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On the Saint-Venant's Problem in Microstretch Elasticity

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Abstract. The aim of this paper is to study the Saint-Venant's problem for right cylinders with general cross-section made of anisotropic elastic microstretch materials. We consider that the cylinders have cross-section inhomogeneity. The solution of the relaxed Saint-Venant's problem is written as the sum of two fields which belong to two classes of semi-inverse solutions. These two classes of semi-inverse solutions for the Saint-Venant's problem are described in terms of the solutions of five generalized plane strain problems.

Keywords: Saint-Venant's problem, semi-inverse solution, anisotropic elastic microstretch materials.

Mathematics Subject Classification (2000): 74B99, 74E05, 74E10, 74G05.

1 Introduction

The theory of elastic bodies with microstretch was introduced by Eringen [4,5]. In the microstretch theory, the particles of the solid can expand and contract independent of translations and the rotations which they execute. This theory can adequately describe the behavior of materials that have internal structure, such as: composite materials reinforced with elastic fibers or porous bodies with pores filled with gas etc. [5], having applications in many fields of engineering such as petroleum industry, material science, biology, etc. A very useful study for the Saint-Venant's problem and Saint-Venant's principle is made by Ieşan [9]. Chiriţă [2] finds two classes of semi-inverse solutions for the Saint-Venant's problem in linear viscoelasticity and in terms of functions of these classes, he gives the general solution for the relaxed Saint-Venant's problem. This method was also used by Chiriţă [1] in the study of Almansi's problem for viscoelastic cylinder. For the inhomogeneous anisotropic elastic materials with voids, Ghiba [7] studied the Saint-Venant's problem for right cylinders. Using these results, Ghiba [7,8] obtained the exact solutions for isotropic and transversal isotropic porous elastic materials.

The deformation of isotropic microstretch elastic cylinders has been studied by Ieşan and Nappa [12,13], Ieşan [10] and by De Cicco and Nappa [3]. The extension, bending and torsion problem for anisotropic microstretch elastic cylinders have been studied by Scalia [14].

In this paper, for the anisotropic and cross-section inhomogeneous elastic materials with microstretch, we find the solution of the relaxed Saint-Venant's problem as a sum of two fields which belong to two classes of semi-inverse solutions. Following the method used by Chiriţă [1,2] and Galeş [6], we analyze the Saint-Venant's problem using some generalized plane strain problems. With the help of these two classes of semi-inverse solutions, we construct the solution for the extension-bending-torsion problem and for the flexure problem, respectively. The results of the present paper can be used to give the exact solutions in the case of isotropic microstretch elastic materials and also for microstretch elastic materials with various anisotropy.

2 Saint Venant's problem

Consider B the interior of a cylinder of length L and we denote by S the lateral boundary of the cylinder. We choose a rectangular Cartesian system $Ox_1x_2x_3$ so that the generator of the cylinder is parallel with the Ox_3 axis and one ends lies in the x_1Ox_2 plane. We denote by ∂B the boundary of B and by $D(x_3) \subset \mathbb{R}^2$ the interior of the bounded cross-section situated at distance x_3 from the x_1Ox_2 plane. In this paper, the Latin subscripts and superscripts are understood to range over the integers $1, 2, 3$, unless we specify else, whereas Greek subscripts and superscripts are confined to the range $1, 2$; summation over repeated subscripts is implied and comma followed by a subscript to denote partial derivative with respect to the corresponding Cartesian coordinate; where no confusion may occur, we suppress the dependence upon the spatial variables.

Let u_i be the components of the displacement vector, φ_i the components of the micro-rotation vector and ψ the microstretch function. We denote by $\mathbf{U}$ the seven-dimensional vector (u_i, φ_i, ψ). In the microstretch theory, the linear strain measures are given by

$$e_{ij}(\mathbf{U}) = u_{j,i} + \varepsilon_{jik}\varphi_k, \qquad \kappa_{ij}(\mathbf{U}) = \varphi_{j,i}, \qquad \gamma_i(\mathbf{U}) = \psi_{,i}. \tag{2.1}$$

The equilibrium equations of the linear theory of microstretch elastic solids, in the absence of body loads, are [4]

$$t_{ji,j}(\mathbf{U}) = 0, \qquad m_{ji,j}(\mathbf{U}) + \varepsilon_{irs}t_{rs}(\mathbf{U}) = 0, \qquad \pi_{i,i}(\mathbf{U}) - \sigma(\mathbf{U}) = 0 \tag{2.2}$$

and the constitutive equations are

$$\begin{aligned} t_{ij}(\mathbf{U}) &= A_{ijrs}e_{rs} + B_{ijrs}\kappa_{rs} + D_{ijr}\gamma_r + a_{ij}\psi, \\ m_{ij}(\mathbf{U}) &= B_{rsij}e_{rs} + C_{ijrs}\kappa_{rs} + E_{ijr}\gamma_r + b_{ij}\psi, \\ \pi_i(\mathbf{U}) &= D_{rsi}e_{rs} + E_{rsi}\kappa_{rs} + c_{ij}\gamma_j + d_i\psi, \\ \sigma(\mathbf{U}) &= a_{ij}e_{ij} + b_{ij}\kappa_{ij} + d_i\gamma_i + \xi\psi, \end{aligned} \tag{2.3}$$

where t_{ij} is the stress tensor, m_{ij} is the couple stress tensor, π_i is the microstress vector, σ is the scalar microstress function, ε_{ijk} is the alternating symbol and $A_{ijrs}, B_{ijrs}, C_{ijrs}, D_{ijr}, E_{ijr}, a_{ij}, b_{ij}, c_{ij}, d_i, \xi$ are constitutive coefficients which satisfy the symmetry relations

$$A_{ijrs} = A_{rsij}, \qquad C_{ijrs} = C_{rsij}, \qquad c_{ij} = c_{ji}. \tag{2.4}$$

In this paper we assume that the microstretch material which fill the cylinder is cross-section inhomogeneous. Thus, we have

$$\begin{gathered}A_{ijrs}=A_{ijrs}(x_1,x_2),\quad B_{ijrs}=B_{ijrs}(x_1,x_2),\quad C_{ijrs}=C_{ijrs}(x_1,x_2),\\ D_{ijr}=D_{ijr}(x_1,x_2),\quad E_{ijr}=E_{ijr}(x_1,x_2),\quad a_{ij}=a_{ij}(x_1,x_2),\\ b_{ij}=b_{ij}(x_1,x_2),\quad c_{ij}=c_{ij}(x_1,x_2),\quad d_i=d_i(x_1,x_2)\quad \xi=\xi(x_1,x_2).\end{gathered}\tag{2.5}$$

The surface loadings at a regular point $\mathbf{x}$ on ∂B, are given by

$$t_i(\mathbf{U})=t_{ji}(\mathbf{U})n_j,\qquad m_i(\mathbf{U})=m_{ji}(\mathbf{U})n_j,\qquad \pi(\mathbf{U})=\pi_i(\mathbf{U})n_i,\tag{2.6}$$

where n_i are the components of the unit outward normal vector to ∂B at $\mathbf{x}$.

Solving the Saint-Venant's problem for B, means to determine the components of the vector $\mathbf{U}$, i.e. the displacement vector, the microrotation vector and the microstretch function, solutions of the equilibrium equations (2.1)–(2.3), when we require

$$\begin{aligned}&t_i(\mathbf{U})=0,\quad m_i(\mathbf{U})=0,\quad \pi(\mathbf{U})=0\qquad \text{on the lateral surface } S,\\ &t_i(\mathbf{U})=t_i^{(1)},\quad m_i(\mathbf{U})=m_i^{(1)},\quad \pi(\mathbf{U})=\pi^{(1)}\qquad \text{on the end } D(0),\\ &t_i(\mathbf{U})=t_i^{(2)},\quad m_i(\mathbf{U})=m_i^{(2)},\quad \pi(\mathbf{U})=\pi^{(2)}\qquad \text{on the end } D(L),\end{aligned}\tag{2.7}$$

where $(t_i^{(1)},m_i^{(1)},\pi^{(1)})$ and $(t_i^{(2)},m_i^{(2)},\pi^{(2)})$ are functions preassigned on $D(0)$ and $D(L)$, respectively.

