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Cubic Dynamical Systems and Ternary Algebras

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Dedicated to Academician Radu Miron on his 80th Birthday

Abstract. With any cubic dynamical system S, modelled by a homogeneous cubic system of differential equations, a commutative ternary algebra $A([\cdot,\cdot,\cdot])$ is associated in a canonical way. It is shown how the properties of S are reflected by the properties of its associated algebra.

Keywords: dynamical system, cubic dynamical system, ternary algebra, commutative ternary algebra.

1 Introduction

Any physical system with a time development is usually called a *dynamical system*. It can be identified with the set of all its states at every instant of its lifetime. Roughly speaking, we say that a dynamical system is one whose states are changing in time. The rule of this time changing is a specific one for different classes of dynamical systems; it is usually called *dynamic*. The notion of *mathematical dynamical system* (or briefly, *dynamical system*) is an abstract representation of dynamical systems as a mathematical model. Consequently, a mathematical dynamical system consists of the space of states of the system together with a rule, its dynamic, for determining the state which corresponds at a given future time to a given present state. Determining such rules for various natural systems is a central problem of the science. Once the dynamic is given, it is the task of mathematical dynamical systems theory to investigate the patterns of how states change in the long run. As natural realizations of some dynamical systems we quote the evolution by natural selection, the evolution of every living individual, etc..

The theory of dynamical systems is an outlined domain of Mathematics, with remarkable achievements in the range of practical applications. In recent years mathematical dynamical systems have become increasingly common as models in biology, enzymology, population dynamics, epidemiology, ecology, physics, chemistry, economics, engineering and other fields. An example of a cubic dynamical system arising in population dynamics is due to Itoh [12]. The possibility of using the differential models or difference models is the final justification for the use of quantitative models in sciences.

Firstly, any cubic differential system (equation) is homogenized by using a standard algebraic procedure. Following the idea of L. Markus [17] (to associate a binary algebra with a homogeneous quadratic dynamical system), a ternary algebra is associated with any homogeneous cubic differential system. This allows us to identify a 1-1 correspondence between classes of (center-) affinely equivalent homogeneous cubic differential systems and classes of isomorphic commutative ternary algebras. In its turn, this correspondence induces a correspondence between the properties of homogeneous cubic differential systems and the properties of their associated algebras. Indeed, the nilpotent elements determine the critical points of the system, while the idempotent elements, if exists at least one, are implied in the stability of the null solution of the analyzed homogeneous system. Moreover, the nilpotents can be also used to construct first integrals for the differential system [30]. The structure of the associated algebra creates the opportunity to find suitable basis (on the ground space of the system) such that the system to receive a simpler form, as well as it exhibits the presence of special prime integrals. Since, in the finite dimensional case, the use of coordinates hides some essential features of the theory, we prefer to use a presentation in the frame of Banach spaces as long as this seems to be appropriately. Some results on quadratic differential equations in Banach spaces were proved in [21]. On another hand, although we have to work exclusively with center affine equivalences we call them shortly affine equivalences.

2 Preliminaries

Cubic dynamical systems. Let $E(\|\cdot\|_1)$ be a Banach space over the field K (here K is $\mathbb{R}$ or $\mathbb{C}$).

Definition 2.1 *a) A **cubic differential equation** (CDEq) on E is every differential equation of the form:*

$$\frac{dX}{dt} = a + L(X) + Q(X) + F(X) \tag{1}$$

where:
$a : E \to E$ - is the constant mapping taking everywhere the value $a \in E$,
$L : E \to E$ - is a continuous linear mapping ,
$Q : E \to E$ - is a continuous quadratic mapping (i.e., $Q(sx) = s^2Q(x), \forall s \in K, \forall x \in E$),
$F : E \to E$- is a continuous cubic mapping (i.e., $F(sx) = s^3F(x), \forall s \in K, \forall x \in E$).

*b) A **homogeneous cubic differential equation** (HCDEq) on E is every differential equation of the form:*

$$\frac{dX}{dt} = F(X) \tag{2}$$

where F:E $\to$ E - is a continuous cubic mapping.

Any dynamical system having as a mathematical model a (H)CDEq is called a (*homogeneous*) *cubic dynamical system.*

With every CDEq (1) on E, a HCDEq is associated on $\overline{E} = E \oplus K$, namely

$$\begin{cases} \dfrac{dX}{dt} = a\lambda^3 + L(X)\lambda^2 + Q(X)\lambda + F(X) \\ \dfrac{d\lambda}{dt} = 0. \end{cases} \tag{3}$$

To the solution $X(t)$ of every Cauchy problem for (1) with the initial condition $X(t_0) = x_0$ corresponds the solution $\overline{X}(t) = X(t) \oplus 1$ of the Cauchy problem (3) with $X(t_0) = x_0, \lambda(t_0) = 1$; more exactly there exists the 1-1 mapping

$$X(t) \Longleftrightarrow X(t) \oplus 1$$

between the set of solutions of (1) and (3), respectively. Actually, the homogenization is only a technical trick. It creates the opportunity to restrict our study on HCDEqs, only.

In the finite dimensional case, i.e., when $E = K^n$, (1) becomes

$$\frac{dx^i}{dt} = a^i + a^i_j x^j + a^i_{jk} x^j x^k + \overset{i}{a}_{jks} x^j x^k x^s, \quad i, j, k, s = 1, 2, ..., n, \tag{4}$$

while (2) becomes

$$\frac{dx^i}{dt} = \overset{i}{a}_{jks} x^j x^k x^s, \quad i, j, k, s = 1, 2, ..., n, \tag{5}$$

where $a^i, a^i_j, a^i_{jk}, a^i_{jks} \in K$, $a^i_{jk} = a^i_{kj}$ and $\overset{i}{a}_{j_1 j_2 j_3} = \overset{i}{a}_{j_{\sigma 1} j_{\sigma 2} j_{\sigma 3}}$, for all $\sigma \in S_3$.

Actually, such a differential system is a *system of cubic differential equations* (SCDEqs, for short), or a *system of homogeneous cubic differential equations* (SHCDEqs, for short), respectively; it is generically denoted by $\mathcal{S}$.

