e. (2.1) has a solution y in K^1 . In addition we have $y \in C^2((0, 1), B)$, $py' \in C([0, 1], B)$ and $(py')' \in L^1([0, 1], B)$ from (2.7) with $\lambda = 1$. \square

Remarks. (i). If B is finite dimensional then (2.6) is automatically satisfied.

(ii). Note that (2.6) holds if $pqf : [0, 1] \times B^2 \rightarrow B$ is completely continuous. To see this let $\Omega \subseteq B_B^1$ be bounded. Then there exists a compact set $Q \subseteq B$ such that $p(s)q(s)f(s, y(s), p(s)y'(s)) \subseteq Q$ for all $s \in [0, 1]$ and $y \in \Omega$. Fix $t \in [0, 1)$ and notice

$$\frac{1}{1-t} \int_t^1 p(s)q(s)f(s, y(s), p(s)y'(s)) ds \in \overline{\text{co}}(Q)$$

which is compact [10]; here $\text{co}(Q)$ is the convex hull of Q .

(iii). For Dirichlet boundary conditions (i.e. $r = 0$ in (2.1)) we need only assume $p > 0$ on $(0, 1)$ in (2.3) for Theorem 2.1 to be valid.

An extra theorem can be proven when $f(t, y, py') \equiv f(t, y)$. Let $C_B([0, 1], B) = \{u \in C([0, 1], B) : u(0) = \mathbf{0}\}$

Theorem 2.2 *Let $f(t, y, py') \equiv f(t, y)$ and suppose (2.2) with $f : [0, 1] \times B \rightarrow B$ and (2.3) are satisfied. In addition assume*

$$(2.8) \quad \int_0^1 \frac{1}{p(u)} \int_u^1 p(s)q(s) ds du < \infty$$

and

$$(2.9) \quad \left\{ \begin{array}{l} \text{for each bounded set } \Omega \subseteq C_B([0, 1], B) \text{ and for each } t \in [0, 1], \text{ the set} \\ \left\{ \int_0^t \frac{1}{p(u)} \int_u^1 p(s)q(s)f(s, y(s)) ds du : y \in \Omega \right\} \\ \text{is relatively compact.} \end{array} \right.$$

Also suppose there is a constant M , independent of λ , such that $|y|_0 \leq M$ for each solution y to (2.4) $_\lambda$, with $f(t, y, py') \equiv f(t, y)$, for each $\lambda \in (0, 1)$. Then (2.1), with $f(t, y, py') \equiv f(t, y)$, has at least one solution in $C([0, 1], B)$.

PROOF: Solving (2.4) $_\lambda$ is equivalent to finding a $y \in C([0, 1], B)$ which satisfies

$$\begin{aligned} y(t) &= -\lambda \int_0^t \frac{1}{p(u)} \int_u^1 p(s)q(s)f(s, y(s)) ds du \\ &\quad - k\lambda \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1} \int_0^1 \frac{1}{p(u)} \int_u^1 p(s)q(s)f(s, y(s)) ds du \int_0^t \frac{ds}{p(u)}. \end{aligned}$$

Define the operator $N_\star : C_B([0, 1], B) \rightarrow C_B([0, 1], B)$ by setting

$$\begin{aligned} (N_\star y)(t) &= - \int_0^t \frac{1}{p(u)} \int_u^1 p(s)q(s)f(s, y(s)) ds du \\ &\quad - k \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1} \int_0^1 \frac{1}{p(u)} \int_u^1 p(s)q(s)f(s, y(s)) ds du \int_0^t \frac{ds}{p(u)}. \end{aligned}$$

Then (2.4) $_\lambda$, with $f(t, y, py') = f(t, y)$ is equivalent to the fixed point problem $y = \lambda N_\star y$. Now N_\star is continuous and completely continuous by the *Arzela-Ascoli theorem*. To see this

let $\Omega \subseteq C_B([0, 1], B)$ be bounded. The boundedness of $N_\star \Omega$ is trivial and the equicontinuity on $[0, 1]$ follows from (here $y \in \Omega$ and $0 \leq s \leq t \leq 1$),

$$\begin{aligned} |N_\star y(t) - N_\star y(s)| &\leq \int_s^t \frac{1}{p(u)} \int_u^1 p(z)q(z)|f(z, y(z))| dz du \\ &\quad + |k| \left| \frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right|^{-1} \int_0^1 \frac{1}{p(u)} \int_u^1 p(z)q(z)|f(z, y(z))| dz du \int_s^t \frac{ds}{p(u)}. \end{aligned}$$

This together with (2.9) imply that N_\star is completely continuous. Set $U = \{u \in C_B([0, 1], B) : \|u\|_0 < M + 1\}$, $K = C_B([0, 1], B)$ and proceed as before. \square

We next obtain an existence principle in the spirit of Theorem 2.1 for problems of the form

$$(2.10) \quad \begin{cases} \frac{1}{p}(py')' = qf(t, y, py'), & 0 < t < 1 \\ \alpha y'(0) + \beta y(0) = 0; & \alpha^2 + \beta^2 > 0, \alpha > 0 \\ ry'(1) - ky(1) = 0; & r^2 + k^2 > 0, r > 0 \end{cases}$$

with

$$(2.11) \quad p \in C^1[0, 1] \text{ and } p > 0 \text{ on } [0, 1]$$

and

$$(2.12) \quad \frac{r\beta}{p(1)} + \frac{k\alpha}{p(0)} - k\beta \int_0^1 \frac{ds}{p(s)} \neq 0.$$

Theorem 2.3 Suppose (2.2), (2.5), (2.6), (2.11) and (2.12) are satisfied. In addition suppose there is a constant M , independent of λ , such that $|y|_1 \leq M$ for each solution y to

$$(2.13)_\lambda \quad \begin{cases} \frac{1}{p}(py')' = qf(t, y, py'), & 0 < t < 1 \\ \alpha y'(0) + \beta y(0) = ry'(1) - ky(1) = 0 \end{cases}$$

for each $\lambda \in (0, 1)$. Then (2.10) has at least one solution in $C^1([0, 1], B)$.

PROOF: Solving (2.13) $_\lambda$ is equivalent to finding a $y \in C^1([0, 1], B)$ which satisfies

$$(2.14) \quad Ty = \lambda Fy$$

where T and F are defined in Theorem 2.1. The same reasoning as in Theorem 2.1 shows that F is continuous and completely continuous. To see that T^{-1} is continuous let $g \in C_1([0, 1], B)$ so the equation $(Ty)(t) = g(t)$ has the general solution

$$y(t) = A \int_0^t \frac{ds}{p(s)} + B + \int_0^t \frac{g(s)}{p(s)} ds$$

where

$$A = \frac{k \left(\frac{-\alpha g(0)}{p(0)} + \beta \int_0^1 \frac{g(s)}{p(s)} ds \right)}{\frac{r\beta}{p(1)} + \frac{k\alpha}{p(0)} - k\beta \int_0^1 \frac{ds}{p(s)}}$$

and

$$B = \frac{A}{k} \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right) - \int_0^1 \frac{g(s)}{p(s)} ds.$$

Clearly T^{-1} is continuous and essentially the same reasoning as in theorem 2.1 yields the result. \square

Remark. We have an analogue of theorem 2.2 for problems (2.10) when $f(t, y, py') = f(t, y)$.