Necessary and sufficient conditions for the existence of the solution of this problem are given by

$$\begin{aligned}&\int_{D(0)}t_i^{(1)}da+\int_{D(L)}t_i^{(2)}da=0,\\ &\int_{D(0)}\left(\varepsilon_{ijk}x_jt_k^{(1)}+m_i^{(1)}\right)da+\int_{D(L)}\left(\varepsilon_{ijk}x_jt_k^{(2)}+m_i^{(2)}\right)da=0.\end{aligned}\tag{2.8}$$

Introducing (2.1) and (2.3) into (2.2) and (2.7) and assuming zero body loads, we obtain the boundary value problem for $\mathbf{U}$ defined by the equations

$$\begin{aligned}\mathcal{T}_i(\mathbf{U})\equiv{}&[A_{jirs}(u_{s,r}+\varepsilon_{srk}\varphi_k)+B_{jirs}\varphi_{s,r}+D_{jir}\psi_{,r}+a_{ji}\psi]_{,j}=0,\\ \mathcal{S}_i(\mathbf{U})\equiv{}&[B_{rsji}(u_{s,r}+\varepsilon_{srk}\varphi_k)+C_{jirs}\varphi_{s,r}+E_{jir}\psi_{,r}+b_{ji}\psi]_{,j}\\ &+\varepsilon_{irs}[A_{rspq}(u_{q,p}+\varepsilon_{qpk}\varphi_k)+B_{rspq}\varphi_{q,p}+D_{rsp}\psi_{,p}+a_{rs}\psi]=0,\\ \mathcal{P}(\mathbf{U})\equiv{}&[D_{rsi}(u_{s,r}+\varepsilon_{srk}\varphi_k)+E_{rsi}\varphi_{s,r}+c_{ij}\psi_{,j}+d_i\psi]_{,i}\\ &-a_{ij}(u_{j,i}+\varepsilon_{jik}\varphi_k)-b_{ij}\varphi_{j,i}-d_i\psi_{,i}-\xi\psi=0\end{aligned}\tag{2.9}$$

in $B=D\times(0,L)$, the lateral boundary conditions

$$\begin{aligned}\mathcal{A}_i(\mathbf{U})&\equiv[A_{\alpha irs}(u_{s,r}+\varepsilon_{srk}\varphi_k)+B_{\alpha irs}\varphi_{s,r}+D_{\alpha ir}\psi_{,r}+a_{\alpha i}\psi]n_\alpha=0,\\ \mathcal{B}_i(\mathbf{U})&\equiv[B_{rs\alpha i}(u_{s,r}+\varepsilon_{srk}\varphi_k)+C_{\alpha irs}\varphi_{s,r}+E_{\alpha ir}\psi_{,r}+b_{\alpha i}\psi]n_\alpha=0,\\ \mathcal{C}(\mathbf{U})&\equiv[D_{rs\alpha}(u_{s,r}+\varepsilon_{srk}\varphi_k)+E_{rs\alpha}\varphi_{s,r}+c_{\alpha j}\psi_{,j}+d_\alpha\psi]n_\alpha=0\end{aligned}\tag{2.10}$$

on $\partial D\times(0,L)$ and the end boundary conditions

$$\begin{aligned}&t_{3i}(\mathbf{U})=t_i^{(1)},\quad m_{3i}(\mathbf{U})=m_i^{(1)},\quad \pi_3(\mathbf{U})=\pi^{(1)}\qquad \text{on } D(0),\\ &t_{3i}(\mathbf{U})=t_i^{(2)},\quad m_{3i}(\mathbf{U})=m_i^{(2)},\quad \pi_3(\mathbf{U})=\pi^{(2)}\qquad \text{on } D(L).\end{aligned}\tag{2.11}$$

The internal energy density

$$\begin{aligned}2\mathcal{W}(\mathbf{U}) = {} & A_{ijrs}e_{ij}(\mathbf{U})e_{rs}(\mathbf{U}) + C_{ijrs}\kappa_{ij}(\mathbf{U})\kappa_{rs}(\mathbf{U}) + c_{ij}\gamma_i(\mathbf{U})\gamma_j(\mathbf{U}) + \xi\psi^2(\mathbf{U}) \\ & +2B_{ijrs}e_{ij}(\mathbf{U})\kappa_{rs}(\mathbf{U}) + 2D_{ijr}e_{ij}(\mathbf{U})\gamma_r(\mathbf{U}) + 2E_{ijr}\kappa_{ij}(\mathbf{U})\gamma_r(\mathbf{U}) \\ & +2a_{ij}e_{ij}(\mathbf{U})\psi(\mathbf{U}) + 2b_{ij}\kappa_{ij}(\mathbf{U})\psi(\mathbf{U}) + 2d_i\gamma_i(\mathbf{U})\psi(\mathbf{U})\end{aligned} \tag{2.12}$$

is assumed to be a positive defined quadratic form in terms of the quantities $e_{ij}, \kappa_{ij}, \gamma_i$ and ψ.

3 The generalized plane strain state

For the interior of the cross section domain $D \subset \mathbb{R}^2$ we define the state of generalized plane strain to be the state in which the components of the displacement vector, of the microrotation vector and the microstretch function, $\mathbf{W} = (w_i, \nu_i, \omega)$, depend only on x_1 and x_2

$$w_i = w_i(x_1, x_2), \quad \nu_i = \nu_i(x_1, x_2), \quad \omega = \omega(x_1, x_2), \qquad (x_1, x_2) \in D \tag{3.1}$$

In view of (3.1), the linear strain measures take the form

$$\begin{aligned}&e_{\alpha j}(\mathbf{W}) = w_{j,\alpha} + \varepsilon_{j\alpha k}\nu_k\ , \quad e_{3\alpha}(\mathbf{W}) = -\varepsilon_{3\alpha\beta}\nu_\beta\ , \quad e_{33}(\mathbf{W}) = 0\ , \\ &\kappa_{\alpha j}(\mathbf{W}) = \nu_{j,\alpha}\ , \quad \kappa_{3j}(\mathbf{W}) = 0\ , \quad \gamma_\alpha(\mathbf{W}) = \omega_{,\alpha}\ , \quad \gamma_3(\mathbf{W}) = 0\ .\end{aligned} \tag{3.2}$$

From (3.2) and (2.3) we deduce that the components of the stress tensor T_{ij}, of the couple stress tensor M_{ij}, of the microstress vector Π_i and the scalar microstress function Σ are independent of x_3, i.e

$$T_{ij} = T_{ij}(x_1, x_2), \quad M_{ij} = M_{ij}(x_1, x_2), \quad \Pi_i = \Pi_i(x_1, x_2), \quad \Sigma = \Sigma(x_1, x_2), \quad (x_1, x_2) \in D. \tag{3.3}$$

In this case, the constitutive equations (2.3) become

$$\begin{aligned}T_{ij}(\mathbf{W}) &= A_{ij\alpha s}e_{\alpha s} + A_{ij3\alpha}e_{3\alpha} + B_{ij\alpha s}\kappa_{\alpha s} + D_{ij\alpha}\gamma_\alpha + a_{ij}\omega, \\ M_{ij}(\mathbf{W}) &= B_{\alpha sij}e_{\alpha s} + B_{3\alpha ij}e_{3\alpha} + C_{ij\alpha s}\kappa_{\alpha s} + E_{ij\alpha}\gamma_\alpha + b_{ij}\omega, \\ \Pi_i(\mathbf{W}) &= D_{\alpha si}e_{\alpha s} + D_{3\alpha i}e_{3\alpha} + E_{\alpha si}\kappa_{\alpha s} + c_{i\alpha}\gamma_\alpha + d_i\omega, \\ \Sigma(\mathbf{W}) &= a_{\alpha j}e_{\alpha j} + a_{3\alpha}e_{3\alpha} + b_{\alpha j}\kappa_{\alpha j} + d_\alpha\gamma_\alpha + \xi\omega.\end{aligned} \tag{3.4}$$

Given the body loads $F_i(x_1, x_2), G_i(x_1, x_2)$ and $H(x_1, x_2)$ on D, the equations of equilibrium in the case of generalized plane strain problems, take the form

$$T_{\alpha i,\alpha}(\mathbf{W}) + F_i = 0,\ M_{\alpha i,\alpha}(\mathbf{W}) + \varepsilon_{irs}T_{rs}(\mathbf{W}) + G_i = 0,\ \Pi_{\alpha,\alpha}(\mathbf{W}) - \Sigma(\mathbf{W}) + H = 0 \tag{3.5}$$

in D and the boundary conditions are

$$T_{\alpha i}(\mathbf{W})n_\alpha = \widetilde{T}_i, \quad M_{\alpha i}(\mathbf{W})n_\alpha = \widetilde{M}_i, \quad \Pi_\alpha(\mathbf{W})n_\alpha = \widetilde{\Pi} \tag{3.6}$$

on ∂D where $\widetilde{T}_i, \widetilde{M}_i$ and $\widetilde{\Pi}$ are prescribed functions that do not depend on x_3.

The generalized plane strain problem for $D \cup \partial D$ consists in finding a solution $\mathbf{W}$ of the boundary value problem given by the equations (3.5), (3.4) and (3.2) and the boundary conditions (3.6).

