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Integral Operators on a Certain Class of Analytic Functions

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Abstract. Let A be the class of functions $f(z)$ which are analytic in the open unit disk U and $L(\alpha)$ the class of functions $f(z) \in A$ which satisfies the conditions: $f(z) \neq 0$ for $z \in U \backslash \{0\}$, $\left|\left(\frac{z}{f(z)}\right)''\right| \leq \alpha$, $z \in U$, $0 < \alpha \leq 2$. The integral operators $H_{\beta,\gamma}(z)$, $G_\gamma(z)$ for $f(z) \in L(\alpha)$ are considered. In the present paper we obtain univalence conditions of these integral operators $H_{\beta,\gamma}(z)$, $G_\gamma(z)$ for $f(z) \in L(\alpha)$.

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1 Introduction

Let A be the class of the functions $f(z)$ which are analytic in the unit disk $U = \{z \in C : |z| < 1\}$ and $f(0) = f'(0) - 1 = 0$.

We denote by S the class of the functions $f(z) \in A$ which are univalent in U. Yang D., Liu J. [5] defines the class $L(\alpha)$.

We need the following theorems.

Theorem 1.1.[3] *Let α be a complex number, $Re\alpha > 0$ and $f(z) \in A$. If*

$$\frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1 \tag{1.1}$$

for all $z \in U$, then for any complex number $\beta, Re\beta \geq Re\alpha$ the function

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta-1} f'(u) du\right]^{\frac{1}{\beta}} \tag{1.2}$$

is in the class S.

Schwarz Lemma [1]. *Let $f(z)$ the function regular in the disk $U_R = \{z \in C;\ |z| \in R\}$, where $|f(z)| < M,\ M$ fixed. If $f(z)$ has in $z = 0$ one zero with multiply $\geq m, m \in N$ then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m,\ \ z \in U_R \tag{1.3}$$

the equality (in the inequality (1.3) for $z \neq 0$) hold in the case $f(z) = e^{i\theta}\frac{M}{R^m}z^m$.

2 Main Results

Theorem 2.1. *Let $g(z) \in L(\alpha)$, $0 < \alpha \leq 2$, $g(z) = z + \sum_{k=3}^{\infty} a_k z^k$, $\gamma \in C$, $Re\gamma > 0$, $M > 1$ and*

$$|\gamma| Re\, \gamma \geq (\alpha + 1)\, M + 1 \tag{2.1}$$

If $|g(z)| \leq M,\ z \in U$, then for every complex number β, $Re\beta \geq Re\gamma$ the function

$$H_{\beta,\gamma}(z) = \left[\beta \int_0^z u^{\beta-1} \left(\frac{g(u)}{u}\right)^{\frac{1}{\gamma}} du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is in the class S.

Proof. Let us consider the function

$$p(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\frac{1}{\gamma}} du \tag{2.3}$$

The function $p(z)$ is regular in U. From (2.3) we have

$$p'(z) = \left(\frac{g(z)}{z}\right)^{\frac{1}{\gamma}},\ p''(z) = \frac{1}{\gamma}\left(\frac{g(z)}{z}\right)^{\frac{1}{\gamma}-1} \frac{zg'(z) - g(z)}{z^2}$$

and

$$\frac{1 - |z|^{2\, Re\gamma}}{Re\gamma} \left|\frac{zp''(z)}{p'(z)}\right| = \frac{1 - |z|^{2\, Re\gamma}}{Re\gamma} \frac{1}{|\gamma|} \left|\frac{zg'(z)}{g(z)} - 1\right| \tag{2.4}$$

for all $z \in U$. From (2.4) we get

$$\frac{1 - |z|^{2\, Re\gamma}}{Re\gamma} \left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1 - |z|^{2\, Re\gamma}}{|\gamma| Re\gamma} \left|\frac{zg'(z)}{g(z)}\right| + \frac{1 - |z|^{2\, Re\gamma}}{|\gamma|\, Re\gamma} \tag{2.5}$$

for all $z \in U$. Hence, we have

$$\frac{1 - |z|^{2\, Re\gamma}}{Re\, \gamma} \left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1 - |z|^{2\, Re\gamma}}{|\gamma|\, Re\gamma} \left(\left|\frac{z^2 g'(z)}{g^2(z)}\right| \frac{|g(z)|}{|z|} + 1\right) \tag{2.6}$$

for all $z \in U$.

By the Schwarz Lemma also $|g(z)| \leq M|z|,\ z \in U$ and using (2.6) we obtain

$$\frac{1 - |z|^{2\, Re\gamma}}{Re\, \gamma} \left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{1 - |z|^{2\, Re\gamma}}{|\gamma|\, Re\, \gamma} \left(\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| M + M + 1\right) \tag{2.7}$$

for all $z \in U$.

Because $g(z) \in L(\alpha)$ we have

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| \leq \alpha |z^2|,\ z \in U \tag{2.8}$$

for all $z \in U$.