The remainder of this section is devoted to the study of the Sturm Liouville boundary value problem

$$(2.15) \quad \begin{cases} Ly = \lambda y, & 0 < t < 1 \\ y(0) = ry'(1) - ky(1) = 0; & r^2 + k^2 > 0, r \geq 0 \text{ and } r > kp(1) \int_0^1 \frac{ds}{p(s)} \end{cases}$$

where $Ly = -\frac{1}{p}(py')'$ and $y : [0, 1] \rightarrow \mathfrak{R}$. Assume (2.3) holds for the remainder of this section.

Remark. (i). We note that $t = 1$ is a regular point but $t = 0$ is a singular point. In fact $t = 0$ is a singular point in the limit circle case [18]. This follows immediately from *Weyl's theorem* [18, Ch.7] since all solutions of $Ly = 0$, $0 < t < 1$ are of the form $y(t) = A \int_0^1 \frac{ds}{p(s)} + B$, where A and B are constants, so they are elements of $L_p^2[0, 1]$ since $\int_0^1 \frac{ds}{p(s)} < \infty$. Thus we have the limit circle case at $t = 0$.

(ii). Because of (2.3) the modern literature refers to (2.15) as a regular Sturm Liouville problem.

Properties of the spectrum of a self adjoint compact operator [7,17,18] will now be used so as to obtain a Rayleigh–Ritz minimization theorem. The details are as follows:

Let $L_p^2[0, 1]$ be the space of functions u such that $\int_0^1 p(t)|u(t)|^2 dt < \infty$; also for $u, v \in L_p^2[0, 1]$ define $\langle u, v \rangle = \int_0^1 p(t)u(t)\overline{v(t)} dt$. Now let

$$D_L = \{w \in C[0, 1] : w \text{ and } pw' \text{ are absolutely continuous on } [0, 1], \\ \frac{1}{p}(pw')' \in L_p^2[0, 1] \text{ with } w(0) = rw'(1) - kw(1) = 0\}.$$

Clearly

$$L : D_L \subseteq C[0, 1] \subseteq L_p^2[0, 1] \rightarrow L_p^1[0, 1].$$

Now for each $h \in L_p^1[0, 1]$ the boundary value problem

$$\begin{cases} Ly = h, & 0 < t < 1 \\ y(0) = ry'(1) - ky(1) = 0 \end{cases}$$

has a unique solution

$$(2.16) \quad L^{-1}h(t) = y(t) = \int_0^t \frac{1}{p(u)} \int_u^1 p(z)h(z) dz du \\ + k \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1} \int_0^1 \frac{1}{p(u)} \int_u^1 p(z)h(z) dz du \left(\int_0^t \frac{ds}{p(s)} \right).$$

It is easy to check since $\int_0^1 \frac{ds}{p(s)} < \infty$ that

$$L^{-1} : L_p^1[0, 1] \rightarrow D_L \subseteq C[0, 1] \subseteq L_p^2[0, 1].$$

We now *claim* that L^{-1} is compact. Let $\Omega \subseteq L_p^1[0, 1]$ be bounded. The result will follow from the *Arzela Ascoli Theorem* once we show $L^{-1}\Omega$ is bounded and equicontinuous on $[0, 1]$. Let $h \in \Omega$ with $\|h\|_1 = \int_0^1 p|h| dt \leq M$ say. The boundedness of $L^{-1}\Omega$ follows immediately from (2.16) and the fact that $\int_0^1 \frac{ds}{p(s)} < \infty$. In addition for $0 \leq s \leq t \leq 1$, we have

$$|L^{-1}h(t) - L^{-1}h(s)| \leq M \int_s^t \frac{du}{p(s)} + M|k| \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1} \int_0^1 \frac{du}{p(u)} \int_s^t \frac{du}{p(u)}.$$

Consequently L^{-1} is compact. Now define the imbedding $j : L_p^2[0, 1] \rightarrow L_p^1[0, 1]$ by $ju = u$. Note j is continuous since Hölder's inequality implies

$$\|ju\|_1 = \int_0^1 p|ju| du = \int_0^1 p|u| du \leq \|u\| \left(\int_0^1 p dt \right)^{\frac{1}{2}}.$$

Consequently

$$L^{-1}j : L_p^2[0, 1] \rightarrow D_L \subseteq L_p^2[0, 1]$$

is compact. In addition for $u, v \in L_p^2[0, 1]$ we have $(L^{-1}ju, v) = (u, L^{-1}jv)$. To see this notice with $A = k \left(\frac{r}{p(1)} - k \int_0^1 \frac{ds}{p(s)} \right)^{-1}$ that

$$\begin{aligned} (L^{-1}ju, v) &= \int_0^1 p(t)L^{-1}ju(t)\overline{v(t)} dt \\ &= \int_0^1 p(t) \left(\int_0^t \frac{1}{p(w)} \int_w^1 p(z)u(z) dz dw + A \int_0^1 \frac{1}{p(w)} \int_w^1 p(z)u(z) dz dw \int_0^t \frac{ds}{p(s)} \right) \overline{v(t)} dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 p(t)u(t) \left(\int_0^t \frac{1}{p(w)} \int_w^1 p(z)\overline{v(z)} dz dw + A \int_0^1 \frac{1}{p(w)} \int_w^1 p(z)\overline{v(z)} dz dw \int_0^t \frac{ds}{p(s)} \right) dt \\
&= (u, L^{-1}jv)
\end{aligned}$$

after interchanging the order of integration three times. Consequently the *Spectral theorem* for compact self adjoint operators [17, Theorem 4.15] or [18, Chapter 6, Section 3] implies that L has an infinite number of real eigenvalues λ_i with corresponding eigenfunctions $y_i \in D_L$. We next note that $\lambda_i > 0$ for all i ; to see this notice

$$\lambda_i \int_0^1 p|y_i|^2 dt = - \int_0^1 y_i (py_i)' dt$$

and so

$$(2.17) \quad \lambda_i = \frac{\int_0^1 p|y_i'|^2 dt - \frac{kp(1)}{r}|y_i(1)|^2}{\int_0^1 p|y_i|^2 dt}.$$

Remark. If $r = 0$ then the term $\frac{kp(1)}{r}|y_i(1)|^2$ does not appear in (2.17). This remark will apply throughout this paper.