Substituting the relation (3.4) into (3.5) and (3.6), we can rewrite the above boundary value problem

$$\begin{aligned}
\mathfrak{T}_i(\mathbf{W}) &\equiv [A_{\alpha i\beta s}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+B_{\alpha i\beta s}\nu_{s,\beta}+D_{\alpha i\beta}\omega_{,\beta}+a_{\alpha i}\omega-\varepsilon_{3\beta\rho}A_{\alpha i3\beta}\nu_\rho]_{,\alpha}+F_i=0,\\
\mathfrak{S}_i(\mathbf{W}) &\equiv [B_{\beta s\alpha i}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+C_{\alpha i\beta s}\nu_{s,\beta}+E_{\alpha i\beta}\omega_{,\beta}+b_{\alpha i}\omega-\varepsilon_{3\beta\rho}B_{3\beta\alpha i}\nu_\rho]_{,\alpha}\\
&\quad +\varepsilon_{irs}[A_{rs\beta q}(w_{q,\beta}+\varepsilon_{s\beta k}\nu_k)+B_{rs\beta q}\nu_{q,\beta}+D_{rs\beta}\omega_{,\beta}+a_{rs}\omega-\varepsilon_{3\beta\rho}A_{rs3\beta}\nu_\rho]+G_i=0,\\
\mathfrak{P}(\mathbf{W}) &\equiv [D_{\beta s\alpha}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+E_{\beta s\alpha}\nu_{s,\beta}+c_{\alpha\beta}\omega_{,\beta}+d_\alpha\omega-\varepsilon_{3\beta\rho}D_{3\beta\alpha}\nu_\rho]_{,\alpha}\\
&\quad -[a_{\beta j}(w_{j,\beta}+\varepsilon_{j\beta k}\nu_k)+b_{\beta j}\nu_{j,\beta}+d_\beta\omega_{,\beta}+\xi\omega-\varepsilon_{3\beta\rho}a_{3\beta}\nu_\rho]+H=0 \quad \text{in } D,
\end{aligned} \tag{3.7}$$

with boundary conditions

$$\begin{aligned}
\mathfrak{A}_i(\mathbf{W}) &\equiv [A_{\alpha i\beta s}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+B_{\alpha i\beta s}\nu_{s,\beta}+D_{\alpha i\beta}\omega_{,\beta}+a_{\alpha i}\omega-\varepsilon_{3\beta\rho}A_{\alpha i3\beta}\nu_\rho]n_\alpha=\widetilde{T}_i,\\
\mathfrak{B}_i(\mathbf{W}) &\equiv [B_{\beta s\alpha i}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+C_{\alpha i\beta s}\nu_{s,\beta}+E_{\alpha i\beta}\omega_{,\beta}+b_{\alpha i}\omega-\varepsilon_{3\beta\rho}B_{3\beta\alpha i}\nu_\rho]n_\alpha=\widetilde{M_i},\\
\mathfrak{C}(\mathbf{W}) &\equiv [D_{\beta s\alpha}(w_{s,\beta}+\varepsilon_{s\beta k}\nu_k)+E_{\beta s\alpha}\nu_{s,\beta}+c_{\alpha\beta}\omega_{,\beta}+d_\alpha\omega-\varepsilon_{3\beta\rho}D_{3\beta\alpha}\nu_\rho]n_\alpha=\widetilde{\Pi} \quad \text{on } \partial D.
\end{aligned} \tag{3.8}$$

We assume that $F_i, G_i, H \in C^\infty(D)$ and $\widetilde{T}_i, \widetilde{M_i}, \widetilde{\Pi} \in C^\infty(\partial D)$.

The above generalized plane strain problem has solutions (Scalia [14]) belonging to $C^\infty(\overline{D})$ if and only if

$$\begin{aligned}
&\int_D F_i da + \int_{\partial D} \widetilde{T}_i ds = 0,\\
&\int_D (\varepsilon_{3\alpha\beta}x_\alpha F_\beta + G_3)da + \int_{\partial D} (\varepsilon_{3\alpha\beta}x_\alpha \widetilde{T}_\beta + \widetilde{M_3})ds = 0.
\end{aligned} \tag{3.9}$$

4 Analysis of Saint-Venant's problem by plane strain solutions

In this section, following the method developed by Chiriţă [1,2], we will reduce the Saint-Venant's problem to a plane strain problem, taking x_3 as a parameter. We consider the problem (2.9) and the boundary conditions (2.10) for the cross-section $D \cup \partial D$

$$\mathcal{T}_i(\mathbf{U}) = 0, \quad \mathcal{S}_i(\mathbf{U}) = 0, \quad \mathcal{P}(\mathbf{U}) = 0, \quad \text{in } D \tag{4.1}$$

and

$$\mathcal{A}_i(\mathbf{U}) = 0, \quad \mathcal{B}_i(\mathbf{U}) = 0, \quad \mathcal{C}(\mathbf{U}) = 0, \quad \text{on } \partial D, \tag{4.2}$$

considering $x_3 \in (0, L)$ as parameter.

The components of the resultant force and the resultant momentum of the traction about the origin of the system $Ox_1x_2x_3$, acting on the cross-section D are defined by

$$\mathcal{R}_i(\mathbf{U}) = \int_D t_{3i}(\mathbf{U})da, \quad \mathcal{M}_i(\mathbf{U}) = \int_D (\varepsilon_{ijk}x_j t_{3k}(\mathbf{U}) + m_{3i}(\mathbf{U}))da. \tag{4.3}$$

We remark that

$$\begin{aligned}
\mathcal{M}_\alpha(\mathbf{U}) &= \int_D (\varepsilon_{3\alpha\beta}x_\beta t_{33}(\mathbf{U}) + m_{3\alpha}(\mathbf{U}))da - x_3\varepsilon_{3\alpha\beta}\mathcal{R}_\beta(\mathbf{U}),\\
\mathcal{M}_3(\mathbf{U}) &= \int_D (\varepsilon_{3\alpha\beta}x_\alpha t_{3\beta}(\mathbf{U}) + m_{33}(\mathbf{U}))da.
\end{aligned} \tag{4.4}$$

We can rewrite the boundary value problem (4.1) and (4.2) in the form

$$\begin{aligned}
\mathcal{T}_i(\mathbf{U}) =& [A_{\alpha i\beta s}(u_{s,\beta} + \varepsilon_{s\beta k}\varphi_k) + B_{\alpha i\beta s}\varphi_{s,\beta} + D_{\alpha i\beta}\psi_{,\beta} + a_{\alpha i}\psi - \varepsilon_{3\beta\rho}A_{\alpha i3\beta}\varphi_\rho]_{,\alpha} \\
&+[A_{\alpha i3s}u_{s,3} + B_{\alpha i3s}\varphi_{s,3} + D_{\alpha i3}\psi_{,3}]_{,\alpha} + t_{3i,3}(\mathbf{U}) = 0\,, \\
\mathcal{S}_i(\mathbf{U}) =& [B_{\beta s\alpha i}(u_{s,\beta} + \varepsilon_{s\beta k}\varphi_k) + C_{\alpha i\beta s}\varphi_{s,\beta} + E_{\alpha i\beta}\psi_{,\beta} + b_{\alpha i}\psi - \varepsilon_{3\beta\rho}B_{3\beta\alpha i}\varphi_\rho]_{,\alpha} \\
&+\varepsilon_{irs}[A_{rs\beta q}(u_{q,\beta} + \varepsilon_{s\beta k}\varphi_k) + B_{rs\beta q}\varphi_{q,\beta} + D_{rs\beta}\psi_{,\beta} + a_{rs}\psi - \varepsilon_{3\beta\rho}A_{rs3\beta}\varphi_\rho] \\
&+[B_{3s\alpha i}u_{s,3} + C_{\alpha i3s}\varphi_{s,3} + E_{\alpha i3}\psi_{,3}]_{,\alpha} + \varepsilon_{irs}[A_{rs3q}u_{q,3} + B_{rs3q}\varphi_{q,3} + D_{rs3}\psi_{,3}] \\
&+m_{3i,3}(\mathbf{U}) = 0\,, \\
\mathcal{P}(\mathbf{U}) =& [D_{\beta s\alpha}(u_{s,\beta} + \varepsilon_{s\beta k}\varphi_k) + E_{\beta s\alpha}\varphi_{s,\beta} + c_{\alpha\beta}\psi_{,\beta} + d_\alpha\psi - \varepsilon_{3\beta\rho}D_{3\beta\alpha}\varphi_\rho]_{,\alpha} \\
&-[a_{\beta j}(u_{j,\beta} + \varepsilon_{j\beta k}\varphi_k) + b_{\beta j}\varphi_{j,\beta} + d_\beta\psi_{,\beta} + \xi\psi\varepsilon_{3\beta\rho}a_{3\beta}\varphi_\rho] \\
&+[D_{3s\alpha}u_{s,3} + E_{3s\alpha}\varphi_{s,3} + c_{\alpha 3}\psi_{,3}] \\
&-[a_{3j}u_{j,3} + b_{3j}\varphi_{j,3} + d_3\psi_{,3}] + \pi_{3,3}(\mathbf{U}) = 0 \quad \text{in } D
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
\mathcal{A}_i(\mathbf{U}) =& [A_{\alpha i\beta s}(u_{s,\beta} + \varepsilon_{s\beta k}\varphi_k) + B_{\alpha i\beta s}\varphi_{s,\beta} + D_{\alpha i\beta}\psi_{,\beta} + a_{\alpha i}\psi - \varepsilon_{3\beta\rho}A_{\alpha i3\beta}\varphi_\rho]n_\alpha \\
=& -[A_{\alpha i3s}u_{s,3} + B_{\alpha i3s}\varphi_{s,3} + D_{\alpha i3}\psi_{,3}]n_\alpha\,, \\
\mathcal{B}_i(\mathbf{U}) =& [B_{\beta s\alpha i}(u_{s,\beta} + \varepsilon_{s\beta k}\varphi_k) + C_{\alpha i\beta s}\varphi_{s,\beta} + E_{\alpha i\beta}\psi_{,\beta} + b_{\alpha i}\psi - \varepsilon_{3\beta\rho}B_{3\beta\alpha i}\varphi_\rho]n_\alpha \\
=& -[B_{3s\alpha i}u_{s,3} + C_{\alpha i3s}\varphi_{s,3} + E_{\alpha i3}\psi_{,3}]n_\alpha\,, \\
\mathcal{C}(\mathbf{U}) =& [D_{\beta s\alpha}(w_{s,\beta} + \varepsilon_{s\beta k}\nu_k) + E_{\beta s\alpha}\nu_{s,\beta} + c_{\alpha\beta}\omega_{,\beta} + d_\alpha\omega - \varepsilon_{3\beta\rho}D_{3\beta\alpha}\nu_\rho]n_\alpha \\
=& -[D_{3s\alpha}u_{s,3} + E_{3s\alpha}\varphi_{s,3} + c_{\alpha 3}\psi_{,3}]n_\alpha \quad \text{on } \partial D\,.
\end{aligned} \tag{4.6}$$