After applying the homogenization procedure, (4) becomes

$$\begin{cases} \dfrac{dx^i}{dt} = a^i \lambda^3 + a^i_j x^j \lambda^2 + a^i_{jk} \lambda x^j x^k + \overset{i}{a}_{jks} x^j x^k x^s, \quad i, j, k, s = 1, 2, ..., n, \\ \dfrac{d\lambda}{dt} = 0. \end{cases} \tag{6}$$

Itoh's Example. By considering a Lotka-Volterra system and taking into account the interaction of any neighboring three individuals in it, Itoh [12] obtains the following cubic system

$$\frac{d}{dt} p(t) = k_1 (p(t) \circ p(t) - p(t)) + k_2 ((p(t) \circ p(t)) \circ p(t) - p(t))$$

where $p(t) \in \mathbb{R}^n$ (for any t) and "$\circ$" denote a binary composition on $\mathbb{R}^n$ defined, in the natural basis $(e_1, e_2, ..., e_n) \subset \mathbb{R}^n$, by

$$e_i \circ e_j = (\frac{1}{2} + a_{ij}) e_i + (\frac{1}{2} + a_{ji}) e_j;$$

here $a_{ij} = -a_{ji}$ and $-\frac{1}{2} \leq a_{ij} \leq \frac{1}{2}$. Actually, $\mathbb{R}^n(\circ)$ is usually called Lotka-Volterra algebra. It is a commutative (but nonassociative) algebra with a basis consisting in idempotent elements ($e_i^2 = e_i, i = 1, 2, ..., n$), only.

Ternary algebras. Let $E([\cdot, \cdot, \cdot])$ be a ternary algebra. For every pair $(x, y) \in E \times E$, we can consider the endomorphisms:

$$L_{x,y}, C_{x,y}, R_{x,y} : E \to E$$
$$L_{x,y}(z) = [x, y, z], C_{x,y}(z) = [x, z, y], R_{x,y}(z) = [z, x, y];$$

they are called, respectivelly, *left/medium/right multiplications* by (x, y). Actually, the ternary composition $[\cdot,\cdot,\cdot] : E \times E \times E \to E$ is a trilinear vector mapping and can be naturally identified with a (1,3)-tensor on E, which will be denoted by $G \in E^* \otimes E^* \otimes E^* \otimes E$ (namely, $G(x, y, z) = [x, y, z]$). Further, the set of all ternary algebras on E is identifiable with the tensor product $E^{*\otimes 3} \otimes E$.

The algebra $E([\cdot,\cdot,\cdot])$ can have several properties which either mim the properties of binary algebras or are suggested by them. For example, if the ternary composition is the superposition of the composition "$\cdot$" of a binary associative algebra, i.e., $G(x, y, z) = (x\cdot y)\cdot z$, then

$$G(G(x, y, z), u, v) = G(x, G(y, z, u), v) = G(x, y, G(z, u, v)), \quad \forall x, y, z, u, v \in E.$$

That is why, a ternary algebra $E([\cdot,\cdot,\cdot])$ is called *associative* if its composition satisfies the axiom:

$$[[x, y, z], u, v] = [x, [y, z, u], v] = [x, y, [z, u, v]], \qquad \forall x, y, z, u, v \in E.$$

Consequently, $E([\cdot,\cdot,\cdot])$ is associative if and only if one of the following five identities hold:

$$\begin{array}{l} L_{x,y} \circ L_{z,u} = L_{[x,y,z],u} = L_{x,[y,z,u]}, \\ C_{[x,y,z],u} = C_{x,u} \circ L_{y,z} = L_{x,y} \circ C_{z,u} \\ R_{u,z} \circ L_{x,y} = C_{x,z} \circ C_{y,u} = L_{x,y} \circ R_{u,z} \\ R_{u,y} \circ C_{x,z} = C_{x,y} \circ R_{z,u} = C_{x,[z,u,y]} \\ R_{u,v} \circ R_{y,z} = R_{[y,z,u],v} = R_{y,[z,u,v]}, \end{array}$$

for all $x, y, z, u \in E$.

A ternary algebra $E([\cdot,\cdot,\cdot])$ is called *commutative* or *symmetric* if

$$[x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}], \ \ \forall x_1, x_2, x_3 \in E, \ \ \forall \sigma \in S_3.$$

For every element x of any ternary algebra $E([\cdot,\cdot,\cdot])$ we can define inductively *left odd powers* x^{2n+1}, *medium odd powers* $x^{\{2n+1\}}$ and *right odd powers* $x^{[2n+1]}$, by

$$x^3 = [x, x, x], \quad x^{2n+1} = [x^{2n-1}, x, x], \quad \forall x \in E, n \geq 2,$$

$$x^{\{3\}} = [x, x, x], \quad x^{\{2n+1\}} = [x, x^{\{2n-1\}}, x], \quad \forall x \in E, n \geq 2,$$

or

$$x^{[3]} = [x, x, x], \quad x^{[2n+1]} = [x, x, x^{[2n-1]}], \quad \forall x \in E, n \geq 2,$$

respectively.

If $E([\cdot,\cdot,\cdot])$ is an associative algebra, then for all $x \in E$,

$$x^{2n+1} = x^{\{2n+1\}} = x^{[2n+1]}, \forall n \in \mathbb{N}$$

holds and, further,

$$[x^{2n+1}, x^{2m+1}, x^{2k+1}] = x^{2(n+m+k+1)+1}, \quad \forall n, m, k \in \mathbb{N},$$

i.e., every element has associative powers. A ternary algebra for which every element has associative powers, i.e.,

$$[x^{2n+1}, x^{2m+1}, x^{2k+1}] = x^{2(n+m+k+1)+1}, \forall n, m, k \in \mathbb{N}, \forall x \in E$$

is said to be *mono-associative* or *power-associative*. Such algebras can be characterized by identities connecting some of their multiplications; for instance, the left multiplications must satisfy the characteristic identities

$$\begin{aligned}&(L_{x,x} \circ L_{x,x})(x) = L_{x^3,x}(x) = L_{x,x^3}(x), \quad \forall x \in E,\\ &L^n_{x,x}(x) = (L^{n-1}_{x,x} \circ L_{x,x})(x) = L_{x^{2n-1},x}(x) = L_{x,x^{2n-1}}(x), \quad \forall x \in E, \forall n > 2.\end{aligned}$$

An element $e \in E$ is called *identity element* for E if

$$[x, e, e] = [e, x, e] = [e, e, x] = x, \quad \forall x \in E,$$

i.e., $L_{e,e} = C_{e,e} = R_{e,e} = id_E$.

It must be remarked that, for ternary algebras, the associativity do not implies the existence of an identity element (unlike with binary algebra case).