If $k \leq 0$ then $\lambda_i \geq 0$ for all i from (2.17); it is now easy to check that $\lambda_i > 0$ for all i . On the other hand if $k > 0$, then Hölder's inequality implies

$$|y_i(1)|^2 = \left| \int_0^1 \sqrt{p(s)} y_i'(s) \frac{1}{\sqrt{p(s)}} ds \right|^2 \leq \int_0^1 \frac{ds}{p(s)} \left(\int_0^1 p|y_i'|^2 dt \right)$$

and so

$$(2.18) \quad \frac{(-k)p(1)|y_i(1)|^2}{r} \geq \frac{(-k)p(1)}{r} \int_0^1 \frac{ds}{p(s)} \left(\int_0^1 p|y_i'|^2 dt \right).$$

Putting (2.18) into (2.17) yields

$$\lambda_i > \frac{\left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right) \int_0^1 p|y_i'|^2 dt}{\int_0^1 p|y_i|^2 dt} > 0$$

since $r > kp(1) \int_0^1 \frac{ds}{p(s)}$.

Theorem 2.4 (*Rayleigh–Ritz Minimization Theorem*)

Suppose (2.3) is satisfied. Let μ be the first (positive) eigenvalue of (2.15). Then

$$(*) \quad \mu \left(\int_0^1 p(t)|y(t)|^2 dt \right) \leq \int_0^1 p(t)|y'(t)|^2 dt - \frac{kp(1)|y(1)|^2}{r}$$

for all functions $y \in D_L$.

Remark. Note when $r = 0$ the term $\frac{kp(1)|y(1)|^2}{r}$ does not appear in (*).

PROOF: This follows immediately from [18, Ch.5, theorem 5] or [12, section 31, theorem 1]. We first show that L defined on D_L is bounded below. It is of interest to do this directly. To see this notice for $u \in D_L$ we have

$$(2.19) \quad \langle Lu, u \rangle = \frac{(-k)p(1)|u(1)|^2}{r} + \int_0^1 p|u'|^2 dt.$$

In addition Hölder's inequality yields for $x \in [0, 1]$ that

$$(2.20) \quad |u(x)|^2 = \left| \int_0^x \sqrt{p(s)}u'(s) \frac{1}{\sqrt{p(s)}} ds \right|^2 \leq \int_0^1 \frac{ds}{p(s)} \left(\int_0^1 p|u'|^2 dt \right).$$

Consequently

$$(2.21) \quad \int_0^1 p|u|^2 dt \leq \int_0^1 \frac{ds}{p(s)} \int_0^1 p(s) ds \left(\int_0^1 p|u'|^2 dt \right)$$

and putting this into (2.19) yields

$$(2.22) \quad \langle Lu, u \rangle \geq \frac{(-k)p(1)|u(1)|^2}{r} + \frac{\int_0^1 p|u|^2}{\int_0^1 \frac{ds}{p(s)} \int_0^1 p(s) ds}.$$

If $k \leq 0$ then (2.22) implies

$$\langle Lu, u \rangle \geq \frac{\langle u, u \rangle}{\int_0^1 \frac{ds}{p(s)} \int_0^1 p(s) ds}$$

so L is bounded below and the result follows. On the other hand if $k > 0$, then (2.20) and (2.19) yield

$$\langle Lu, u \rangle \geq \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right) \int_0^1 p|u'|^2 dt.$$

This together with (2.21) yields

$$\langle Lu, u \rangle \geq \frac{\left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right) \langle u, u \rangle}{\int_0^1 \frac{ds}{p(s)} \int_0^1 p(s) ds}$$

so L is bounded below.

Finally we need to show that $\inf_{u \in D_L} \frac{\langle Lu, u \rangle}{\langle u, u \rangle}$ is attained. Let ϕ_1, ϕ_2, \dots be corresponding eigenvectors to $\lambda_1, \lambda_2, \dots$. From [18, theorem 5, Ch.6] any $u \in D_L$ can be written in the form $u = \sum_{i=1}^{\infty} \langle u, \phi_i \rangle \phi_i$. Consequently

$$Lu = \sum_1^{\infty} \lambda_i |\langle u, \phi_i \rangle|^2 \geq \mu \sum_1^{\infty} |\langle u, \phi_i \rangle|^2 = \mu \|u\|^2.$$

Thus $\frac{\langle Lu, u \rangle}{\langle u, u \rangle} \geq \mu$ for $u \neq 0$ with equality if $u = \phi$, the eigenfunction corresponding to μ . \square

We conclude this section with the corresponding result of theorem 2.4 in a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$.

Theorem 2.5 *Let (2.3) hold and let $u : [0, 1] \rightarrow H$ and $pu' : [0, 1] \rightarrow H$ be absolutely continuous with $u(0) = ru'(1) - ku(1) = 0$. In addition assume μ is the first (positive) eigenvalue of (2.15). Then*

$$(\star\star) \quad \mu \left(\int_0^1 p(t)|u(t)|^2 dt \right) \leq \int_0^1 p(t)|u'(t)|^2 dt - \frac{kp(1)|u(1)|^2}{r}.$$

PROOF: The compact set $S = u([0, 1]) \cup pu'([0, 1])$ is separable in H as is the smallest closed subspace H' of H which contains S . Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis for H' . By Parseval's relation, theorem 2.4 and two applications of the monotone convergence theorem,

$$\begin{aligned} \mu \left(\int_0^1 p(t) |u(t)|^2 dt \right) &= \mu \int_0^1 p(t) \sum_n |\langle u(t), e_n \rangle|^2 dt = \sum_n \int_0^1 p(t) |\langle u(t), e_n \rangle|^2 dt \\ &\leq \sum_n \left(\int_0^1 p(t) |\langle u'(t), e_n \rangle|^2 dt - \frac{kp(1) |\langle u(1), e_n \rangle|^2}{r} \right) \\ &= \int_0^1 \sum_n p(t) |\langle u'(t), e_n \rangle|^2 dt - \frac{kp(1)}{r} \sum_n |\langle u(1), e_n \rangle|^2 \\ &= \int_0^1 p(t) |u'(t)|^2 dt - \frac{kp(1) |u(1)|^2}{r}. \quad \square \end{aligned}$$

3 Existence Theorems.

In this section existence results are established for the second order problem

$$(3.1) \quad \begin{cases} \frac{1}{p}(py')' = qf(t, y, py'), & 0 < t < 1 \\ y(0) = ry'(1) - ky(1) = 0; & r^2 + k^2 > 0, r \geq 0 \text{ and } r > kp(1) \int_0^1 \frac{ds}{p(s)} \end{cases}$$

where $f : [0, 1] \times H^2 \rightarrow H$ is continuous, H is a real Hilbert space. We first examine the case

$$(3.2) \quad q \text{ is continuous on } [0, 1].$$

Theorem 3.1 Suppose (2.2) with $B = H$, (2.3), (2.6) and (3.2) are satisfied and in addition assume f has the decomposition $f(t, u, v) = g(t, u, v) + h(t, u, v)$ with $g, h : [0, 1] \times H^2 \rightarrow H$ continuous. Also suppose

$$(3.3) \quad \begin{cases} |h(t, u, v)| \leq K \{|u|^\alpha + |v|^\beta + 1\} \text{ for constants } K, \alpha, \beta \\ \text{with } 0 \leq \alpha, \beta < 1 \end{cases}$$

and

$$(3.4) \quad \begin{cases} \text{for } a, b \in \mathfrak{R}, \langle u, g(t, u, v) \rangle \geq a|u|^2 + b|\langle u, v \rangle| \text{ for } (t, u, v) \in [0, 1] \times H^2 \text{ and} \\ |g(t, u, v)| \leq A(t, u)|v|^2 + B(t, u) \text{ where } A(t, u) \text{ and } B(t, u) \\ \text{are bounded on bounded sets.} \end{cases}$$