From (2.7) and (2.8) we obtain

$$\frac{1-|z|^{2\,Re\gamma}}{Re\,\gamma}\left|\frac{zp''(z)}{p'(z)}\right| \leq \frac{\alpha M+M+1}{|\gamma|\,Re\,\gamma}\left(1-|z|^{2\,Re\gamma}\right) \leq \frac{(\alpha+1)M+1}{|\gamma|\,Re\,\gamma} \tag{2.9}$$

for all $z \in U$.

Since $|\gamma| Re\,\gamma \geq (\alpha+1)\,M+1$ we conclude that

$$\frac{1-|z|^{2\,Re\gamma}}{Re\,\gamma}\left|\frac{zp''(z)}{p'(z)}\right| \leq 1,\; z \in U \tag{2.10}$$

Now (2.10) and Theorem 1.1 imply that the function $H_{\beta,\gamma}(z)$ is in the class S.

Theorem 2.2 *Let* $g(z) \in L(\alpha),\; 0 < \alpha \leq 2,\; g(z) = z + \sum_{k=3}^{\infty} a_k z^k,\; \gamma \in C,\; Re\,\gamma > 0,\; M > 1$ *and*

$$\frac{|\gamma-1|}{Re\,\gamma} \leq \frac{1}{(\alpha+1)M+1.} \tag{2.11}$$

If $|g(z)| < M,\; z \in U$, *then the function*

$$G_\gamma(z) = \left[\gamma \int_0^z g^{\gamma-1}(u)\,du\right]^{\frac{1}{\gamma}} \tag{2.12}$$

is in the class S.

Proof. From (2.12) we have

$$G_\gamma(z) = \left[\gamma \int_0^z u^{\gamma-1}\left(\frac{g(u)}{u}\right)^{\gamma-1} du\right]^{\frac{1}{\gamma}}. \tag{2.13}$$

Let us consider the function

$$h(z) = \int_0^z \left(\frac{g(u)}{u}\right)^{\gamma-1} du. \tag{2.14}$$

The function $h(z)$ is regular in U.

From (2.14) we obtain

$$h'(z) = \left(\frac{g(z)}{z}\right)^{\gamma-1},\; h''(z) = (\gamma-1)\left(\frac{g(z)}{z}\right)^{\gamma-2}\frac{zg'(z)-g(z)}{z^2}$$

and

$$\frac{1-|z|^{2\,Re\gamma}}{Re\gamma}\left|\frac{zh''(z)}{h'(z)}\right| = \frac{1-|z|^{2\,Re\gamma}}{Re\gamma}|\gamma-1|\left(\left|\frac{zg'(z)}{g(z)}\right|+1\right) \tag{2.15}$$

We see that

$$\frac{1-|z|^{2\,Re\gamma}}{Re\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\frac{1-|z|^{2\,Re\gamma}}{|\gamma|Re\gamma}\left(\left|\frac{z^2g'(z)}{g^2(z)}\right|\frac{|g(z)|}{|z|}+1\right) \tag{2.16}$$

and, hence by the Schwarz Lemma also $|g(z)| \leq M|z|$, we have

$$\frac{1-|z|^{2\,Re\gamma}}{Re\,\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\frac{1-|z|^{2\,Re\gamma}}{|\gamma|\,Re\gamma}\left(\left|\frac{z^2g'(z)}{g^2(z)}-1\right|M+M+1\right) \tag{2.17}$$

Because $g(z) \in L(\alpha)$ we have

$$\left|\frac{z^2g'(z)}{g^2(z)}-1\right| \leq \alpha|z^2|,\ z \in U \tag{2.18}$$

From (2.17) and (2.18) we obtain

$$\frac{1-|z|^{2\,Re\gamma}}{Re\,\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq |\gamma-1|\frac{(\alpha+1)M+1}{Re\,\gamma},\ z \in U \tag{2.19}$$

and using (2.11), (2.19) we get

$$\frac{1-|z|^{2\,Re\gamma}}{Re\,\gamma}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1,\ z \in U \tag{2.20}$$

Thus, using Theorem 1.1 for $\beta = \gamma$, we have $G_\gamma(z) \in S$.

Remark. For $0 < M \leq 1$, Theorem 2.1 and Theorem 2.2 hold only in the case $g(z) = kz,\ |k| = 1$.

References

[1] O. Mayer, *The functions theory of one complex variable*, Bucureşti, 1981.

[2] Z. Nehari, *Conformal mapping*, Mc Graw-Hill Book Comp., New York, 1952 (Dover. Publ. Inc., 1975).

[3] N. N. Pascu, *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilov (Braşov, 1987), University of Braşov, 1987, 43-48.

[4] V. Pescar, *New univalence criteria*, "Transilvania" University of Braşov, Braşov, 2002.

[5] D. Yang and J. Liu, *On a class of univalent functions*, Internat. J. Math. & Math. Sci., Vol. 22, no. 3(1999), 605-610.