One can observe that the boundary value problem (4.5) and (4.6) can be viewed as a generalized plane strain boundary value problem with

$$\begin{aligned}
F_i(\mathbf{U}) =& [A_{\alpha i3s}u_{s,3} + B_{\alpha i3s}\varphi_{s,3} + D_{\alpha i3}\psi_{,3}]_{,\alpha} + t_{3i,3}(\mathbf{U})\,, \\
G_i(\mathbf{U}) =& [B_{3s\alpha i}u_{s,3} + C_{\alpha i3s}\varphi_{s,3} + E_{\alpha i3}\psi_{,3}]_{,\alpha} \\
&+\varepsilon_{irs}[A_{rs3q}u_{q,3} + B_{rs3q}\varphi_{q,3} + D_{rs3}\psi_{,3}] + m_{3i,3}(\mathbf{U})\,, \\
H(\mathbf{U}) =& [D_{3s\alpha}u_{s,3} + E_{3s\alpha}\varphi_{s,3} + c_{\alpha 3}\psi_{,3}] \\
&-[a_{3j}u_{j,3} + b_{3j}\varphi_{j,3} + d_3\psi_{,3}] + \pi_{3,3}(\mathbf{U})\,, \\
\widetilde{T}_i(\mathbf{U}) =& -[A_{\alpha i3s}u_{s,3} + B_{\alpha i3s}\varphi_{s,3} + D_{\alpha i3}\psi_{,3}]n_\alpha\,, \\
\widetilde{M}_i(\mathbf{U}) =& -[B_{3s\alpha i}u_{s,3} + C_{\alpha i3s}\varphi_{s,3} + E_{\alpha i3}\psi_{,3}]n_\alpha\,, \\
\widetilde{\Pi}(\mathbf{U}) =& -[D_{3s\alpha}u_{s,3} + E_{3s\alpha}\varphi_{s,3} + c_{\alpha 3}\psi_{,3}]n_\alpha\,.
\end{aligned} \tag{4.7}$$

The necessary and sufficient conditions (3.9) become

$$\int_D t_{3i,3}(\mathbf{U})da = 0, \quad \int_D [\varepsilon_{3\alpha\beta}x_\alpha t_{3\beta,3}(\mathbf{U}) + m_{33,3}(\mathbf{U})]da = 0. \tag{4.8}$$

It is easy to observe that, under hypothesis (2.5), the constitutive equations (2.3) give

$$t_{3i,3}(\mathbf{U}) = t_{3i}(\mathbf{U}_{,3})\,, \quad m_{3i,3}(\mathbf{U}) = m_{3i}(\mathbf{U}_{,3})\,, \tag{4.9}$$

so that the relation (4.8) takes the form

$$\int_D t_{3i}(\mathbf{U}_{,3})da = 0, \quad \int_D [\varepsilon_{3\alpha\beta} x_\alpha t_{3\beta}(\mathbf{U}_{,3}) + m_{33}(\mathbf{U}_{,3})]da = 0 \tag{4.10}$$

which means that a sufficient condition for expressing a solution of Saint-Venant's problem in terms of a generalized plane strain is that $\mathcal{R}_i(\mathbf{U})$ and $\mathcal{M}_3(\mathbf{U})$ are independent of x_3. The above relations help us to conclude with the following result.

Proposition 4.1 *Let* $\mathbf{U}$ *be a solution to Saint-Venant's problem. If relation (4.10) holds true then* $\mathbf{U}$ *can be express in term of a state of generalized plane strain.*

Corollary 4.1 *Let* $\mathbf{U}$ *be a solution to Saint-Venant's problem for which (4.10) holds true. Then*

$$(\mathcal{M}_\alpha(\mathbf{U}))_{,3} = 0. \tag{4.11}$$

Proof. Using the equations (2.2), (2.6), (2.7) and the divergence theorem, we get

$$\begin{aligned} \left(\int_D x_\beta t_{33}(\mathbf{U})da\right)_{,3} &= \int_D (x_\beta t_{i3,i}(\mathbf{U}) - x_\beta t_{\alpha 3,\alpha}(\mathbf{U}))da = -\int_D x_\beta t_{\alpha 3,\alpha}(\mathbf{U})da \\ &= -\int_D (x_\beta t_{\alpha 3}(\mathbf{U}))_{,\alpha} da + \int_D t_{\beta 3}(\mathbf{U})da = \int_D t_{\beta 3}(\mathbf{U})da \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} \left(\int_D m_{3\alpha}(\mathbf{U})da\right)_{,3} &= \int_D m_{i\alpha,i}(\mathbf{U})da - \int_D m_{\beta\alpha,\beta}(\mathbf{U})da \\ = -\int_D \varepsilon_{\alpha rs} t_{rs}(\mathbf{U})da - \int_{\partial D} m_{\beta\alpha}(\mathbf{U})n_\beta ds &= \int_D \varepsilon_{3\alpha\beta} t_{3\beta}(\mathbf{U})da - \int_D \varepsilon_{3\alpha\beta} t_{\beta 3}(\mathbf{U})da. \end{aligned} \tag{4.13}$$

From the relation $(4.4)_1$, the hypothesis and the above two relations, we obtain

$$(\mathcal{M}_\alpha(\mathbf{U}))_{,3} = 0 \tag{4.14}$$

and the proof is complete. □

Remark 4.1 *Relations (4.9), (4.12) and (4.13) combined with (4.10) yield*

$$\int_D (\varepsilon_{3\alpha\beta} x_\beta t_{33}(\mathbf{U}_{,33}) + m_{3\alpha}(\mathbf{U}_{,33}))da = 0. \tag{4.15}$$

In what follows, the relations (4.10) and (4.15) allow us to point out two classes of seven-dimensional vector fields which can be expressed in terms of a state of generalized plain strain. These classes will be called classes of semi-inverse solutions to Saint-Venant's problem.

The Class C_I. Inspired by (4.10), we denote by

$$\begin{aligned} C_I = \{\mathbf{U}^0 = (u_i^0, \varphi_i^0, \psi^0) \,|\, u_{i,3}^0 = \alpha_i + \varepsilon_{ijk}\beta_j x_k \,,\, \varphi_{i,3}^0 = \beta_i \,,\, \psi_{,3}^0 = 0 \\ \text{with } \alpha_i, \beta_i \text{ arbitrary constants}\}. \end{aligned} \tag{4.16}$$

It is easy to see that this class characterize the rigid motion of the microstretch elastic materials.

Let $\mathbf{U}^0 \in C_I$, from relations (4.3),(4.4) and (4.10) we can deduce that

$$\big(\mathcal{R}_i(\mathbf{U}^0)\big)_{,3} = 0\,, \quad \big(\mathcal{M}_i(\mathbf{U}^0)\big)_{,3} = 0 \tag{4.17}$$

and by a direct integration we obtain

$$\begin{gathered} u^0_\alpha = -\frac{1}{2} a_\alpha x_3^2 - \varepsilon_{3\alpha\beta} a_4 x_\beta x_3 + w_\alpha(x_1,x_2)\,,\ u^0_3 = (a_1x_1 + a_2x_2 + a_3)x_3 + w_3(x_1,x_2)\,, \\ \varphi^0_\alpha = \varepsilon_{3\alpha\beta} a_\beta x_3 + \nu_\alpha(x_1,x_2)\,, \quad \varphi^0_3 = a_4 x_3 + \nu_3(x_1,x_2)\,, \quad \psi^0 = \omega(x_1,x_2)\,. \end{gathered} \tag{4.18}$$

except for an additive rigid motion, where $\mathbf{W} = (w_i, \nu_i, \omega)$ is an arbitrary vector field independent of x_3 and we have used the notations $a_\alpha = \varepsilon_{3\rho\alpha}\beta_\rho$, $a_3 = \alpha_3$ and $a_4 = \beta_3$. The relation (4.18) tells us that a solution which belongs to the class C_I can be written in terms of four constants and seven functions independent of x_3.