Then (3.1) has a solution in each of the following cases (here μ is as in Theorem 2.5):

(i). $a \geq 0, b \geq 0$

(ii). $a < 0, b \geq 0$ and $|a|N_0 < \mu$ where $N_0 = \sup_{[0,1]} q(t)$

(iii). $a \geq 0, b < 0$ and $|b|N_1 \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} < \sqrt{\mu}$ if $k \leq 0$ whereas

$|b|N_1 < \sqrt{\mu} \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)$ if $k > 0$; here $N_1 = \sup_{[0,1]} p(t)q(t)$

(iv). $a < 0, b < 0$ and $|a|N_0 + |b|N_1\sqrt{\mu} \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} < \mu$ if $k \leq 0$ whereas

$|a|N_0 \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right) + |b|N_1\sqrt{\mu} < \mu \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)$ if $k < 0$.

Remark. Note when $r = 0$ the term $\frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}$ does not appear in (iii) or (iv) above.

Remark. For $a, b \in H$ we will write $\langle a, b \rangle$ as $a.b$ for the remainder of this paper.

PROOF: To prove existence of a solution we apply Theorem 2.1. Let y be a solution to (2.4) $_\lambda$ with $r > kp(1) \int_0^1 \frac{ds}{p(s)}$. Now $\int_0^1 (py')'.y dt = \frac{kp(1)}{r} |y(1)|^2 - \int_0^1 p(t) |y'(t)|^2 dt$ and this together with $(py')' = \lambda pqf(t, y, py')$ yields

$$(3.5) \quad \|y'\|^2 \leq \frac{kp(1)}{r} |y(1)|^2 - \lambda \int_0^1 pqy.g(t, y, py') dt + \int_0^1 pq|y||h(t, y, py')| dt.$$

Remark. When $r = 0$ the term $\frac{kp(1)}{r} |y(1)|^2$ does not appear in (3.5).

In addition (3.3) yields

$$(3.6) \quad \int_0^1 pq|y||h(t, y, py')| dt \leq K \int_0^1 pq|y|^{\alpha+1} dt + K \int_0^1 pq|y||py'|^\beta dt + K \int_0^1 pq|y| dt.$$

Also for $t \in [0, 1]$, Hölder's inequality yields

$$(3.7) \quad |y(t)| = \left| \int_0^t \sqrt{p(s)} y'(s) \frac{1}{\sqrt{p(s)}} ds \right| \leq \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\|.$$

Thus

$$(3.8) \quad \int_0^1 pq|y|^{\alpha+1} dt \leq K_\alpha \|y'\|^{\alpha+1} \text{ and } \int_0^1 pq|y| dt \leq K_0 \|y'\|$$

where $K_\alpha = \int_0^1 p(t)q(t) \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{\alpha+1}{2}} dt$. In addition Hölder's inequality and (3.7) implies

$$(3.9) \quad \begin{aligned} \int_0^1 pq|y||py'|^\beta dt &= \int_0^1 p^{\frac{\beta}{2}} |y|^\beta p^{\frac{\beta+2}{2}} q|y| dt \\ &\leq \|y'\|^\beta \left(\int_0^1 (p(t))^{\frac{\beta+2}{2-\beta}} (q(t))^{\frac{2}{2-\beta}} |y|^{\frac{2}{2-\beta}} dt \right)^{\frac{2-\beta}{2}} \leq L_0 \|y'\|^{\beta+1} \end{aligned}$$

where $L_0 = \left(\int_0^1 (p(t))^{\frac{\beta+2}{2-\beta}} (q(t))^{\frac{2}{2-\beta}} \left(\int_t^1 \frac{ds}{p(s)} \right)^{\frac{1}{2-\beta}} dt \right)^{\frac{2-\beta}{2}}$. Putting (3.8) and (3.9) into (3.6) yields

$$(3.10) \quad \|y'\|^2 \leq \frac{kp(1)}{r} |y(1)|^2 - \lambda \int_0^1 pqy.g(t, y, py') dt + K K_\alpha \|y'\|^{\alpha+1} + K L_0 \|y'\|^{\beta+1} + K K_0 \|y'\|.$$

(I). $k \leq 0$.

CASE(i): $a \geq 0, b \geq 0$.

Then $y.g(t, y, py') \geq 0$ and so (3.10), with $\frac{kp(1)}{r} |y(1)|^2$ term deleted, yields

$$\|y'\|^2 \leq K K_\alpha \|y'\|^{\alpha+1} + K L_0 \|y'\|^{\beta+1} + K K_0 \|y'\|.$$

Thus there exists a constant M_0 independent of λ such that

$$(3.11) \quad \|y'\| \leq M_0$$

for each solution y to (2.4) _{λ} .

CASE(ii): $a < 0$, $b \geq 0$.

Then $y \cdot g(t, y, py') \geq a|y|^2$ and so

$$(3.12) \quad -\lambda \int_0^1 pqy \cdot g(t, y, py') dt \leq (-a)N_0 \int_0^1 p|y|^2 dt \leq (-a)N_0 \left(\frac{1}{\mu} \|y'\|^2 - \frac{kp(1)}{r\mu} |y(1)|^2 \right)$$

from Theorem 2.5. Putting (3.12) into (3.10) yields

$$\begin{aligned} \left(1 + \frac{aN_0}{\mu} \right) \|y'\|^2 &\leq \frac{kp(1)}{r} \left(1 + \frac{aN_0}{\mu} \right) |y(1)|^2 + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\| \\ &\leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\| \end{aligned}$$

since $k \leq 0$ and $|a|N_0 < \mu$. Once again we have (3.11).

CASE(iii): $a \geq 0$, $b < 0$.

Then $y \cdot g(t, y, py') \geq b|y \cdot py'|$ and so Hölder's inequality and Theorem 2.5 yields

$$(3.13) \quad \begin{aligned} -\lambda \int_0^1 pqy \cdot g(t, y, py') dt &\leq (-b)N_1 \int_0^1 \sqrt{p}|y| \sqrt{p}|y'| dt \leq (-b)N_1 \|y\| \|y'\| \\ &\leq (-b)N_1 \left(\frac{1}{\mu} \|y'\|^2 - \frac{kp(1)}{r\mu} |y(1)|^2 \right)^{\frac{1}{2}} \|y'\|. \end{aligned}$$

This together with (3.7) and the fact that $k \leq 0$ yields

$$(3.14) \quad -\lambda \int_0^1 pqy \cdot g(t, y, py') dt \leq \frac{(-b)N_1}{\sqrt{\mu}} \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\|^2.$$

Putting this into (3.10), with $\frac{kp(1)}{r}|y(1)|^2$ term deleted, yields

$$\left(1 + \frac{bN_1}{\sqrt{\mu}} \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \right) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and we again have (3.11).