Examples.
1. The ternary algebra defined on $\mathbb{R}^2$, and having in the basis (a, b) the multiplication table

$$\begin{aligned}&[a, a, a] = b,\\ &[a, a, b] = [a, b, a] = [b, a, a] = a,\\ &[a, b, b] = [b, b, a] = [b, a, b] = b,\\ &[b, b, b] = a\end{aligned}$$

is an associative, commutative algebra having no identity element.
2. The ternary algebra defined on $\mathbb{R}^2$, and having in the basis (a, b) the multiplication table

$$\begin{aligned}&[a, a, a] = a,\\ &[a, a, b] = [a, b, a] = [b, a, a] = b,\\ &[a, b, b] = [b, b, a] = [b, a, b] = a,\\ &[b, b, b] = a\end{aligned}$$

is an associative, commutative algebra having a and b as identity elements.

Any ternary algebra $E([., ., .])$ having no identity element can be naturally embedded into a ternary algebra with identity element; more exactly, $\overline{E} = E \oplus K$ endowed with ternary composition

$$[x \oplus \lambda, y \oplus \mu, z \oplus \nu] = ([x, y, z] + \mu\nu x + \lambda\nu y + \lambda\mu z) \oplus (\lambda\mu\nu), \tag{7}$$

has the identity element $0 \oplus 1$.

Any element $e \in E \setminus \{0\}$ having the property $[e, e, e] = e$ is called *idempotent element* of E. An element $e \in E \setminus \{0\}$ having the property $[e, e, e] = 0$ is called *nilpotent* element of E; by definition, we consider $0 \in E$ be an nilpotent, too.

3 B-algebra associated with a HCDEq

Let $E(\|\cdot\|_1)$ be a Banach space over the field K (here K is $\mathbb{R}$ or $\mathbb{C}$) and (2) be a HCDEq on E.

The polar form of F is the symmetric trilinear vector form $G : E \times E \times E \to E$, defined by

$$\begin{gathered} G(X,Y,Z) = \frac{1}{6}\left[T(X+Y+Z)- \right. \\ \left. -T(X+Y) - T(Y+Z) - T(X+Z) + T(X) + T(Y) + T(Z)\right]. \end{gathered} \tag{1}$$

Let us denote by $E([\cdot,\cdot,\cdot])$ the symmetric ternary algebra defined on E by means of the ternary operation

$$[x,y,z] = G(x,y,z).$$

In general, $E([\cdot,\cdot,\cdot])$ is a nonassociative commutative algebra (more exactly, it is not necessarily an associative algebra).

Since G is a continuous mapping, there exists its usually defined norm $\|G\|_1$ (induced by the norm of E) such that the inequality $\|[x,y,z]\|_1 \le \|x\|_1 \cdot \|y\|_1 \cdot \|z\|_1$ holds for all $x, y, z \in E$. Let us define the new norm $\|\cdot\|$ on E by

$$\|x\| = \sqrt{\|G\|_1} \cdot \|x\|_1. \tag{2}$$

Then, one gets

$$\|[x,y,z]\| \le \|x\| \cdot \|y\| \cdot \|z\|, \quad \forall x \in E.$$

This norm on E will be mainly used in Section 5. The algebra $E([\cdot,\cdot,\cdot])$, endowed with the norm $\|\cdot\|$, becomes a normed algebra; it is called the *B-algebra associated* to HCDEq (2), or a *B-algebra*, for short. If no confusion is possible, we shall denote this B-algebra by E.

Consequently, a ternary symmetric (B-)algebra is naturaly associated with any HCDEq. This suggest to put $F(X) = X^3$, so that (2) gets the form

$$\frac{dX}{dt} = X^3. \tag{3}$$

4 The affine equivalence of two HCDEqs

Let us consider another HCDEq on the Banach space E', namely

$$\frac{dY}{dt} = F_1(Y), \tag{1}$$

where $F_1 : E' \to E'$ is a continuous cubic vector form on E'. We shall denote by G_1 the polar form for F_1. The ternary operation "$\{\cdot,\cdot,\cdot\}$" defined on E' by

$$\{x,y,z\} = G_1(x,y,z), \quad \forall x,y,z \in E'$$

endows E' with the algebraic structure of a commutative ternary algebra.

Definition 4.1 *The HCDEq* (2) *is said to be* ***center affinely equivalent*** (*or briefly,* ***equivalent***) *to HCDEq* (1) *if and only if there exists an invertible continuous linear mapping* $h : E' \to E$ *such that* $X = h(Y)$ *is a solution for* (2) *as long as* Y *is a solution for* (1). h *is called an* ***center affine equivalence*** *of* (*2*) *with* (*1*)

Convention. Although h is really a center affine equivalence, we shall name it everywhere in what follows, shorter and more comfortable, an *affine equivalence*.

Proposition 4.1 *The equation* (2) *is affinely equivalent to* (1) *if and only if there exists an invertible continuous linear mapping* $h : E' \to E$ *such that*

$$h \circ F_1 = F \circ h. \tag{2}$$

Proof. Let $y_o \in E'$ be an arbitrarily chosen (but fixed) element and $Y(t)$ be the solution of the Cauchy problem

$$\frac{dY}{dt} = F_1(Y), \qquad\qquad Y(t_0) = y_0. \tag{3}$$

Since (2) is equivalent to (1) there exists an invertible continuous linear transformation $h : E' \to E$ such that $X(t) = h(Y(t))$ is the solution of (2) with the initial condition $x_o = h(y_o)$. Consequently, the following equations hold:

$$\begin{aligned}\frac{dX(t)}{dt} &= F(X(t)) = \\ &= F(h(Y(t))) = h\left(\frac{dY(t)}{dt}\right) = h(F_1(Y(t)), \ \ \forall t \in I_1,\end{aligned} \tag{4}$$

i.e., $(F \circ h)(y_o) = (h \circ F_1)(y_o)$ (here I_1 is the domain of $Y(t)$). As y_o was arbitrarily chosen in E it follows that (2), holds. Conversely, if (2) holds and $Y(t)$ is the solution of problem (1), then the equations

$$\begin{aligned}\frac{dX(t)}{dt} &= h\left(\frac{dY(t)}{dt}\right) = (h \circ F_1)(Y(t)) = \\ &= (F \circ h)(Y(t))) = F(X(t)),\end{aligned} \tag{5}$$

also hold, i.e., $X(t)$ is a solution of equation (2) with the same domain as $Y(t)$.

As (2) is equivalent to $F_1 \circ h^{-1} = h^{-1} \circ F$ and h^{-1} is also a continuous mapping, it follows:

Corollary 4.2 *If* (2) *is equivalent to* (1), *then* (1) *is also equivalent to* (2).