If we put into the constitutive equations (2.3) the relations (4.18), we deduce that the components of the stress tensor, couple stress tensor, microstress vector and scalar microstress function are

$$\begin{aligned} t_{ij}(\mathbf{U}^0) &= A_{ij33}(a_1x_1 + a_2x_2 + a_3) + (B_{ij33} - \varepsilon_{3\alpha\beta} A_{ij3\alpha} x_\beta) a_4 + \varepsilon_{3\alpha\beta} B_{ij3\alpha} a_\beta + T_{ij}(\mathbf{W}), \\ m_{ij}(\mathbf{U}^0) &= B_{33ij}(a_1x_1 + a_2x_2 + a_3) + (C_{ij33} - \varepsilon_{3\alpha\beta} B_{3\alpha ij} x_\beta) a_4 + \varepsilon_{3\alpha\beta} C_{ij3\alpha} a_\beta + M_{ij}(\mathbf{W}), \\ \pi_i(\mathbf{U}^0) &= D_{33i}(a_1x_1 + a_2x_2 + a_3) + (E_{33i} - \varepsilon_{3\alpha\beta} D_{3\alpha i} x_\beta) a_4 + \varepsilon_{3\alpha\beta} E_{3\alpha i} a_\beta + \Pi_i(\mathbf{W}), \\ \sigma(\mathbf{U}^0) &= a_{33}(a_1x_1 + a_2x_2 + a_3) + (b_{33} - \varepsilon_{3\alpha\beta} a_{3\alpha} x_\beta) a_4 + \varepsilon_{3\alpha\beta} b_{3\alpha} a_\beta + \Sigma(\mathbf{W}). \end{aligned} \tag{4.19}$$

where $T_{ij}(\mathbf{W})$, $M_{ij}(\mathbf{W})$, $\Pi_i(\mathbf{W})$ and $\Sigma(\mathbf{W})$ are given by the relations (3.4). Therefore, the boundary value problem defined by (4.1) and (4.2) becomes

$$\begin{aligned} \mathcal{T}_i(\mathbf{U}^0) &= \mathfrak{T}_i(\mathbf{W}) + [A_{\alpha i33} a_\beta x_\beta + A_{\alpha i33} a_3 + (B_{\alpha i33} - \varepsilon_{3\rho\beta} A_{\alpha i3\rho} x_\beta) a_4 + \varepsilon_{3\rho\beta} B_{\alpha i3\rho} a_\beta]_{,\alpha} = 0, \\ \mathcal{S}_i(\mathbf{U}^0) &= \mathfrak{S}_i(\mathbf{W}) + [B_{33\alpha i} a_\beta x_\beta + B_{33\alpha i} a_3 + (C_{\alpha i33} - \varepsilon_{3\rho\beta} B_{3\rho\alpha i} x_\beta) a_4 + \varepsilon_{3\rho\beta} C_{\alpha i3\rho} a_\beta]_{,\alpha} \\ &\quad + \varepsilon_{irs} A_{rs33}(a_1x_1 + a_2x_2 + a_3) + \varepsilon_{irs}(B_{rs33} - \varepsilon_{3\alpha\beta} A_{rs3\alpha} x_\beta) a_4 + \varepsilon_{irs}\varepsilon_{3\alpha\beta} B_{rs3\alpha} a_\beta = 0, \\ \mathcal{P}(\mathbf{U}^0) &= \mathfrak{P}(\mathbf{W}) + [D_{33\alpha} a_\beta x_\beta + D_{33\alpha} a_3 + (E_{33\alpha} - \varepsilon_{3\rho\beta} D_{3\rho\alpha} x_\beta) a_4 + \varepsilon_{3\rho\beta} E_{3\rho\alpha} a_\beta]_{,\alpha} \\ &\quad - [a_{33} a_\beta x_\beta + a_{33} a_3) + (b_{33} - \varepsilon_{3\alpha\beta} a_{3\alpha} x_\beta) a_4 + \varepsilon_{3\alpha\beta} b_{3\alpha} a_\beta] = 0. \end{aligned} \tag{4.20}$$

in D and the boundary conditions are

$$\begin{aligned} \mathcal{A}_i(\mathbf{U}^0) &= \mathfrak{A}_i(\mathbf{W}) + [A_{\alpha i33} a_\beta x_\beta + A_{\alpha i33} a_3 + (B_{\alpha i33} - \varepsilon_{3\rho\beta} A_{\alpha i3\rho} x_\beta) a_4 + \varepsilon_{3\rho\beta} B_{\alpha i3\rho} a_\beta] n_\alpha = 0, \\ \mathcal{B}_i(\mathbf{U}^0) &= \mathfrak{B}_i(\mathbf{W}) + [B_{33\alpha i} a_\beta x_\beta + B_{33\alpha i} a_3 + (C_{\alpha i33} - \varepsilon_{3\rho\beta} B_{3\rho\alpha i} x_\beta) a_4 + \varepsilon_{3\rho\beta} C_{\alpha i3\rho} a_\beta] n_\alpha = 0, \\ \mathcal{C}(\mathbf{U}^0) &= \mathfrak{C}(\mathbf{W}) + [D_{33\alpha} a_\beta x_\beta + D_{33\alpha} a_3 + (E_{33\alpha} - \varepsilon_{3\rho\beta} D_{3\rho\alpha} x_\beta) a_4 + \varepsilon_{3\rho\beta} E_{3\rho\alpha} a_\beta] n_\alpha = 0. \end{aligned} \tag{4.21}$$

on ∂D, where $\mathfrak{T}_i(\mathbf{W})$, $\mathfrak{S}_i(\mathbf{W})$, $\mathfrak{P}(\mathbf{W})$, $\mathfrak{A}_i(\mathbf{W})$, $\mathfrak{B}_i(\mathbf{W})$ and $\mathfrak{C}(\mathbf{W})$ are given by (3.7) and (3.8).

Considering the Proposition 4.1 and (4.17), we remark that the necessary and sufficient conditions to solve the above problem are satisfied for any constants a_s, $s = 1,2,3,4$. Thus, the functions defined by (4.18) are solutions of the Saint-Venant's problem if $\mathbf{W} = (w_i, \nu_i, \omega)$ is the solution of the plane boundary value problem defined by (4.20) and (4.21).

We denote by $\mathbf{W}^{(j)} = (w_i^{(j)}, \nu_i^{(j)}, \omega^{(j)})$ a solution of the above boundary value problem when $a_i = \delta_{ij}$, $a_4 = 0$ and by $\mathbf{W}^{(4)} = (w_i^{(4)}, \nu_i^{(4)}, \omega^{(4)})$ a solution of the above boundary value problem when $a_i = 0$, $a_4 = 1$. Therefore, $\mathbf{W}^{(j)}$, $s = 1, 2, 3, 4$ are characterized by the equations

$$\mathfrak{T}_i(\mathbf{W}^{(s)}) + f_i^{(s)} = 0\,, \quad \mathfrak{S}_i(\mathbf{W}^{(s)}) + g_i^{(s)} = 0\,, \quad \mathfrak{P}(\mathbf{W}^{(s)}) + h^{(s)} = 0\,, \tag{4.22}$$

in D and the boundary conditions

$$\mathfrak{A}_i(\mathbf{W}^{(s)}) = \widetilde{T}_i^{(s)} \quad \mathfrak{M}_i(\mathbf{W}^{(s)}) = \widetilde{M}_i^{(s)}\,, \quad \mathfrak{P}(\mathbf{W}^{(s)}) = \widetilde{\Pi}^{(s)}\,, \tag{4.23}$$

on ∂D, where

$$\begin{aligned}
f_i^{(\gamma)} &= (A_{\alpha i33}x_\gamma + \varepsilon_{3\rho\gamma}B_{\alpha i3\rho})_{,\alpha}\,, \quad f_i^{(3)} = (A_{\alpha i33})_{,\alpha}\,, \quad f_i^{(4)} = (\varepsilon_{3\beta\rho}A_{\alpha i3\rho}x_\beta + B_{\alpha i33})_{,\alpha}\\
g_i^{(\gamma)} &= (B_{33\alpha i}x_\gamma + \varepsilon_{3\rho\gamma}C_{\alpha i3\rho})_{,\alpha} + \varepsilon_{irs}(A_{rs33}x_\gamma + \varepsilon_{3\alpha\gamma}B_{rs3\alpha})\,, \quad g_i^{(3)} = (B_{33\alpha i})_{,\alpha} + \varepsilon_{irs}A_{rs33}\,,\\
g_i^{(4)} &= (\varepsilon_{3\beta\rho}B_{3\rho\alpha i}x_\beta + C_{\alpha i33})_{,\alpha} + \varepsilon_{irs}(\varepsilon_{3\beta\alpha}A_{rs3\alpha}x_\beta + B_{rs33})\\
h^{(\gamma)} &= (D_{33\alpha}x_\gamma + \varepsilon_{3\rho\gamma}E_{3\rho\alpha})_{,\alpha} - (a_{33}x_\gamma + \varepsilon_{3\alpha\gamma}b_{3\alpha})\,, \quad h^{(3)} = (D_{33\alpha})_{,\alpha} - a_{33}\,,\\
h^{(4)} &= (\varepsilon_{3\beta\rho}D_{3\rho\alpha}x_\beta + E_{33\alpha})_{,\alpha} - (\varepsilon_{3\beta\alpha}a_{3\alpha}x_\beta + b_{33})
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\widetilde{T}_i^{(\gamma)} &= -(A_{\alpha i33}x_\gamma + \varepsilon_{3\rho\gamma}B_{\alpha i3\rho})n_\alpha\,, \quad \widetilde{T}_i^{(3)} = -A_{\alpha i33}n_\alpha\,,\\
\widetilde{T}_i^{(4)} &= -(\varepsilon_{3\beta\rho}A_{\alpha i3\rho}x_\beta + B_{\alpha i33})n_\alpha \quad \widetilde{M}_i^{(\gamma)} = -(B_{33\alpha i}x_\gamma + \varepsilon_{3\rho\gamma}C_{\alpha i3\rho})n_\alpha\,,\\
\widetilde{M}_i^{(3)} &= -B_{33\alpha i}n_\alpha\,, \quad \widetilde{M}_i^{(4)} = -(\varepsilon_{3\beta\rho}B_{3\rho\alpha i}x_\beta + C_{\alpha i33})n_\alpha\\
\widetilde{\Pi}^{(\gamma)} &= -(D_{33\alpha}x_\gamma + \varepsilon_{3\rho\gamma}E_{3\rho\alpha})n_\alpha\,, \quad \widetilde{\Pi}^{(3)} = -D_{33\alpha}n_\alpha\,,\\
\widetilde{\Pi}^{(4)} &= -(\varepsilon_{3\beta\rho}D_{3\rho\alpha}x_\beta + E_{33\alpha})n_\alpha
\end{aligned} \tag{4.25}$$