CASE(iv): $a < 0$, $b < 0$.

Combining case(ii) and case(iii) will again yield (3.11).

(II). $k > 0$.

CASE(i): $a \geq 0$, $b \geq 0$.

Now since $k > 0$, (3.10) together with (3.7) yields

$$\begin{aligned} \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right) \|y'\|^2 &\leq -\lambda \int_0^1 pqy \cdot g(t, y, py') dt + KK_\alpha \|y'\|^{\alpha+1} \\ &\quad + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|. \end{aligned}$$

Also $y \cdot g(t, y, py') \geq 0$ so we have

$$\left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} \right) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and once again (3.11) holds since $r > kp(1) \int_0^1 \frac{ds}{p(s)}$.

CASE(ii): $a < 0, b \geq 0$.

Then (3.12) together with (3.10) yields

$$\left(1 + \frac{aN_0}{\mu}\right) \|y'\|^2 \leq \frac{kp(1)}{r} \left(1 + \frac{aN_0}{\mu}\right) |y(1)|^2 + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|.$$

Now (3.7) together with $k > 0$ and $|a|N_0 < \mu$ yields

$$\left(1 + \frac{aN_0}{\mu}\right) \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and once again (3.11) holds.

CASE(iii): $a \geq 0, b < 0$.

Then (3.13) together with (3.10) yields

$$\begin{aligned} \|y'\|^2 \leq & \frac{kp(1)}{r} |y(1)|^2 + (-b)N_1 \left(\frac{1}{\mu} \|y'\|^2 - \frac{kp(1)}{r\mu} |y(1)|^2\right)^{\frac{1}{2}} \|y'\| + KK_\alpha \|y'\|^{\alpha+1} \\ & + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|. \end{aligned}$$

Now since $k > 0$, we have

$$\|y'\|^2 \leq \frac{kp(1)}{r} |y(1)|^2 + \frac{(-b)N_1}{\sqrt{\mu}} \|y'\|^2 + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|.$$

Then (3.7) implies

$$\left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)} + \frac{bN_1}{\sqrt{\mu}}\right) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and we again have (3.11) holding.

CASE(iv): $a < 0, b < 0$.

Combining case(ii) and case(iii) will again yield (3.11).

Thus in all cases there exists a constant M_0 independent of λ with (3.11) holding i.e. $\|y'\| \leq M_0$. In addition (3.7) yields

$$(3.15) \quad \sup_{[0,1]} |y(t)| \leq \left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} M_0 \equiv M_1.$$

This bound together with (3.2) and (3.3) implies that there exist constants A and B , independent of λ , with $|(py')'| \leq Apq|py'|^2 + Bpq$. Consequently

$$(3.16) \quad \begin{aligned} \int_0^1 |(py')'| ds & \leq A\|y'\|^2 \sup_{[0,1]} p^2 q + B \int_0^1 pq dt \\ & \leq AM_0^2 \sup_{[0,1]} p^2 q + B \int_0^1 pq dt \equiv M_2. \end{aligned}$$

Fix z in H with norm 1 and set $k(t) = \langle y(t), z \rangle$, $0 \leq t \leq 1$. Now $\sup_{[0,1]} |k(t)| \leq M_1$ and there exists $t_0 \in (0, 1)$ such that $|k'(t_0)| = |k(1) - k(0)| \leq 2M_1$. Consequently for $t \in [0, 1]$, we have

$$\begin{aligned} |p(t)k'(t)| &\leq |p(t_0)k'(t_0)| + \left| \int_{t_0}^t |(py')' ds \right| \leq 2M_1 \sup_{[0,1]} p(t) + |z| \int_0^1 |(py')'| ds \\ &\leq 2M_1 \sup_{[0,1]} p(t) + M_2 \equiv M_3. \end{aligned}$$

Thus $|\langle p(t)y'(t), z \rangle| \leq M_3$. If $p(t)y'(t) \neq 0$ set $z = \frac{p(t)y'(t)}{|p(t)y'(t)|}$ to get $|p(t)y'(t)| \leq M_3$, which also holds if $p(t)y'(t) = 0$. Consequently

$$(3.17) \quad \sup_{(0,1]} |p(t)y'(t)| \leq M_3.$$

Thus (3.15), (3.17) together with Theorem 2.1 yields the result. \square

We next discuss the situation where q may be singular at $t = 0$ and/or $t = 1$.

Theorem 3.2 Suppose (2.2) with $B = H$, (2.3) and (2.6) are satisfied. Also suppose $f = g + h$ with (3.3) satisfied and in addition assume

$$(3.18) \quad \begin{cases} \text{for } a, b \in \mathfrak{R}, \langle u, g(t, u, v) \rangle \geq a|u|^2 + b|\langle u, v \rangle| \text{ for } (t, u, v) \in [0, 1] \times H^2 \text{ and} \\ \text{there exists } \gamma, 0 \leq \gamma < 2 \text{ with } |g(t, u, v)| \leq A(t, u)|v|^\gamma + B(t, u) \\ \text{where } A(t, u) \text{ and } B(t, u) \text{ are bounded on bounded sets} \end{cases}$$

and

$$(3.19) \quad \int_0^1 (p(t))^{\frac{\tau+2}{2-\tau}} (q(t))^{\frac{2}{2-\tau}} dt < \infty \text{ where } \tau = \max\{\beta, \gamma\}.$$

Then (3.1) has a solution in each of the following cases:

(i). $a \geq 0, b \geq 0$

(ii). $a < 0, b \geq 0$ with $|a|Q_0 < 1$ if $k \leq 0$ whereas $|a|Q_0 < \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)$ if $k > 0$; here $Q_0 = \int_0^1 \frac{ds}{p(s)} \int_0^1 pq ds$

(iii). $a \geq 0, b < 0$ with $|b|Q_1 < 1$ if $k \leq 0$ whereas $|b|Q_1 < \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)$ if $k > 0$; here $Q_1 = \left(\int_0^1 p^3 q^2 ds\right)^{\frac{1}{2}} \left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}}$

(iv). $a < 0, b < 0$ with $|a|Q_0 + |b|Q_1 < 1$ if $k \leq 0$ whereas $|a|Q_0 + |b|Q_1 < \left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right)$ if $k > 0$.

Remark. Note when $r = 0$ the term $\frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}$ does not appear in (ii), (iii) or (iv) above.

PROOF: Let y be a solution to (2.4) $_\lambda$ with $r > kp(1) \int_0^1 \frac{ds}{p(s)}$. Reasoning exactly as in Theorem 3.1 yields (with K_α and L_0 as in Theorem 3.1),

$$(3.20) \quad \|y'\|^2 \leq \frac{kp(1)}{r} |y(1)|^2 - \lambda \int_0^1 pqy \cdot g(t, y, py') dt + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

with

$$(3.21) \quad \sup_{[0,1]} |y(t)| \leq \left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} \|y'\|.$$

Remark. When $r = 0$ the term $\frac{kp(1)}{r}|y(1)|^2$ does not appear in (3.20).