Obviously, E' can be identified, via h, with E, so that it is enough to analyze the set of all HCDEqs on E, only. That is why, in what follows we tackle only HCDEqs defined on a fixed Banach space E.

Remark 4.3 *The binary relation defined by Definition 4.1 on the set of all HCDEqs on E is a reflexive, symmetric and transitive one so that it is really an equivalence relation.*

Theorem 4.4 *The equations* (2) *and* (1) *are equivalent if and only if the ternary algebras* $E([\cdot,\cdot,\cdot])$ *and* $E'(\{\cdot,\cdot,\cdot\})$ *are continuously isomorphic.*

Proof. If (2) and (1) are equivalent equations, then there exists an invertible continuous linear mapping $h : E' \to E$ which satisfies (2). By passing to the polar forms for F and F_1, it follows that h is necessarily an algebra isomorphism between $E'(\{\cdot,\cdot,\cdot\})$ and $E([\cdot,\cdot,\cdot])$. Conversely, if $h : E'(\{\cdot,\cdot,\cdot\}) \to E([\cdot,\cdot,\cdot])$ is a continuous algebra isomorphism, then (2) holds, i.e., h is an equivalence of the HCDEs (2) and (1).

Remark 4.5 *Theorem 4 ensures that there exists a bijection between the classes of affinely equivalent HCDEqs on* E *and the classes of isomorphic commutative ternary algebras on* E*. Consequently, there exists a correspondence between certain qualitative properties of a HCDEq* (2) *and the invariance properties under a continuous isomorphism of its associated commutative algebra.*

Remark 4.6 *In what follows, instead to say "*(2) *is a HCDEq on* E*" we prefer to say "*(2) *is a HCDEq on the ternary algebra* E*" considering implicitly that* E *is just the* B*-algebra associated with* (2) *by means of the construction before presented. The correspondence exhibited in Theorem 4 is the support of this convention.*

Example

The HCDSs associated with algebras from Examples 1, 2, Sect. 2, are

$$\begin{cases} \dfrac{dx}{dt} = 3x^2y + y^3 \\ \dfrac{dy}{dt} = x^3 + 3xy^2, \end{cases} \quad \text{and, respectively} \quad \begin{cases} \dfrac{dx}{dt} = x^3 + 3xy^2 \\ \dfrac{dy}{dt} = 3x^2y + y^3. \end{cases}$$

These two systems are not affinely equivalent because their associated ternary algebras are not isomorphic each other (indeed, only one of them has identity element).

It what follows we can always suppose that the associated B-algebra to a HCDEq has an identity element. Indeed, if the algebra $E([\cdot,\cdot,\cdot])$ has no identity element, then it can be embedded into the ternary algebra $\overline{E}([\cdot,\cdot,\cdot])$ with identity element $0 \oplus 1$ (see Section 2). Further, $\overline{E}([\cdot,\cdot,\cdot])$ becomes a B-algebra relative to the norm defined by $\|x \oplus \lambda\| = \|x\| + |\lambda|$. Thus, the trilinear symmetric form $\overline{G} : \overline{E} \times \overline{E} \times \overline{E} \to \overline{E}$, defined by

$$\overline{G}(x \oplus \lambda, y \oplus \mu, z \oplus \nu) = ([x,y,z] + \mu\nu x + \lambda\nu y + \lambda\mu z) \oplus (\lambda\mu\nu)$$

for all $x \oplus \lambda, y \oplus \mu, z \oplus \nu \in \overline{E}$, has the corresponding cubic form

$$\overline{F}(x \oplus \lambda) = (F(x) + 3\lambda^2 x) \oplus \lambda^3, \qquad \forall x \oplus \lambda \in \overline{E}, \tag{6}$$

and $\overline{F}$ is just the cubic form associated with the HCDEq

$$\begin{cases} \dfrac{dX}{dt} = F(X) + 3\lambda^2 X \\ \dfrac{d\lambda}{dt} = \lambda^3. \end{cases} \tag{7}$$

In this way, another Cauchy problem is associated with the Cauchy problem

$$\frac{dX}{dt} = F(X), \qquad X(t_0) = X_0 \tag{8}$$

namely:

$$\begin{cases} \frac{dX}{dt} = F(X) + 3\lambda^2 X \\ \frac{d\lambda}{dt} = \lambda^3. \end{cases} \qquad \begin{cases} X(t_0) = X_0 \\ \lambda(t_0) = 0. \end{cases} \tag{9}$$

There exists a bijection between the set of solutions $X(t)$ for problem (8) and the set of solutions for problem (9), namely

$$X(t) \Leftrightarrow X(t) \oplus 0$$

The B-algebra $\overline{E}([\cdot,\cdot,\cdot])$ inherits the most part of the properties of $E([\cdot,\cdot,\cdot])$. For instance, if E is an associative ternary algebra, then $\overline{E}([\cdot,\cdot,\cdot])$ is associative, too. Moreover, in the associative case, any ideal $\overline{I} \subset \overline{E}$ is an ideal in $E \oplus 0 \approx E$.

5 The analyticity of solutions of a CDEq

Since F is a continuous linear mapping, the Cauchy problem (8) has a unique solution. Cauchy-Kovalevskaja Theorem assures us that (8) has a unique analytic solution. We shall prove that any saturated solution is necessarily an analytic one by exploiting the particularities of our problem. This approach will be useful later in connection with the problem of solving Cauchy problems for a HCDEq whose associated ternary algebra satisfies to a weak associativity axiom.

Let $X : I \to E$ $(I \subset K)$ be the unique saturated solution of (8). It results that $X(t)$ is a continuous differentiable function on I and the following equalities hold :

$$\frac{dX(t)}{dt} = X^3(t), \quad X(t_0) = x_0, \quad \frac{dX(t_0)}{dt} = X^3(t_0).$$

As F is a continuous cubic form and $X(t)$ is a continuous differentiable mapping on I it results, by using a complete induction, that $X(t)$ is a C^∞-mapping on I. We shall consider the Taylor series associated to $X(t)$

$$X(t_0) + \frac{t-t_0}{1!}\frac{dX(t_0)}{dt} + \frac{(t-t_0)^2}{2!}\frac{d^2X(t_0)}{dt^2} + ... \tag{1}$$

$$... + \frac{(t-t_0)^n}{n!}\frac{d^nX(t_0)}{dt^n} + ... \ .$$

It can be inductively proved that

$$\left\| \frac{d^nX(t_0)}{dt^n} \right\| \le (2n-1)!! \cdot \|x_0\|^{2n+1}, \quad \forall n \ge 1. \tag{2}$$