In view of the linearity of the previous four problems, we have

$$\mathbf{W} = \sum_{s=1}^{4} a_s \mathbf{W}^{(s)}. \tag{4.26}$$

In what follows, we assume that the solutions $\mathbf{W}^{(s)}$ of the above problems are known. Then the solution $\mathbf{U}^0 \in C_I$ can be written in the form

$$\mathbf{U}^0 = \sum_{s=1}^{4} a_s \mathbf{U}^{(s)} \tag{4.27}$$

where

$$\begin{aligned}
&u_\alpha^{(\gamma)} = -\tfrac{1}{2}x_3^2\delta_{\alpha\gamma} + w_\alpha^{(\gamma)}, && u_\alpha^{(3)} = w_\alpha^{(3)}, && u_\alpha^{(4)} = -\varepsilon_{3\alpha\beta}x_\beta x_3 + w_\alpha^{(4)},\\
&u_3^{(\gamma)} = x_\gamma x_3 + w_3^{(\gamma)}, && u_3^{(3)} = x_3 + w_3^{(3)}, && u_3^{(4)} = w_3^{(4)},\\
&\varphi_\alpha^{(\gamma)} = \varepsilon_{3\alpha\gamma}x_3 + \nu_\alpha^{(\gamma)}, && \varphi_\alpha^{(3)} = \nu_\alpha^{(3)}, && \varphi_\alpha^{(4)} = \nu_\alpha^{(4)}\\
&\varphi_3^{(\gamma)} = \nu_3^{(\gamma)}, && \varphi_3^{(3)} = \nu_3^{(3)}, && \varphi_3^{(4)} = x_3 + \nu_3^{(4)}.
\end{aligned} \tag{4.28}$$

From (4.19) and (4.27), it follows that

$$\begin{aligned} t_{ij}(\mathbf{U}^0) &= \sum_{s=1}^{4} a_s t_{ij}(\mathbf{U}^{(s)}), \quad m_{ij}(\mathbf{U}^0) = \sum_{s=1}^{4} a_s m_{ij}(\mathbf{U}^{(s)}), \\ \pi_i(\mathbf{U}^0) &= \sum_{s=1}^{4} a_s \pi_i(\mathbf{U}^{(s)}), \quad \sigma(\mathbf{U}^0) = \sum_{s=1}^{4} a_s \sigma(\mathbf{U}^{(s)}), \end{aligned} \tag{4.29}$$

where

$$\begin{aligned} t_{ij}(\mathbf{U}^{(\gamma)}) &= T_{ij}(\mathbf{W}^{(\gamma)}) + A_{ij33}x_\gamma + \varepsilon_{3\alpha\gamma}B_{ij3\alpha}, \quad t_{ij}(\mathbf{U}^{(3)}) = T_{ij}(\mathbf{W}^{(3)}) + A_{ij33}, \\ t_{ij}(\mathbf{U}^{(4)}) &= T_{ij}(\mathbf{W}^{(4)}) + \varepsilon_{3\beta\alpha}A_{ij3\alpha}x_\beta + B_{ij33}, \\ m_{ij}(\mathbf{U}^{(\gamma)}) &= M_{ij}(\mathbf{W}^{(\gamma)}) + B_{33ij}x_\gamma + \varepsilon_{3\alpha\gamma}C_{ij3\alpha}, \quad m_{ij}(\mathbf{U}^{(3)}) = M_{ij}(\mathbf{W}^{(3)}) + B_{33ij}, \\ m_{ij}(\mathbf{U}^{(4)}) &= M_{ij}(\mathbf{W}^{(4)}) + \varepsilon_{3\beta\alpha}B_{3\alpha ij}x_\beta + C_{ij33}, \\ \pi_i(\mathbf{U}^{(\alpha)}) &= \Pi_i(\mathbf{W}^{(\gamma)}) + D_{33i}x_\gamma + \varepsilon_{3\alpha\gamma}E_{3\alpha i}, \quad \pi_i(\mathbf{U}^{(3)}) = \Pi_i(\mathbf{W}^{(3)}) + D_{33i}, \\ \pi_i(\mathbf{U}^{(4)}) &= \Pi_i(\mathbf{W}^{(4)}) + \varepsilon_{3\beta\alpha}D_{3\alpha i}x_\beta + E_{33i}, \\ \sigma(\mathbf{U}^{(\gamma)}) &= \Sigma(\mathbf{W}^{(\gamma)}) + a_{33}x_\gamma + \varepsilon_{3\alpha\gamma}b_{3\alpha}, \quad \sigma(\mathbf{U}^{(3)}) = \Sigma(\mathbf{W}^{(3)}) + a_{33}, \\ \sigma(\mathbf{U}^{(4)}) &= \Sigma(\mathbf{W}^{(4)}) + \varepsilon_{3\beta\alpha}a_{3\alpha}x_\beta + b_{33}. \end{aligned} \tag{4.30}$$

From the relations (4.22), (4.23) and (4.30), it is evident that $\mathbf{U}^{(s)}, s = 1, 2, 3, 4$ are the solutions of the generalized plane boundary problem

$$t_{\alpha i,\alpha}(\mathbf{U}^{(s)}) = 0, \quad m_{\alpha i,\alpha}(\mathbf{U}^{(s)}) + \varepsilon_{irs}t_{rs}(\mathbf{U}^{(s)}) = 0, \quad \pi_{\alpha,\alpha}(\mathbf{U}^{(s)}) - \sigma(\mathbf{U}^{(s)}) = 0, \quad \text{in}\, D \tag{4.31}$$

and

$$t_{\alpha i}(\mathbf{U}^{(s)})n_\alpha = 0, \quad m_{\alpha i}(\mathbf{U}^{(s)})n_\alpha = 0, \quad \pi_{\alpha,\alpha}(\mathbf{U}^{(s)})n_\alpha = 0, \quad \text{on}\, \partial D \tag{4.32}$$

Using the divergence theorem, these relations imply

$$\begin{aligned} \mathcal{R}_\alpha(\mathbf{U}^0) &= \int_D t_{3\alpha}(\mathbf{U}^0)da = \sum_{s=1}^{4} a_s \int_D t_{3\alpha}(\mathbf{U}^{(s)})da \\ &= \sum_{s=1}^{4} a_s \int_D \Big(t_{3\alpha}(\mathbf{U}^{(s)}) + x_\alpha t_{\rho 3,\rho}(\mathbf{U}^{(s)})\Big)\, da \\ &= \sum_{s=1}^{4} a_s \int_D \Big(t_{3\alpha}(\mathbf{U}^{(s)}) - t_{\alpha 3}(\mathbf{U}^{(s)}) + \Big(x_\alpha t_{\rho 3}(\mathbf{U}^{(s)})\Big)_{,\rho}\Big)\, da \\ &= \sum_{s=1}^{4} a_s \int_D \varepsilon_{3\rho\alpha} m_{\beta\rho,\beta}(\mathbf{U}^{(s)}) + \sum_{s=1}^{4} a_s \int_{\partial D} x_\alpha t_{\rho 3}(\mathbf{U}^{(s)}) n_\rho ds = 0 \end{aligned} \tag{4.33}$$

For a solution $(u_i^0, \varphi_i^0, \psi) \in C_I$, we also have

$$\mathcal{R}_3(\mathbf{U}^0) = \sum_{s=1}^{4} a_s D_{3s}, \quad \mathcal{M}_\alpha(\mathbf{U}^0) = \sum_{s=1}^{4} \varepsilon_{3\alpha\beta} a_s D_{\beta s}, \quad \mathcal{M}_3(\mathbf{U}^0) = \sum_{s=1}^{4} a_s D_{4s}, \tag{4.34}$$

where

$$\begin{gathered} D_{3s} = \int_D t_{33}(\mathbf{U}^{(s)})da, \quad D_{\beta s} = \int_D \left(x_\beta t_{33}(\mathbf{U}^{(s)}) + \varepsilon_{3\rho\beta} m_{3\rho}(\mathbf{U}^{(s)})\right) da, \\ D_{4s} = \int_D \left(\varepsilon_{3\alpha\beta} x_\alpha t_{3\beta}(\mathbf{U}^{(s)}) + m_{33}(\mathbf{U}^{(s)})\right) da, \quad s = 1, 2, 3, 4. \end{gathered} \tag{4.35}$$

We write $u_i^0\{\widehat{\mathbf{a}}\}$, $\varphi_i^0\{\widehat{\mathbf{a}}\}$ and $\psi^0\{\widehat{\mathbf{a}}\}$ to indicate the dependence of the components u_i^0, φ_i^0, ψ^0 of $\widehat{\mathbf{a}} = (a_1, a_2, a_3, a_4)$.