(I). $k \leq 0$.

Then (3.20) yields

$$(3.22) \quad \|y'\|^2 \leq -\lambda \int_0^1 pqy \cdot g(t, y, py') dt + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|.$$

CASE(i): $a \geq 0, b \geq 0$.

Exactly the reasoning as in Theorem 3.1 yields the existence of a constant M_0 independent of λ with

$$(3.23) \quad \|y'\| \leq M_0$$

for each solution y to (2.4) $_\lambda$.

CASE(ii): $a < 0, b \geq 0$.

Then $y \cdot g(t, y, py') \geq a|y|^2$ and this together with (3.21) yields

$$(3.24) \quad -\lambda \int_0^1 pqy \cdot g(t, y, py') dt \leq (-a) \sup_{[0,1]} |y|^2 \int_0^1 pq dt \leq (-a)Q_0 \|y'\|^2.$$

Putting this into (3.22) yields

$$(1 + aQ_0) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and we again have (3.23).

CASE(iii): $a \geq 0, b < 0$.

Then $y \cdot g(t, y, py') \geq b|y \cdot py'|$ and so Hölder's inequality and (3.21) yields

$$(3.25) \quad -\lambda \int_0^1 pqy \cdot g(t, y, py') dt \leq (-b) \sup_{[0,1]} |y| \int_0^1 \sqrt{p}|y'|p^{\frac{3}{2}}q dt \leq (-b)Q_1 \|y'\|^2.$$

Putting this into (3.22) yields

$$(1 + bQ_1) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and we again have (3.23).

CASE(iv): $a < 0, b < 0$.

Combining case(ii) and case(iii) will again yield (3.23).

(II). $k > 0$.

Putting (3.21) into (3.20) yields

$$\left(1 - \frac{kp(1)}{r} \int_0^1 \frac{ds}{p(s)}\right) \|y'\|^2 \leq -\lambda \int_0^1 pqy \cdot g(t, y, py') dt + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|$$

and it is easy to check, using (3.24) and (3.25) where appropriate, that (3.23) holds in each case.

Thus in all cases there exists a constant M_0 independent of λ with (3.23) holding. In addition (3.21) yields

$$(3.26) \quad \sup_{[0,1]} |y(t)| \leq \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} M_0 \equiv M_1.$$

Now (3.2), (3.18) and (3.26) implies that there exist constants A and B , independent of λ , with $|(py')'| \leq Apq|py'|^\tau + Bpq$ where $\tau = \{\gamma, \beta\}$. Consequently Hölder's inequality yields

$$\begin{aligned} \int_0^1 |(py')'| ds &\leq A \int_0^1 p^{\frac{\tau}{2}} |y'|^\tau p^{\frac{2+\tau}{2}} q dt + B \int_0^1 pq dt \\ &\leq A \left(\int_0^1 p |y'|^2 dt \right)^{\frac{\tau}{2}} \left(\int_0^1 p^{\frac{2+\tau}{2-\tau}} q^{\frac{2}{2-\tau}} dt \right)^{\frac{2-\tau}{2}} + B \int_0^1 pq dt \\ &\leq AM_0^{\frac{\tau}{2}} \left(\int_0^1 p^{\frac{2+\tau}{2-\tau}} q^{\frac{2}{2-\tau}} dt \right)^{\frac{2-\tau}{2}} + B \int_0^1 pq dt \equiv M_2 \end{aligned}$$

and exactly the same analysis as in Theorem 3.1 completes the proof. \square

Remark. If $f(t, y, py') \equiv f(t, y)$ then it is possible (using Theorem 2.2 in place of Theorem 2.1) to replace (2.6) and (3.19) with (2.9) and (2.8) and we have an analogue of Theorem 3.2. A similar remark applies to all the other theorem's established in this paper.

Finally in this section we improve Theorem 3.2 when $p \equiv 1$ and $r = 0$ (Dirichlet boundary conditions).

Theorem 3.3 *Let $p \equiv 1$ and suppose (2.2) with $B = H$, (2.6) with $p \equiv 1$, (3.3), (3.18) and (3.19) with $p \equiv 1$, are satisfied. Then (3.1), with $p \equiv 1$ and $r = 0$ has a solution in each of the following cases:*

- (i). $a \geq 0, b \geq 0$
- (ii). $a < 0, b \geq 0$ and $|a|Q_2 < \pi$ where $Q_2 = \int_0^1 q ds$
- (iii). $a \geq 0, b < 0$ and $|b|Q_3 < \sqrt{\pi}$ where $Q_3 = \left(\int_0^1 q^2 ds \right)^{\frac{1}{2}}$
- (iv). $a < 0, b < 0$ and $|a|Q_2 + |b|Q_3\sqrt{\pi} < \pi$.

PROOF: Let y be a solution to (2.4) $_\lambda$ with $p \equiv 1$ and $r = 0$. Reasoning exactly as in Theorem 3.1 we obtain the existence of constants K_α and L_0 such that

$$(3.27) \quad \|y'\|^2 \leq -\lambda \int_0^1 qy.g(t, y, y') dt + KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|.$$

In addition since $y(0) = y(1) = 0$ and $p \equiv 1$, we have

$$(3.28) \quad \sup_{[0,1]} |y(t)|^2 \leq \int_0^1 |y \cdot y'| dt \leq \|y\| \|y'\|.$$

CASE(i): $a \geq 0, b \geq 0$.

Exactly the reasoning as in Theorem 3.1 yields the existence of a constant M_0 independent of λ with

$$(3.29) \quad \|y'\| \leq M_0$$

for each solution y to $(2.4)_\lambda$ with $p \equiv 1$ and $r = 0$.

CASE(ii): $a < 0, b \geq 0$.

Then $y.g(t, y, y') \geq a|y|^2$ and this together with (3.28) yields

$$-\lambda \int_0^1 qy.g(t, y, y') dt \leq (-a) \sup_{[0,1]} |y|^2 \int_0^1 q dt \leq (-a)Q_2 \|y\| \|y'\|.$$

Apply Theorem 2.5 (here $\mu = \pi^2$ and $r = 0$) to obtain

$$-\lambda \int_0^1 qy.g(t, y, y') dt \leq \frac{(-a)Q_2}{\pi} \|y'\|^2$$

and putting this into (3.28) yields

$$\left(1 + \frac{aQ_2}{\pi}\right) \|y'\|^2 \leq KK_\alpha \|y'\|^{\alpha+1} + KL_0 \|y'\|^{\beta+1} + KK_0 \|y'\|.$$

Consequently (3.29) is again true.

CASE(iii): $a \geq 0, b < 0$.