Thus the series

$$\|x_0\| + \frac{|t-t_0|}{1!}\left\|\frac{dX(t_0)}{dt}\right\| + \frac{|t-t_0|^2}{2!}\left\|\frac{d^2X(t_0)}{dt^2}\right\| + ...$$

is upper bounded by the Taylor series for

$$\frac{\|x_0\|}{\sqrt{1-2(t-t_0)|x_0\|}}$$

namely,

$$\|x_0\| + \frac{|t-t_0|}{1!}\|x_0\|^3 + ... + \frac{(2n-1)!!}{n!}|t-t_0|^n\|x_0\|^{2n+1} + .. = \frac{\|x_0\|}{\sqrt{1-2|t-t_0|\cdot|x_0\|}}.$$

Consequently, the series (1) is absolutely and uniformly convergent for $2|t-t_0|\cdot\|x_0\| < 1$; it means that

$$X(t) = X(t_0) + \frac{t-t_0}{1!}\frac{dX(t_0)}{dt} + \frac{(t-t_0)^2}{2!}\frac{d^2X(t_0)}{dt^2} + ... \tag{3}$$

for any $t \in I$ such that $|t-t_0| < \frac{1}{2\|x_0\|}$. If t_1 belongs to the interior of the existence domain of $X(t)$ and $x_1 = X(t_1)$, then $X(t)$ is also the solution for the Cauchy problem

$$\frac{dX(t)}{dt} = X^3(t), \quad X(t_1) = x_1$$

for every $t \in I$ such that $|t-t_1| < \frac{1}{2\|x_1\|}$. As earlier, it follows that, in a neighborhood of t_1, the solution can be represented as a convergent power series, so that every solution of the Cauchy problem (8) is an analytic function on its whole existence domain.

6 The case of power-associative algebras

Let us suppose now that the B-algebra $E([\cdot,\cdot,\cdot])$ connected to the equation (2) is power-associative (or, mono-associative), i.e., every element $x \in E$ generates an associative subalgebra of E.

If $X(t)$ is the saturated solution of the problem (8), then

$$\frac{dX(t_0)}{dt} = x_0^3 = L_{x_0,x_0}(x_0);$$

it can be inductively proved that

$$\frac{d^nX(t_0)}{dt^n} = (2n-1)!!L^n_{x_0,x_0}(x_0), \quad \forall n > 1 \tag{1}$$

(here $L^n_{x_0,x_0}$ is recursively defined by $L^{n-1}_{x_0,x_0} \circ L_{x_0,x_0}$ for $n \geq 2$). In this case (3) gives

$$X(t) = (I + \frac{t-t_0}{1!} L_{x_0,x_0} + \frac{3!!(t-t_0)^2}{2!} L^2_{x_0,x_0} + ...$$
$$... + \frac{(2n-1)!!(t-t_0)^n}{n!} L^n_{x_0,x_0} + ...)(x_0) = (I - 2(t-t_0)L_{x_0,x_0})^{-1/2}(x_0)$$

for $2|t-t_0| \cdot \|L_{x_0 x_0}\| < 1$ (here I denotes the identity mapping on E). Let us consider the analytic mapping

$$Y(t) = (I - 2(t-t_0)L_{x_0,x_0})^{-1/2}(x_0) \tag{2}$$

defined for every t such that $2^{-1}(t-t_0)^{-1}$ is not in the spectrum of L_{x_0,x_0}. By applying Theorem 6.1 in Ch.II §4, [27], the equalities

$$G(Y(t), Y(t), Y(t)) =$$
$$= G\left(\sum_{i=0}^{\infty} \frac{(2i-1)!!(t-t_0)^i}{i!} L^i_{x_0,x_0}, \sum_{j=0}^{\infty} \frac{(2j-1)!!(t-t_0)^j}{j!} L^j_{x_0,x_0},\right.$$
$$\left.\sum_{k=0}^{\infty} \frac{(2k-1)!!(t-t_0)^k}{k!} L^k_{x_0,x_0}\right) =$$
$$= \sum_{p=1}^{\infty} \frac{(2p-1)!!(t-t_0)^{p-1}}{(p-1)!} L^p_{x_0,x_0}(x_0) = \frac{dY(t)}{dt}$$

hold, i.e., $Y(t)$ is a solution of the problem (8) for every t such that $2^{-1}(t-t_0)^{-1}$ is not in the spectrum of $L_{x_0 x_0}$. Consequently, the following result has been proved:

Theorem 6.1 *If the B-algebra $E([\cdot,\cdot,\cdot])$, associated (as before) to equation (2), is a power-associative algebra, then the saturated solution of problem (8) is (2) for every arbitrarily chosen element $x_o \in E$ and for every $t \in K$ such that $2^{-1}(t-t_0)^{-1}$ is not in the spectrum of L_{x_0,x_0}.*

Remark 6.2 *The computations performed in checking that $Y(t)$ is a solution for equation (2) can be similarly applied to the case when x_o has associative powers, only. Consequently, if x_o has associative powers, then (2) is the solution to the problem (8). That is why, in the real situations, it is necessary to check whether an element x_o has or has not associative powers. For commutative binary algebras this can be done by using the Albert's criterion (see [2]). Unfortunately, for a commutative ternary algebra such a criterion is not known yet.*

Remark 6.3 *If only a finite number of the powers of x_o are linearly independent, the algebra $K(x_o)$ spanned by these powers will be a finite-dimensional one. Let $(x_o, x_1, ..., x_s)$ be the base of $K(x_o)$ consisting of the first independent "powers" of x_o (for definition of powers of an element in a nonassociative algebra see Section 2). In this case, the solution to (8) takes the form*

$$X(t) = f^o(t)x_o + f^1(t)x_1 + ... + f^s(t)x_s, \tag{3}$$

where $f^i(t)_{i=0,1,2,...,s}$ satisfies the cubic differential system

$$\frac{df^i(t)}{dt} = \sum_{j,k,m}^{s} C^i_{jkm} f^j f^k f^m, \quad i,j,k,m = 1,2,...,s \tag{4}$$

with the initial conditions

$$f^0(t_0) = 1, \quad f^1(t_0) = 0, \quad , ..., f^s(t_0) = 0; \tag{5}$$

here C^i_{jkm} are the structure constants of the subalgebra $K(x_o)$ in base $(x_o, x_1, . .., x_s)$, i.e., they are defined by

$$[x_j, x_k, x_m] = \sum_{j,k,m=0}^{s} C^i_{jkm} x_i, \qquad j, k, m = 0, 1, ..., s.$$

This statement is obtained by imposing to $X(t)$ in (3) *to be a solution for* (2).