In view of Remark 4.1, we will introduce the following class of semi-inverse solutions of Saint-Venant's problem.

The Class C_{II}. We denote by C_{II} the class of seven-dimensional vectors $\mathbf{U}^* = (u_i^*, \varphi_i^*, \psi^*)$ for which the conditions (4.10) hold true and moreover the expression of $u^*_{3,33}$ is the same of that of a rigid motion.

For $(u_i^*, \varphi_i^*, \psi^*) \in C_{II}$ it follows that $(u^*_{i,3}, \varphi^*_{i,3}, \psi^*_{,3}) \in C_I$ and using the notations introduced above, we have

$$\mathbf{u}^*_{,3} = \mathbf{u}^0\{\widehat{\mathbf{b}}\}, \quad \boldsymbol{\varphi}^*_{,3} = \boldsymbol{\varphi}^0\{\widehat{\mathbf{b}}\}, \quad \psi^*_{,3} = \psi^0\{\widehat{\mathbf{b}}\} \tag{4.36}$$

and after integration we deduce

$$\begin{aligned} \mathbf{u}^* &= \int_0^{x_3} \mathbf{u}^0\{\widehat{\mathbf{b}}\}dx_3 + \mathbf{u}^0\{\widehat{\mathbf{c}}\} + \mathbf{w}^*(x_1, x_2), \\ \boldsymbol{\varphi}^* &= \int_0^{x_3} \boldsymbol{\varphi}^0\{\widehat{\mathbf{b}}\}dx_3 + \boldsymbol{\varphi}^0\{\widehat{\mathbf{c}}\} + \boldsymbol{\nu}^*(x_1, x_2), \\ \psi^* &= \int_0^{x_3} \psi^0\{\widehat{\mathbf{b}}\}dx_3 + \psi^0\{\widehat{\mathbf{c}}\} + \omega^*(x_1, x_2), \end{aligned} \tag{4.37}$$

where $\widehat{\mathbf{b}}$ and $\widehat{\mathbf{c}}$ are arbitrary four-dimensional vectors, $\mathbf{w}^*$, $\boldsymbol{\nu}^*$ and ω^* are independent functions of x_3.

The components of the stress tensor, couple stress tensor, microstress vector and scalar microstress function corresponding to $\mathbf{U}^* = (u_i^*, \varphi_i^*, \psi^*)$ defined by (4.37) have the form

$$\begin{aligned} t_{ij}(\mathbf{U}^*) &= \sum_{s=1}^4 (c_s + x_3 b_s) t_{ij}(\mathbf{U}^{(s)}) + k_{ij} + T_{ij}(\mathbf{W}^*), \\ m_{ij}(\mathbf{U}^*) &= \sum_{s=1}^4 (c_s + x_3 b_s) m_{ij}(\mathbf{U}^{(s)}) + h_{ij} + M_{ij}(\mathbf{W}^*), \\ \pi_i(\mathbf{U}^*) &= \sum_{s=1}^4 (c_s + x_3 b_s) \pi_i(\mathbf{U}^{(s)}) + p_i + \Pi_i(\mathbf{W}^*), \\ \sigma(\mathbf{U}^*) &= \sum_{s=1}^4 (c_s + x_3 b_s) \sigma(\mathbf{U}^{(s)}) + s_i + \Sigma(\mathbf{W}^*), \end{aligned} \tag{4.38}$$

where

$$\begin{aligned} k_{ij} &= \sum_{s=1}^{4}(A_{ij3l}w_l^{(s)}b_s + B_{ij3l}\nu_l^{(s)}b_s + D_{ij3}\omega^{(s)}b_s), \\ h_{ij} &= \sum_{s=1}^{4}(B_{3lij}w_l^{(s)}b_s + C_{ij3l}\nu_l^{(s)}b_s + E_{ij3}\omega^{(s)}b_s), \\ p_i &= \sum_{s=1}^{4}(D_{3li}w_l^{(s)}b_s + E_{3li}\nu_l^{(s)}b_s + c_{i3}\omega^{(s)}b_s), \\ s &= \sum_{s=1}^{4}(a_{3l}w_l^{(s)}b_s + b_{3l}\nu_l^{(s)}b_s + d_3\omega^{(s)}b_s) \end{aligned} \tag{4.39}$$

and we denote $\mathbf{W}^* = (w_i^*, \nu_i^*, \omega^*)$.

Because $(u^*_{i,3}, \varphi^*_{i,3}, \psi^*_{,3}) \in C_I$, in view of relations (4.33)–(4.35) and Corollary 4.1, we must have

$$\sum_{s=1}^{4} b_s D_{3s} = 0\,, \quad \sum_{s=1}^{4} b_s D_{4s} = 0 \tag{4.40}$$

Using the relations (4.31) and (4.32) in the problem (4.1)–(4.2) for $\mathbf{U}^*$, we can deduce that $\mathbf{W}^*$ is a solution of the following boundary value problem

$$\mathfrak{T}_i(\mathbf{W}^*) + \widetilde{F}_i = 0, \quad \mathfrak{S}_i(\mathbf{W}^*) + \widetilde{G}_i = 0, \quad \mathfrak{P}(\mathbf{W}^*) + \widetilde{H} = 0 \quad \text{in}\, D \tag{4.41}$$

and

$$\mathfrak{A}_i(\mathbf{W}^*) + \widetilde{T}_i = 0, \quad \mathfrak{B}_i(\mathbf{W}^*) + \widetilde{M}_i = 0, \quad \mathfrak{C}(\mathbf{W}^*) + \widetilde{\Pi}_i = 0 \quad \text{on}\, \partial D. \tag{4.42}$$

where

$$\begin{aligned} &\widetilde{F}_i = k_{\alpha i,\alpha} + \sum_{s=1}^{4} b_s t_{3i}(\mathbf{U}^{(s)}), \quad \widetilde{G}_i = h_{\alpha i,\alpha} + \varepsilon_{irs}k_{rs} + \sum_{s=1}^{4} b_s m_{3i}(\mathbf{U}^{(s)}), \\ &\widetilde{H} = p_{\alpha,\alpha} - s + \sum_{s=1}^{4} b_s \pi_3(\mathbf{U}^{(s)}), \quad \widetilde{T}_i = -k_{\alpha i}n_\alpha, \quad \widetilde{M}_i = -h_{\alpha i}n_\alpha, \quad \widetilde{\Pi} = -p_\alpha n_\alpha. \end{aligned} \tag{4.43}$$

The necessary and sufficient conditions for the existence of a solution of the above problem are satisfied on the basis of relations (4.10), (4.31), (4.36) and (4.40).

Proposition 4.2 *If* $\mathbf{U}^* \in C_{II}$, *then has the form (4.37), where* $\widehat{\mathbf{b}}$ *satisfied the conditions (4.40) and* $\mathbf{W}^*$ *is the solution of the generalized plane strain problem given by (4.41)–(4.42).*

Remark 4.2 *Let* $\mathbf{U}^* \in C_{II}$. *Then, from relations (4.3), (4.4), (4.12), (4.13) and (4.38), we have*

$$
\begin{aligned}
&\mathcal{R}_\alpha(\mathbf{U}^*) = \sum_{s=1}^{4} b_s D_{\alpha s}, \quad \mathcal{R}_3(\mathbf{U}^*) = \sum_{s=1}^{4} c_s D_{3s} + \int_D (k_{33} + T_{33}(\mathbf{W}^*))\, da,\\
&\mathcal{M}_\alpha(\mathbf{U}^*) = \sum_{s=1}^{4} \varepsilon_{3\alpha\beta} c_s D_{\beta s} + \int_D (\varepsilon_{3\alpha\beta} x_\beta (k_{33} + T_{33}(\mathbf{W}^*)) + h_{3\alpha} + M_{3\alpha}(\mathbf{W}^*))\, da,\\
&\mathcal{M}_3(\mathbf{U}^*) = \sum_{s=1}^{4} c_s D_{4s} + \int_D (\varepsilon_{3\alpha\beta} x_\alpha (k_{3\beta} + T_{3\beta}(\mathbf{W}^*)) + h_{33} + M_{33}(\mathbf{W}^*))\, da,
\end{aligned}
\tag{4.44}
$$

with the components of the vector fields $\widehat{\mathbf{b}}$ satisfying the conditions (4.40).