Then $y.g(t, y, y') \geq b|y.y'|$ and applying Hölder's inequality, (3.28) and Theorem 2.5 yields

$$\begin{aligned} -\lambda \int_0^1 qy.g(t, y, y') dt &\leq (-b) \sup_{[0,1]} |y| \int_0^1 q|y'| dt \leq (-b) \|y'\| \sup_{[0,1]} |y| \left(\int_0^1 q^2 ds \right)^{\frac{1}{2}} \\ &\leq (-b)Q_3 (\|y\| \|y'\|)^{\frac{1}{2}} \|y'\| \leq \frac{(-b)Q_3}{\sqrt{\pi}} \|y'\|^2. \end{aligned}$$

Putting this into (3.27) yields (3.29) again.

CASE(iv): $a < 0, b < 0$.

Combining case(ii) and case(iii) will again yield (3.29).

Exactly the same reasoning as in Theorem 3.2 completes the proof. \square

4 Pure Sturm Liouville Boundary Data.

Existence of solutions to the boundary value problem (2.10) will be established in this section.

Theorem 4.1 *Suppose (2.2) with $B = H$, (2.6), (2.11) and (2.12) are satisfied. In addition assume f has the decomposition $f = g + h$ and suppose (3.3), (3.18) and (3.19) are satisfied. Then (2.10) has a solution in each of the following cases:*

(I). $k \leq 0, \beta > 0$

(i). $a \geq 0, b \geq 0$ and there exists a $\delta > 0$ with $\alpha > \beta p(0)(1 + \delta) \int_0^1 \frac{ds}{p(s)}$ and

$$\frac{(-k)p(1)}{r} > \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)$$

(ii). $a < 0, b \geq 0$ and there exists a $\delta > 0$ with $\alpha + \alpha a(1 + \delta)Q_0 > \beta p(0)(1 + \delta) \int_0^1 \frac{ds}{p(s)}$

and $\frac{(-k)p(1)}{r} + a \left(1 + \frac{1}{\delta}\right) L_1 > \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)$; here $L_1 = \int_0^1 pq dt$ and $Q_0 = L_1 \left(\int_0^1 \frac{ds}{p(s)}\right)$

(iii). $a \geq 0, b < 0$ and there exists a $\delta > 0$ with $\alpha + b\alpha \left(\left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} + \frac{\delta}{2}\right) L_2$

$> \beta p(0)(1 + \delta) \int_0^1 \frac{ds}{p(s)}$ and $\frac{(-k)p(1)}{r} + \frac{bL_2}{2\delta} > \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)$; here $L_2 = \left(\int_0^1 p^3 q^2 dt\right)^{\frac{1}{2}}$
 (iv). $a < 0, b < 0$ and there exists a $\delta > 0$ with $\alpha + \alpha a(1 + \delta) Q_0 + b\alpha \left(\left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} + \frac{\delta}{2}\right) L_2$
 $> \beta p(0)(1 + \delta) \int_0^1 \frac{ds}{p(s)}$ and $\frac{(-k)p(1)}{r} + a\left(1 + \frac{1}{\delta}\right) L_1 + \frac{bL_2}{2\delta} > \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)$.

(II). $k > 0, \beta \leq 0$

Then the results of (i), (ii), (iii) and (iv) of (I) hold with r replacing α , k replacing β , α replacing r , β replacing k and $p(1)$ replacing $p(0)$.

(III). $k \leq 0, \beta \leq 0$

(i). $a \geq 0, b \geq 0$

(ii). $a < 0, b \geq 0$ and there exists a $\delta > 0$ with $\frac{|a|}{4}(1 + \delta) Q_0 < 1$,
 $\frac{(-k)p(1)}{r} + \frac{a}{4}\left(2 + \frac{1}{\delta} + \delta\right) L_1 > 0$ and $\frac{(-\beta)p(0)}{\alpha} + \frac{a}{4}\left(1 + \frac{2}{\delta} + \frac{1}{\delta^2}\right) L_1 > 0$

(iii). $a \geq 0, b < 0$ and there exists a $\delta > 0$ with $\frac{|b|L_2}{2} \left(\left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} + \frac{\delta}{2}\right) < 1$,
 $\frac{(-k)p(1)}{r} + \frac{b}{4}\left(1 + \frac{1}{\delta}\right) L_2 > 0$ and $\frac{(-\beta)p(0)}{\alpha} + \frac{b}{4}\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right) L_2 > 0$

(iv). $a < 0, b < 0$ and there exists a $\delta > 0$ with $\frac{|a|}{4}(1 + \delta) Q_0 + \frac{|b|L_2}{2} \left(\left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} + \frac{\delta}{2}\right)$
 < 1 , $\frac{(-k)p(1)}{r} + \frac{a}{4}\left(2 + \frac{1}{\delta} + \delta\right) L_1 + \frac{b}{4}\left(1 + \frac{1}{\delta}\right) L_2 > 0$ and $\frac{(-\beta)p(0)}{\alpha} + \frac{a}{4}\left(1 + \frac{2}{\delta} + \frac{1}{\delta^2}\right) L_1$
 $+ \frac{b}{4}\left(\frac{1}{\delta} + \frac{1}{\delta^2}\right) L_2 > 0$.

PROOF: Let y be a solution of (2.13) $_{\lambda}$. The differential equation immediately yields

$$(4.1) \quad \|y'\|^2 + \frac{(-k)p(1)}{r}|y(1)|^2 + \frac{(-\beta)p(0)}{\alpha}|y(0)|^2 \leq -\lambda \int_0^1 pqy \cdot g(t, y, py') dt + \int_0^1 pq|y||h(t, y, py')| dt.$$

(I). $k \leq 0, \beta > 0$.

Hölder's inequality implies for $t \in [0, 1]$ that

$$(4.2) \quad |y(t)| \leq \int_0^1 |y'(s)| ds + |y(1)| \leq \left(\int_0^1 \frac{ds}{p(s)}\right)^{\frac{1}{2}} \|y'\| + |y(1)|$$

and since $|ab| \leq \frac{\delta}{2}|a|^2 + \frac{1}{2\delta}|b|^2$ for any $\delta > 0$, we have

$$(4.3) \quad \sup_{[0,1]} |y(t)|^2 \leq (1 + \delta) \|y'\|^2 \int_0^1 \frac{ds}{p(s)} + \left(1 + \frac{1}{\delta}\right) |y(1)|^2.$$

Putting this into (4.1) gives

$$(4.4) \quad \left(1 - \frac{\beta p(0)}{\alpha}(1 + \delta) \int_0^1 \frac{ds}{p(s)}\right) \|y'\|^2 + \left(\frac{(-k)p(1)}{r} - \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)\right) |y(1)|^2 \leq -\lambda \int_0^1 pqy \cdot g(t, y, py') dt + \int_0^1 pq|y||h(t, y, py')| dt.$$

Following the ideas of Theorem 3.1 (where we use (4.2) in place of (3.7)) we obtain the existence of a constant C_1 , independent of λ , with

$$(4.5) \quad \begin{aligned} & \left(1 - \frac{\beta p(0)}{\alpha} (1 + \delta) \int_0^1 \frac{ds}{p(s)}\right) \|y'\|^2 + \left(\frac{(-k)p(1)}{\alpha} - \frac{\beta p(0)}{\alpha} \left(1 + \frac{1}{\delta}\right)\right) |y(1)|^2 \\ & \leq -\lambda \int_0^1 pqy.g(t, y, py') dt + C_1 \left(\|y'\|^{\alpha+1} + |y(1)|^{\alpha+1} + \|y'\|^{\beta+1} + |y(1)|^{\beta+1} + 1\right). \end{aligned}$$

CASE(i): $a \geq 0, b \geq 0$.

