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An Application of Fryszkowski's Selection Theorem to the Darboux Problem for Third Order Hyperbolic Inclusions

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Abstract. In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x,y,z,u)$, where $F : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$ is a lower semicontinuous multifunction, whose values are non-empty closed and not necessarily convex subsets of $\mathbb{R}^n$. We prove an existence theorem of a local solution for the specified Darboux Problem using the Fryszkowski's Selection Theorem and Schauder's Fixed Point Theorem.

Keywords: multifunction, measurable multifunction, decomposable multifunction, upper and lower semi-continuity of multifunction, selection, hyperbolic inclusion, initial values, absolutely continuous in Carathéodory's sense function.

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1 Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x,y,z)}{\partial x\,\partial y\,\partial z} \in F(x,y,z,u),\ (x,y,z) \in D = [0,a]\times[0,b]\times[0,c] \subset \mathbb{R}^3,\ u\in \mathbb{R}^n, \tag{1.1}$$

with the initial values

$$\begin{cases} u(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a] \times [0,b], \\ u(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b] \times [0,c], \\ u(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a] \times [0,c], \end{cases} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense functions [5, §565 - §570], $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x), & x \in [0,a], \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y), & y \in [0,b], \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z), & z \in [0,c], \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0. \end{cases} \tag{1.3}$$

$F : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$ is a lower semi-continuous multifunction, whose values are non-empty closed and not necessarily convex subsets of $\mathbb{R}^n$. We prove an existence theorem of a local solution for the specified Darboux Problem (1.1) + (1.2) using Fryszkowski's Selection Theorem and Schauder's Fixed Point Theorem.

In [35] we considered the Darboux Problem (1.1) + (1.2) where $F : D \times \Omega \to 2^{\mathbb{R}^n}$ were a multifunction with compact and non-empty values and $\Omega \subset \mathbb{R}^n$ were an open subset. Under suitable assumptions, we proved an existence theorem for a local selection of the Darboux Problem (1.1) + (1.2), using the Kakutani-Ky Fan Fixed Point Theorem, and we also proved that the set of its solutions is compact in Banach space $C(D_0; \mathbb{R}^n)$, $D_0 = [0,x_0] \times [0,y_0] \times [0,z_0] \subseteq D$; moreover, as a function of the initial values, this set defines an upper semi-continuous multifunction.

In [36] we proved a theorem of prolongation for the solutions of the considered problem and also an existence theorem for a saturated solution.

In [37] we proved a characterization theorem for the solutions of Darboux Problem (1.1)+ (1.2) using the Aumann integral [2] defined for multifunctions.

In [38], using the notion of uniform convergence on compact sets defined by Arrigo Cellina [9], [10] for a sequence of single-valued functions, $f_k : \Lambda \to \mathbb{R}^n$ such that $f_k \to F$, where F is a multifunction, we considered a sequence of approximating univalued equations of the form $u_{xyz} = f_k(x,y,z,u)$ and we proved that they have a unique solution, using Schauder's Fixed Point Theorem. Using a characterization theorem for the solutions of the Darboux Problem for the specified inclusion [37], we proved that the sequence of solutions to the approximating univalued equations uniformly converges, on compact sets, to a solution of the Darboux Problem (1.1) + (1.2) for the considered inclusion, hence we obtained a global solution of this problem as the uniform limit of the sequence of the solutions for the approximating equations.

In [39] we considered the Darboux Problem (1.1) + (1.2), where $F : D \times B \to \operatorname{comp} A$ satisfies the Carathéodory type conditions, and whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$. We proved an existence theorem of a continuous selection, and also an existence theorem of an absolutely continuous solution for Darboux Problem (1.1) + (1.2).

In [40] we considered the Darboux Problem (1.1) + (1.2), where $F : D \times B \to \operatorname{comp} A$ is a continuous multifunction whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$. We proved a theorem which establishes the existence of a continuous selection for each of the functions $(x,y,z) \to F(x,y,z,u(x,y,z))$ with respect to a given family of continuous functions $(x,y,z) \to u(x,y,z)$. Using this result and Schauder's Fixed Point Theorem it is obtained an existence theorem of an absolutely continuous solution for Darboux Problem (1.1) + (1.2).