A solution of the Saint-Venant's problem residing within C_I corresponds to the end loads

$$
\begin{aligned}
t_i(L) &= -t_i(0) = \sum_{s=1}^{4} a_s t_{3i}(\mathbf{U}^{(s)}),\\
m_i(L) &= -m_i(0) = \sum_{s=1}^{4} a_s m_{3i}(\mathbf{U}^{(s)}),\\
\pi(L) &= -\pi(0) = \sum_{s=1}^{4} a_s \pi_3(\mathbf{U}^{(s)})
\end{aligned}
\tag{4.45}
$$

and a solution of the Saint-Venant's problem residing within C_{II} corresponds to the end loads

$$
\begin{aligned}
t_i(0) &= -\sum_{s=1}^{4} c_s t_{3i}(\mathbf{U}^{(s)}) - k_{3i} - T_{3i}(\mathbf{W}^*),\\
t_i(L) &= \sum_{s=1}^{4} (c_s + L b_s) t_{3i}(\mathbf{U}^{(s)}) + k_{3i} + T_{3i}(\mathbf{W}^*),\\
m_i(0) &= -\sum_{s=1}^{4} c_s m_{3i}(\mathbf{U}^{(s)}) - h_{3i} - M_{3i}(\mathbf{W}^*),\\
m_i(L) &= \sum_{s=1}^{4} (c_s + L b_s) m_{3i}(\mathbf{U}^{(s)}) + h_{3i} + M_{3i}(\mathbf{W}^*),\\
\pi(0) &= -\sum_{s=1}^{4} c_s \pi_3(\mathbf{U}^{(s)}) - p_3 - \Pi_3(\mathbf{W}^*),\\
\pi(L) &= \sum_{s=1}^{4} (c_s + L b_s) \pi_3(\mathbf{U}^{(s)}) + p_3 + \Pi_3(\mathbf{W}^*),
\end{aligned}
\tag{4.46}
$$

where we have used relations $(2.7)_{2,3}$, (4.29) and (4.38).

5 The relaxed Saint-Venant's problem

The relaxed Saint Venant's problem consist in the determination of a equilibrium displacement field $\mathbf{U} = (u_i, \varphi_i, \psi)$ that satisfies the boundary lateral conditions

$$t_i(\mathbf{U}) = 0, \quad m_i(\mathbf{U}) = 0, \quad \pi(\mathbf{U}) = 0 \quad \text{on } \partial D \times (0, L) \tag{5.1}$$

and

$$\mathcal{R}_i(\mathbf{U}) = -R_i, \quad \mathcal{M}_i(\mathbf{U}) = -M_i \quad \text{on } x_3 = 0, \tag{5.2}$$

where R_i and M_i are preassigned functions. Similar conditions are assumed on the end located at $x_3 = L$.

In this section we find a solution of the relaxed Saint Venant's problem following the method developed by Ieşan [9] and Chiriţă [1,2]. The method consists in the decomposition of the relaxed Saint-Venant's problem into the problems $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ characterized by

$$(\mathcal{P}_1) \text{ (extension − bending − torsion) : } \quad R_\alpha = 0, \tag{5.3}$$

$$(\mathcal{P}_2) \text{ (flexure) : } \quad R_3 = 0, M_i = 0. \tag{5.4}$$

The solution of problem $(\mathcal{P}_1)$. Considering the results of the previous Section, a solution of the problem $(\mathcal{P}_1)$ has the form

$$\mathbf{U}_I = \mathbf{U}^0 = \sum_{s=1}^{4} a_s \mathbf{U}^{(s)}. \tag{5.5}$$

This solution corresponds to the pointwise values of the couple stress tensor and of the microstress vector on the ends of the cylinder which have the form $(4.45)_{2,3}$.

In view of relations (4.34) and (5.1)–(5.3), we have the following system

$$\sum_{s=1}^{4} a_s D_{3s} = -R_3, \quad \sum_{s=1}^{4} a_s D_{\alpha s} = \varepsilon_{3\alpha\beta} M_\beta, \quad \sum_{s=1}^{4} a_s D_{4s} = -M_3 \tag{5.6}$$

which gives us the unknown constants $a_s, s = 1, 2, 3, 4$.

Following the method used by Ieşan [9] and by Ieşan and Ciarletta [11], Scalia [14] proved that

$$D_{sr} = D_{rs}, \quad s, r = 1, 2, 3, 4 \tag{5.7}$$

and, that the system (5.6) uniquely determines the constants $a_s, s = 1, 2, 3, 4$.

Conclusion: the solution of the problem $(\mathcal{P}_1)$ has the form (5.5), where $\mathbf{U}^{(s)}, s = 1, 2, 3, 4$ are the solutions of the problems defined by (4.31)–(4.32) and $a_s, s = 1, 2, 3, 4$ are the solutions of the algebraic system (5.6).

The solution of problem $(\mathcal{P}_2)$. Inspired by the results of the previous Section, we will seek for a solution of the problem $(\mathcal{P}_2)$ in the class $\mathcal{C}_{II}$

$$\mathbf{U}_{II} = \mathbf{U}^* = (u_i^*, \varphi^*, \psi^*) \tag{5.8}$$

where, using (4.37), we have that

$$\begin{aligned}
u_\alpha^* &= -\frac{1}{6}b_\alpha x_3^3 - \frac{1}{2}c_\alpha x_3^2 - \frac{1}{2}\varepsilon_{3\alpha\beta}b_4 x_\beta x_3^2 - \varepsilon_{3\alpha\beta}c_4 x_\beta x_3 + \sum_{s=1}^{4}(c_s + b_s x_3)w_\alpha^{(s)} + w_\alpha^*,\\
u_3^* &= \frac{1}{2}(b_1x_1 + b_2x_2 + b_3)x_3^2 + (c_1x_1 + c_2x_2 + c_3)x_3 + \sum_{s=1}^{4}(c_s + b_s x_3)w_3^{(s)} + w_3^*,\\
\varphi_\alpha^* &= \frac{1}{2}\varepsilon_{3\alpha\beta}b_\beta x_3^2 + \varepsilon_{3\alpha\beta}c_\beta x_3 + \sum_{s=1}^{4}(c_s + b_s x_3)\nu_\alpha^{(s)} + \nu_\alpha^*, \qquad (5.9)\\
\varphi_3^* &= \frac{1}{2}b_4 x_3^2 + c_4 x_3 + \sum_{s=1}^{4}(c_s + b_s x_3)\nu_3^{(s)} + \nu_3^*,\\
\psi^* &= \sum_{s=1}^{4}(c_s + b_s x_3)\omega^{(s)} + \omega^*.
\end{aligned}$$

The solution U_{II} corresponds to the pointwise values of the couple stress tensor and of the microstress vector on the ends of the cylinder given by $(4.46)_{3,4,5,6}$.

The unknown constants $b_s, s = 1, 2, 3, 4$ satisfy the conditions (4.40) and from (4.44) and (5.2), we get

$$\sum_{s=1}^{4} b_s D_{\alpha s} = -R_\alpha. \qquad (5.10)$$

The system formed by the relations (4.40) and (5.10) give us the possibility to uniquely determinate the constants $b_s, s = 1, 2, 3, 4$. Knowing b_s, we can determinate w_i^*, ν_i^* and ω^* from the generalized plane strain problem defined by (4.41)–(4.43). From (4.44) and (5.2), we have that the unknown constants $c_s, s = 1, 2, 3, 4$, can be uniquely determinated by the following system

$$\begin{aligned}
\sum_{s=1}^{4} c_s D_{\beta s} &= -\int_D \left[k_{33} + T_{33}(\mathbf{W}^*) + \varepsilon_{3\alpha\beta}\left(h_{3\alpha} + M_{3\alpha}(\mathbf{W}^*)\right)\right] da,\\
\sum_{s=1}^{4} c_s D_{3s} &= -\int_D \left[k_{33} + T_{33}(\mathbf{W}^*)\right] da, \qquad (5.11)\\
\sum_{s=1}^{4} c_s D_{4s} &= -\int_D \left[\varepsilon_{3\alpha\beta}x_\alpha\left(k_{3\beta} + T_{3\beta}(\mathbf{W}^*)\right) + h_{33} + M_{33}(\mathbf{W}^*)\right] da.
\end{aligned}$$

Conclusion: a solution of the problem $(\mathcal{P}_2)$ has the form (5.9) where the constants b_s and $c_s, s = 1, 2, 3, 4$ are solutions of the system (4.40), (5.10)–(5.11) and with w_i^*, ν_i^* and ω^* determined from the generalized plane strain problem defined by (4.41)–(4.43).

Remark 5.1 *The relaxed Saint-Venant's problem has a solution of the form*

$$\mathbf{U} = \mathbf{U}_I + \mathbf{U}_{II} \qquad (5.12)$$

where U_I and U_{II} are defined by the relations (5.5), (5.8) and (5.9).

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