This situation is usually met in the case of finite-dimensional B-algebras. It warns us to the necessity to pay a special attention to the algebras with a single generator.

7 Certain properties of B-algebras

The problem of the existence of a correspondence between the qualitative properties of a CDEq and the properties of the associated B-algebra naturally occurs. Unfortunately, this problem is incompletely solved. We shall give here some partial answers.

a) Firstly, we shall show that the *critical points* of HCDEq (2) are in a 1-1 correspondence with the nilpotent elements of the associated algebra.

Definition 7.1 *The element $x_o \in E$ is called a* ***critical*** *(stationary) point for* (2) *if the constant mapping $X : K \to E$, $X(t) = x_o$, $t \in K$ is a solution to* (2).

By a straightforward computation we obtain the result :

Proposition 7.1 *The nonzero element $x_o \in E$ is a critical point for* (2) *if and only if $x_0^3 = 0$, i.e., x_o is a nilpotent element for $E([\cdot, \cdot, \cdot])$.*

If $x_o \in E$ $(x_o \neq 0)$ is a critical point for (2), then λx_o is also a critical point for (2) for every fixed $\lambda \in K$. Recall that, by definition, $0 \in E$ is a critical point for (2); it is an isolated critical point if and only if the algebra $E([\cdot, \cdot, \cdot])$ has no nonzero nilpotent elements.

b) Let $x_o \in E$ be an idempotent element. Then x_o has associative powers and, consequently, the corresponding Cauchy problem (8) can be solved using formula given in Theorem 6. The solution (see (2)) is

$$X(t) = [1 - 2(t - t_0)]^{-1/2} \cdot x_0; \tag{1}$$

it is an unbounded solution; actually it blows up in finite time. Of course, the same result is obtained by applying Remark 6.3. Indeed, in this case, we must find out the solution of the form $X(t) = f(t)x_o$ where $f(t)$ is the solution for the Cauchy problem

$$f'(t) = f^3(t), \qquad f(t_0) = 1;$$

its solution is just (1).

Remark 7.2 *The importance of the idempotent elements consists in the fact that they lead to unbounded solutions for* (2), *so that they give indirect qualitative information about the dynamical system modelled by the considered HCDEq. Further, they give information on the stability of the null solution of the analized HCDEq (see Proposition 8.1).*

c) Let E_0 be a closed ideal of $E([.,.,.])$ and E_1 be a closed vector subspace which is its complement in E (i.e., $E = E_0 \oplus E_1$). If we denote by $p_i : E \to E_i$ $(i = 0, 1)$ the two projectors associated with the direct sum decomposition of E, $X_i = p_i(X)$, $i = 0, 1$, then (2) becomes

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + 3G(X_0, X_0, X_1)+ \\ \qquad\qquad +3G(X_0, X_1, X_1) + (p_0 \circ F)(X_1) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases} \tag{2}$$

In the particular case when E_1 is also an ideal for $E([.,.,.])$, then

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) \\ \dfrac{dX_1}{dt} = F(X_1). \end{cases} \tag{3}$$

If E_1 is only a closed subalgebra of $E([.,.,.])$, then (2) becomes

$$\begin{cases} \dfrac{dX_0}{dt} = F(X_0) + 3G(X_0, X_0, X_1) + 3G(X_0, X_1, X_1) \\ \dfrac{dX_1}{dt} = (p_1 \circ F)(X_1). \end{cases} \tag{4}$$

Remark 7.3 *In every finite dimensional real binary algebra there exists either an idempotent element* (*at least one*) *or a nilpotent element* (*at least one*) (*see* [*13*], *for example*); *here "or" has not a disjunctive significance. Consequently, if HQDEq has* 0 *as an isolated critical point, then the associated algebra is a NN-algebra (i.e., algebra without nonzero nilpotents), it has at least an idempotent element and the HQDE has unbounded solutions. Moreover, if* $E([\cdot,\cdot,\cdot])$ *is either an associative algebra or a power-associative one, the idempotent element is closely connected with its structure by creating the possibility to give a Peirce decomposition* (*see* [*26*]) *of the basic space of this algebra. Unfortunately, there exists ternary commutative algebras which has no nilpotent or idempotent element. Indeed, the ternary algebra* $\mathbb{R}^2[\cdot,\cdot,\cdot]$ *having in basis* (a, b) *the multiplication table:*

$$\begin{array}{ll} [a,a,a] = a - b, & [b,b,b] = a + 3b, \\ [a,a,b] = [a,b,a] = [b,a,a] = a + \frac{1}{3}b, & [a,b,b] = [b,a,b] = [b,b,a] = a - \frac{1}{3}b, \end{array}$$

is a commutative algebra for which

$$[xa + yb, xa + yb, xa + yb] = (x^3 + 3x^2y + 3xy^2 + y^3)a + (-x^3 + x^2y - xy^2 + 3y^3)b.$$

By imposing $[xa + yb, xa + yb, xa + yb] = 0$ *and* $[xa + yb, xa + yb, xa + yb] = xa + yb$ *one gets two algebraic systems which has no nonzero solutions. Therefore,* $\mathbb{R}^2[\cdot,\cdot,\cdot]$ *has neither a nilpotent nor an idempotent element.*

The finite dimensional case will be analyzed later.

8 Qualitative properties of solutions

Since the function F in autonomous differential equation (2) is continuous, the existence and uniqueness theorem assures the existence of a flow $\Phi_t(x)$ which has the property $\Phi_t(ax) = a\Phi_{at}(x)$ for all $a \in \mathbb{R}$.

Let us consider the Cauchy problem (2)+(X(0)=P), where $P^3 = aP$, $a \in \mathbb{R}^*$. Then P has associative powers and, after using formula (2), one finds that

$$X_P(t) = \Phi_t(P) = a[1 - 2(t - t_0)]^{-1/2} \cdot P \tag{1}$$

is the solution of (2)+(X(0)=P). This solution is unbounded and blows up in finite positive time for $a > 0$ and in finite negative time for $a < 0$.

Proposition 8.1 *If $E([.,.,.])$ has an idempotent, then the origin $0 \in E$ is an unstable critical (or, steady state) point for (2).*

Proof. Let $e \in E$ be a nonzero idempotent. Taking $P = \alpha \cdot e$ with $\alpha > 0$ and using (1) it results that, in dependence with the absolute value of α, every neighborhood of the origin has a solution that starts in that neighborhood and that blow up. That is why the origin is unstable.