Then $y.g(t, y, py') \geq 0$ and putting this into (4.5) yields the existence of a constant M_0 independent of λ with

$$(4.6) \quad \|y'\| \leq M_0 \text{ and } |y(1)| \leq M_0$$

for each solution y to (2.13) $_{\lambda}$.

CASE(ii): $a < 0, b \geq 0$.

Then $y.g(t, y, py') \geq a|y|^2$ and this together with (4.3) yields

$$\begin{aligned} -\lambda \int_0^1 pqy.g(t, y, py') dt & \leq (-a) \sup_{[0,1]} |y|^2 \int_0^1 pq dt \\ & \leq (-a)L_1 \left((1 + \delta) \|y'\|^2 \int_0^1 \frac{ds}{p(s)} + \left(1 + \frac{1}{\delta}\right) |y(1)|^2 \right). \end{aligned}$$

Putting this into (4.5) will again yield (4.6).

CASE(iii): $a \geq 0, b < 0$.

Then $y.g(t, y, py') \geq b|y.py'|$ so Hölder's inequality together with (4.2) yields

$$\begin{aligned} -\lambda \int_0^1 pqy.g(t, y, py') dt & \leq (-b) \sup_{[0,1]} |y| \int_0^1 p^2 q |y'| dt \leq (-b)L_2 \|y'\| \sup_{[0,1]} |y| \\ & \leq (-b)L_2 \|y'\| \left(\left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\| + |y(1)| \right) \\ & \leq (-b)L_2 \|y'\|^2 \left(\left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} + \frac{\delta}{2} \right) + \frac{(-b)L_2}{2\delta} |y(1)|^2. \end{aligned}$$

Putting this into (4.5) will again yield (4.6).

CASE(iv): $a < 0, b < 0$.

Combining case(ii) and case(iii) will again yield (4.6).

Consequently (4.2) yields

$$\sup_{[0,1]} |y(t)| \leq \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} M_0 + M_0 \equiv M_1.$$

Exactly the same analysis as in Theorem 3.2 will complete the proof in this case.

A similar analysis proves (II).

(III). $k \leq 0, \beta \leq 0$.

For $t \in [0, 1]$ we have

$$|y(t)| \leq \frac{1}{2} \int_0^1 |y'(s)| ds + \frac{1}{2} (|y(0)| + |y(1)|)$$

and this together with Hölder's inequality yields

$$(4.7) \quad \sup_{[0,1]} |y(t)| \leq \frac{1}{2} \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\| + \frac{1}{2} (|y(0)| + |y(1)|).$$

Now $|ab| \leq \frac{\delta}{2}|a|^2 + \frac{1}{2\delta}|b|^2$ for any $\delta > 0$, so we have

$$(4.8) \quad \sup_{[0,1]} |y(t)|^2 \leq \frac{1}{4} (1 + \delta) \|y'\|^2 \int_0^1 \frac{ds}{p(s)} + \frac{1}{4} \left(2 + \frac{1}{\delta} + \delta \right) |y(1)|^2 + \frac{1}{4} \left(1 + \frac{2}{\delta} + \frac{1}{\delta^2} \right) |y(0)|^2.$$

Following the ideas of Theorem 3.1 (where we use (4.7) in place of (3.7)) we obtain the existence of a constant C_2 , independent of λ , with

$$(4.9) \quad \|y'\|^2 + \frac{(-k)p(1)}{\alpha} |y(1)|^2 + \frac{(-\beta)p(0)}{\alpha} |y(0)|^2 \leq -\lambda \int_0^1 pqy.g(t, y, py') dt + C_2 \left(\|y'\|^{\alpha+1} + |y(0)|^{\alpha+1} + |y(1)|^{\alpha+1} + \|y'\|^{\beta+1} + |y(0)|^{\beta+1} + |y(1)|^{\beta+1} + 1 \right).$$

CASE(i): $a \geq 0, b \geq 0$.

Then $y.g(t, y, py') \geq 0$ and this together with (4.9) and the fact that $k \leq 0, \beta \leq 0$ yields the existence of a constant M_0 independent of λ with

$$(4.10) \quad \|y'\| \leq M_0, |y(0)| \leq M_0 \text{ and } |y(1)| \leq M_0$$

for each solution y to (2.13) $_{\lambda}$.

CASE(ii): $a < 0, b \geq 0$.

Then $y.g(t, y, py') \geq a|y|^2$ and this together with (4.8) yields

$$\begin{aligned} -\lambda \int_0^1 pqy.g(t, y, py') dt &\leq \frac{(-a)L_1}{4} (1 + \delta) \|y'\|^2 \int_0^1 \frac{ds}{p(s)} + \frac{(-a)L_1}{4} \left(2 + \frac{1}{\delta} + \delta \right) |y(0)|^2 \\ &\quad + \frac{(-a)L_1}{4} \left(1 + \frac{2}{\delta} + \frac{1}{\delta^2} \right) |y(1)|^2. \end{aligned}$$

Putting this into (4.9) will again yield (4.10).

CASE(iii): $a \geq 0, b < 0$.

Then $y.g(t, y, py') \geq b|y.py'|$ so Hölder's inequality together with (4.7) implies

$$\begin{aligned} -\lambda \int_0^1 pqy.g(t, y, py') dt &\leq (-b)L_2 \|y'\| \left(\frac{1}{2} \left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\| + \frac{1}{2} (|y(0)| + |y(1)|) \right) \\ &\leq \frac{(-b)L_2}{2} \left(\left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} \|y'\|^2 + (|y(0)| + |y(1)|) \|y'\| \right) \\ &\leq \frac{(-b)L_2}{2} \|y'\|^2 \left(\left(\int_0^1 \frac{ds}{p(s)} \right)^{\frac{1}{2}} + \frac{\delta}{2} \right) + \frac{(-b)L_2}{4} \left(1 + \frac{1}{\delta} \right) |y(1)|^2 \end{aligned}$$

$$+ \frac{(-b)L_2}{4} \left(\frac{1}{\delta} + \frac{1}{\delta^2} \right) |y(0)|^2.$$

Putting this into (4.9) will again yield (4.10).

CASE(iv): $a < 0$, $b < 0$.

Combining case(ii) and case(iii) will again yield (4.10).

Exactly the same analysis as in Theorem 3.2 completes the proof. \square

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