LIBERTAS MATEMATICA, vol XXVII (2007)

# About a Combinatorial Probability Limit

**Laurenţiu MODAN**

**Abstract.** The *binomial distribution* $\mathcal{B}(n, p_n)$, for $p_n \in [0, 1]$, $(\forall)\, n \in I\!N$, and respectively *Poisson* $(\lambda)$ *distribution*, for $\lambda \in I\!R_+^*$ are integrant parts of *Combinatorial Probabilities* (see [1]). The present note remembers us that in some given conditions, $\mathcal{B}(n, p_n)$ goes to Poisson $(\lambda)$, this property being frequently applied in the quotidian life, when are used events whose appearance is rare.

**Keywords:** average, distribution, limit.

**Mathematics Subject Classification (2000):** 05A10, 60C10.

## 1 Introduction

*Binomial distribution* for a random variable $X$ (see [3], [4]) was introduced in 1735, by the Swiss mathematician Jacques Bernoulli, and it is given by:

$$X : \left( \begin{array}{c} k \\ \left( \begin{array}{c} n \\ k \end{array} \right) p^k q^{n-k} \end{array} \right), \quad k \in \{0, 1, \ldots, n\}, p, q \in [0, 1]\,, p + q = 1.$$

Generally, this distribution is noted $\mathcal{B}(n, p)$ and for a some random variable $X$, we note $X \in \mathcal{B}(n, p)$.

*Poisson* $(\lambda)$ *distribution* for a random variable $Y$ (see [3], [4]), known as *the rare events law*, was introduced in 1837 by the French mathematician Siméon Poisson. So, for a same random variable $Y \in$ *Poisson* $(\lambda)$, this is given by:

$$Y : \left( \begin{array}{c} k \\ e^{-\lambda} \frac{\lambda^k}{k!} \end{array} \right), \quad \lambda \in I\!R_+^*, k \in I\!N\,.$$

## 2 Main Result

As a preliminary fact, we compute the *average* for a random variable, *binomial distributed.*

**Proposition**. *Let the random variable* $X \in \mathcal{B}(n, p), n \in I\!N$, $p \in [0, 1]$. *Then, its average is:*

$$m = M(X) = np\,.$$

**Proof.** Using the average definition for a random variable (see [2]), we have:

$$m = M(X) = \sum_{k=0}^{n} k \begin{pmatrix} n \\ k \end{pmatrix} p^k q^{n-k} = \sum_{k=1}^{n} \frac{n(n-1)\dots(n-k+1)}{(k-1)!} p^k q^{n-k} =$$

$$= np \sum_{l=0}^{n-1} \frac{(n-1)(n-2)\dots(n-l)}{l!} p^l q^{n-(l+1)} =$$

$$= np \sum_{l=0}^{n-1} \begin{pmatrix} n-1 \\ l \end{pmatrix} p^l q^{n-1-l} = np(p+q)^{n-1} = np\,. \square$$

In the following, we consider the sequence $(X_n)_{n \in \mathbb{N}}$, of the *random binomial variables* and we shall study its limit behaviour.

**Lemma**. *Let* $(X_n)_{n \in \mathbb{N}^*}$, *a sequence whose terms are random variables* $\mathcal{B}(n, p_n)$, *with* $M(X_n) = \lambda \in \mathbb{R}_+^*$ *and* $p_n = p(n) \in [0,1], (\forall)\, n \in \mathbb{N}^*$. *Then, we have:*

$$\lim_{n \to \infty} \mathcal{B}(n, p_n) = \text{ Poisson}\,(\lambda).$$

**Proof**. From anterior Proposition, we know that $M(X_n) = n \cdot p_n$. Because $M(X_n) = \lambda$, from the hypothesis, it follows:

$$np_n = \lambda \Leftrightarrow p_n = \frac{\lambda}{n}\,.$$

Therefore it occurs:

$$\lim_{n \to \infty} \mathcal{B}(n, p_n) = \lim_{n \to \infty} \begin{pmatrix} n \\ k \end{pmatrix} p_n^k q_n^{n-k} = \lim_{n \to \infty} \frac{n!}{k!(n-k)!} \left( \frac{\lambda}{n} \right)^k \left( 1 - \frac{\lambda}{n} \right)^{n-k} =$$

$$= \frac{\lambda^k}{k!} \lim_{n \to \infty} \frac{n!}{(n-k)! n^k} \left( 1 - \frac{\lambda}{n} \right)^{n-k} =$$

$$= \frac{\lambda^k}{k!} \lim_{n \to \infty} \frac{n-k+1}{n} \cdot \frac{n-k+2}{n} \cdot \ldots \cdot \frac{n}{n} \left( 1 - \frac{\lambda}{n} \right)^{n-k} =$$

$$= \frac{\lambda^k}{k!} \lim_{n \to \infty} \left[ \left( 1 - \frac{\lambda}{n} \right)^{-\frac{n}{\lambda}} \right]^{-\frac{\lambda(n-k)}{n}} = \frac{\lambda^k}{k!}\, e^{-\lambda} . \square$$

**Remark.** The anterior Lemma is a guarantee that for little probabilities $p_n = \frac{\lambda}{n}, n \in \mathbb{N}^*$, on limit, the distribution $\mathcal{B}(n, p_n)$ has a behaviour as Poisson $(\lambda)$, $\lambda \in \mathbb{R}_+^*$. Usually, we say that in the anterior conditions, Poisson $(\lambda)$ approximates $\mathcal{B}(n, p_n)$. We remind here, that the best approximations appear when $n \geq 30$ and $np \leq 5$ (see [1]).

A chain of Lemma applications in the *Risk Theory*, can be found in [3]. $\square$

## References

[1] P. Erdös and J. Spencer, *Probabilistic Methods in Combinatorics*, Akádemiai Kiadó, Budapest, 1974.

[2] L. Modan, *Discrete Aspects of the Mathematical Risk* (in Romanian, to appear.)

[3] J. Neveu, *Les bases mathématiques du calcul des probabilitiés*, Masson, Paris, 1964.

[4] O. Onicescu, *Probabilities and Random Processes* (in Romanian), Ed. Şt. şi Enciclop., Bucharest, 1977.