If the origin is stable for (2) then E([.,.,.]) has no idempotent; in particular, it has no identity element.

Proposition 8.2 *If the origin $0 \in E$ is stable and $E([.,.,.])$ has a (nonzero) nilpotent, then the origin is not asymptotically stable steady state point for (2).*

Proof. If $n \neq 0$ is a nilpotent, i.e., $n^3 = 0$, then $N_\lambda = \lambda \cdot n$ with $\lambda > 0$ is also a nilpotent. For any neighborhood V of 0 there exists $\lambda_0 > 0$ such that $N_{\lambda_0} \in V$. Since $lim_{t\to\infty} \parallel N_{\lambda_0} - 0 \parallel = \lambda_0 \parallel n \parallel \neq 0$, it results that 0 is not asymptotically stable solution.

Remark 8.3 *The results just proved before, as well as their proofs, generalize the similar results for quadratic differential systems ($QDSs$) stated and proved by $Kinyon\&Sagle$ [14]. For the results which follow the proofs are adaptations of the proofs for similar results obtained in [14] for QDSs. That is why, we shall give the statements of these results and avoid some proofs.*

It can be proved

Corollary 8.4 *If $E([.,.,.])$ is a commutative finite dimensional ternary algebra, which has either an (nonzero) idempotent or a nonzero nilpotent, then the origin $0 \in E$ is not asymptotically stable for (2).*

Theorem 8.5 *Let (2) be such that its associated algebra $E([.,.,.])$ has a symmetric positive definite, bilinear form $H : E \times E \to \mathbb{R}$. If H satisfies*

$$H(X, X^3) = 0, \quad \forall X \in E,$$

(or $H(X, X^3) \leq 0,\ \forall X \in E$) the origin $0 \in E$ is stable.

Obviously, the existence of H implies the nonexistence of an idempotent (indeed, if $e^3 = e$ and $H(e, e^3) = 0 = H(e, e)$, then necessarily $e = 0$).

Theorem 8.6 *Let $E([., ., .])$ be the ternary algebra associated with HCDEq* (2).
(1) *The trajectory through $P \in E$ does not pass through aP for any $a \notin [0, 1]$. If P lies on a periodic trajectory, the trajectory through P does not pass through aP for any $a \neq 1$.*
(2) *If $\gamma \subset E$ is a periodic orbit with least period τ, then $a\gamma = \{aP | P \in \gamma\}$ is a periodic trajectory with least period for $a \neq 0$. Thus, scalar multiples of periodic orbits are periodic, and solutions of any period exist, provided that one periodic orbit exists.*
(3) *The periodic trajectories lie on cones.*

Proof. **(1)** Suppose that for some $r \in \mathbb{R}$, $\Phi_r = aP$, where $a \notin [0, 1]$. Then $0 < \dfrac{a}{a-1} < 1$ for any $a < 0$ and the following equalities hold:

$$\Phi_{\frac{ar}{a-1}}(P) = \Phi_{\frac{r}{a-1}+r}(P) = \Phi_{\frac{r}{a-1}}(\Phi_r(P)) = \Phi_{\frac{r}{a-1}}(aP) = a\Phi_{\frac{ar}{a-1}}(P).$$

This implies $\Phi_{\frac{ar}{a-1}}(P) = 0$ which is possible only when $P = 0$. The above performed computations can be similarly performed when $a > 1$. We deal now with the case $0 < a < 1$ in the particular case when P belongs to a periodic trajectory. If $P \neq 0$ lies on a periodic trajectory and $\tau > 0$ is the least positive number such that $\Phi_{t+\tau}(x) = \Phi_t(x)$, then let take $p \in \mathbb{N}$ such that $p > \dfrac{r}{(a-1)\tau}$. Thus,

$$\Phi_{\frac{ar}{a-1}}(P) = \Phi_{\frac{ar}{a-1}+p\tau}(P) = \Phi_{\frac{br}{b-1}}(P) = b\Phi_{\frac{br}{b-1}}(P)$$

where $b = \dfrac{ar + p\tau(a-1)}{p\tau(a-1) + r} > 1$. The proof of **(1)** is finished.
The proofs for **(2)** and **(3)** can be performed in the same way as the proofs of the similar results for QDSs, stated in Theorem 3.10, [14].

Theorem 8.7 *Let $E([., ., .])$ be the ternary algebra associated with HCDEq* (2). *Then no periodic orbit is an attractor.*

Proof. See proof of Theorem 3.12, [14].

Theorem 8.8 *Let* (2) *be a HCDEq for which its associated ternary algebra $E([., ., .])$ is power-associative. Then* (2) *has no periodic (nonconstant) solution.*

Proof. This time, the corresponding flux is $\Phi_t(x) = (I - 2tL_{x,x})^{-\frac{1}{2}}(x)$. Let P such that $\Phi_t(P)$ is periodic of period $\tau > 0$. Then $(I - 2\tau L_{P,P})^{-\frac{1}{2}}(P) = P$ and $(I - 2\tau L_{P,P})^{-1}(P) = (I - 2\tau L_{P,P})^{-\frac{1}{2}}(P) = P$, i.e., $P = (I - 2\tau L_{P,P})(P)$. Thus $P^3 = 0$, which is impossible (otherwise P would be a critical point).

9 Finite dimensional cubic differential systems

Definition 9.1 *Any system of the form*

$$\frac{dx^i}{dt} = a^i_{jkm} x^j x^k x^m, \qquad i, j, k, m = 1, 2, ..., n \tag{1}$$

where $a^i_{jkm} \in \mathbb{R}$ *and* $a^i_{i_1,i_2,i_3} = a^i_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}}$, $\forall \sigma \in S_3$ *and* x^i *are real functions is called* ***homogeneous cubic differential system*** *(shortly, HCDS) on* $\mathbb{R}^n$.

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ an invertible linear mapping, $\mathcal{B} = \{e_1, e_2, ..., e_n\}$ a basis for $\mathbb{R}^n$ and $[T]_{\mathcal{B}} = [t^i_j]$ be the matrix of T in basis $\mathcal{B}$. If $y^i = t^i_j x^j$ are the equations of T, then HCDS (1) is transformed in

$$\frac{dy^i}{dt} = \widetilde{a}^i_{jkm} y^j y^k y^m, \qquad i, j, k, m = 1, 2, ..., n \tag{2}$$

where

$$\widetilde{a}^i_{jkm} = t^i_s a^s_{pqr} \widetilde{t}^p_j \widetilde{t}^q_k \widetilde{t}^r_m \tag{3}$$

where $[\widetilde{t}^i_j]$ is the inverse matrix for $[t^i_j]$.

Definition 9.2 *It is said that two HCDSs are affinely equivalent if there exists an invertible linear mapping which transform a system in the other.*

It must be remarked that the coefficients of HCDS (1) are, as well as the structure constants of a ternary algebra on $\mathbb{R}^n$ in a fixed (arbitrarily choosen) basis $\mathcal{B} = \{e_1, e_2, ..., e_n\}$, the components of a (1,3) affine tensor. This suggests to associate with HCDS (1) the ternary algebra on $\mathbb{R}^n$ which has, in the basis $\mathcal{B}$, the structure constants a^i_{jkm} (i.e., $[e_j, e_k, e_m] = a^i_{jkm} e_i$). This procedure achives a bijective correspondence between the classes of affinely equivalent HCDSs and the classes of isomorphic symmetric ternary algebras on $\mathbb{R}^n$. Consequently, the problem of classification up to an affine equivalence of HCDSs is equivalent with the problem of classification up to un isomorphism of symmetric ternary algebras.

Remark 9.1 *There exists a wider class of differential systems which can be transformed easily in a HCDS. It is the class of cubic differential systems* (*CDS*) *having the form:*

$$\frac{dx^i}{dt} = a^i + a^i_j x^j + a^i_{jk} x^j x^k + a^i_{jkm} x^j x^k x^m, \qquad i, j, k, m = 1, 2, ..., n \tag{4}$$

where $a^i, a^i_j, a^i_{jk}, a^i_{jkm} \in \mathbb{R}$ *and* $a^i_{jk} = a^i_{kj}$, $a^i_{i_1,i_2,i_3} = a^i_{i_{\sigma(1)},i_{\sigma(2)},i_{\sigma(3)}}$, $\forall \sigma \in S_3$. *Any CDS* (*4*) *is transformed, by the usual procedure of homogenisation, in a HCDS, namely*

$$\begin{cases} \dfrac{dx^i}{dt} = a^i y^3 + a^i_j x^j y^2 + a^i_{jk} x^j x^k y + a^i_{jkm} x^j x^k x^m, \\ \dfrac{dy}{dt} = 0, \end{cases} \tag{5}$$

for i,j,k,m =1,2,...,n. Certainly, we are interested in the solutions of HCDS (*5*) *for which* $y(t_0) = 1$.

Example. Let us consider HCDS din $\mathbb{R}^2$

$$\begin{cases} \dfrac{dx}{dt} = x^3 - 3xy^2 \\ \dfrac{dy}{dt} = 3x^2 y - y^3 \end{cases} \tag{6}$$

The coefficients of (6) are

$$a^1_{111} = 1, \quad a^1_{122} = a^1_{212} = a^1_{221} = -1,$$
$$a^2_{112} = a^2_{121} = a^2_{211} = 1, \quad a^2_{222} = 1.$$

The corresponding ternary algebra has, in the basis $\mathcal{B} = \{e_1, e_2\}$, the multiplication table

$$[e_1, e_1, e_1] = e_1, \quad [e_1, e_2, e_2] = [e_2, e_1, e_2] = [e_2, e_2, e_1] = -e_1,$$
$$[e_1, e_1, e_2] = [e_1, e_2, e_1] = [e_2, e_1, e_1] = e_2, \quad [e_2, e_2, e_2] = e_2.$$

Consequently, we get

$$\begin{aligned}&[x_1e_1 + y_1e_2, x_2e_1 + y_2e_2, x_3e_1 + y_3e_2] =\\ &= (x_1x_2x_3 - x_1y_2y_3 - y_1x_2y_3 - y_1y_2x_3)e_1+\\ &+(x_1x_2y_3 + x_1y_2x_3 + y_1x_2x_3 - y_1y_2y_3)e_2.\end{aligned}$$

The multiplication table assures us that e_1 is an identity element of the algebra. By a straightforward computation, it can be easily checked that this ternary algebra is associative. Indeed, if we denote $x = x_1e_1 + y_1e_2$, $y = x_2e_1 + y_2e_2$, $z = x_3e_1 + y_3e_2$, then $[x, y, z] = (x \cdot y) \cdot z$ where $(x_1e_1 + y_1e_2) \cdot (x_2e_1 + y_2e_2) = (x_1x_2 - y_1y_2)e_1 + (x_1y_2 + y_1x_2)e_2$ is just the complex multiplication on $\mathbb{R}^2$.

According with Theorem 6, every Cauchy problem for (6) has solution which can be determined by formula (2). If $X(t) = (x(t), y(t)), X(t_0) = X_0 = (x_0, y_0)$, one finds

$$L_{X_0,X_0} = \begin{bmatrix} x_0^2 - y_0^2 & -2x_0y_0 \\ 2x_0y_0 & x_0^2 - y_0^2 \end{bmatrix} \tag{7}$$

which have the complex eigenvalues $\lambda_{1,2} = x_0^2 - y_0^2 \pm 2ix_0y_0 = (x_0 \pm iy_0)^2$. If $X_0 = r(\cos\theta + i\sin\theta)$, then $\lambda_{1,2} = r^2(\cos 2\theta + i \sin 2\theta)$ and the solution is

$$\begin{cases} x(t) = \Re\{z_0(1 - 2(t - t_0)z_0^2)^{-\frac{1}{2}}\} \\ y(t) = \Im\{z_0(1 - 2(t - t_0)z_0^2)^{-\frac{1}{2}}\}, \end{cases} \tag{8}$$

where for ”square” is chosen the appropriate determination.

10 Concluding Remarks

There exists a 1-1 correspondence between the classes of affine equivalent cubic dynamical systems (shortly, CDSs) on a fixed Banach space E and classes of isomorphic commutative ternary algebras on E. This correspondence induces a complex correspondence between the properties of CDSs and those of the ternary algebras them associated as before. Indeed, to the distinguished elements of the algebras such as nilpotent, idempotent (in particular, identity) elements correspond not only particular solutions a CDS but some quantum of information about them. The structure of commutative ternary algebras is reflected in particular forms of the associated CDSs. Certainly, this application of ternary algebras impulses the systematic study of such algebras. There exists obvious and essential differences between binary and ternary algebras that was before shown. For instance, an associative ternary algebra has not necessarily an identity or, in case of existence, there exist sometimes more than one identity. But, instead of any appearance concerning possibly differences, there exists many common places for binary and ternary algebras. An important encouraging sign is the possibility, exhibited by Kinyon&Sagle [14], of transform any CDS in a QDS. We deal with this problem in a forthcoming paper.

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