In [41] we considered the same Darboux Problem (1.1) + (1.2), where F is a continuous multifunction, the refined case, and we obtained a similar result, stronger than the one in [38].

This paper has been suggested by [34], and it provides an extension of the result in that article. In a manner used in [1], [24], [39]-[41], we obtain the existence of a local solution for Darboux Problem (1.1) + (1.2).

2 Preliminaries

The definitions and Theorems 2.1-2.10 in this section are recalled from [1]-[33].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x\in X} \Phi(x)$ is the *range* of Φ, $\Phi(X) = \left\{ \bigcup\limits_{x\in X} \Phi(x) \mid x \in X \right\}$.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup\limits_{x\in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of Φ, denoted graph Φ, is the set

$$\operatorname{graph}\Phi = \{(x,y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^Y$. An element $x \in X$ with the property that $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper semi-continuous* if, for any closed $B \subseteq Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If X and Y are two topological spaces, the multifunction $\Phi : X \to 2^Y$ is *lower semi-continuous* if, for every open subset $\Omega \subseteq Y$, the set $\Phi^-(\Omega)$ is open in X.

Definition 2.7. The multifunction $\Phi : X \to 2^Y$ is *continuous* if it is upper semi-continuous and lower semi-continuous.

Definition 2.8. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* (*weakly measurable*), if $\Phi^-(B) \in \mathcal{F}$ for every closed (open) subset $B \subseteq Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^-(B)$ is measurable.

Theorem 2.1 [30]. *Let X and Y be two metric spaces, Y being compact and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi(x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

i) *the multifunction Φ is upper semi-continuous;*

ii) *the graph of Φ is a closed subset of $X \times Y$;*

iii) *any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi(x_n)$ and $y_n \to y$ it follows that $y \in \Phi(x)$.*

Theorem 2.2 [21]. *When X is first countable and Y is a metric space, $\Phi : X \to 2^Y$ is lower semi-continuous if and only if, for every $\tilde{x} \in X$, every sequence $\{x_n\}$ in X converging* *to* $\tilde{x}$ *and every* $\tilde{y} \in \Phi(\tilde{x})$*, there exists a sequence* $\{y_n\}$ *in* Y *converging to* $\tilde{y}$*, such that* $y_n \in \Phi(x_n)$ *for all* $n \in \mathbb{N}$*.*

Definition 2.9 [5], [11], [14]. The function $u : \triangle \to \mathbb{R}^n$, $\triangle \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* [5, §565 - §570] if and only if it is continuous on $\triangle$, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x(x,y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on $\triangle$.

Theorem 2.3. [5], [11], [33] *The function* $u : \triangle \to \mathbb{R}^n$, $\triangle = [0,a] \times [0,b] \subset \mathbb{R}^2$, *is absolutely continuous in Carathéodory's sense on* $\triangle$ *if and only if there exist* $f \in L^1(\triangle;\mathbb{R}^n)$, $g \in L^1([0,a];\mathbb{R}^n)$, $h \in L^1([0,b];\mathbb{R}^n)$ *such that*

$$u(x,y) = \int_0^x \int_0^y f(s,t)\,ds\,dt + \int_0^x g(s)\,ds + \int_0^y h(t)\,dt + u(0,0).$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^*(\triangle;\mathbb{R}^n)$, [14]. In [11], this space is denoted by $AC(\triangle;\mathbb{R}^n)$.

Theorem 2.4. [11] *The space* $C^*(\triangle;\mathbb{R}^n)$ *endowed with the norm*

$$\|u(\cdot,\cdot)\| = \int_0^a \int_0^b \|u_{xy}(s,t)\|\,ds\,dt + \int_0^a \|u_x(s,0)\|\,ds + \int_0^b \|u_y(0,t)\|\,dt + \|u(0,0)\|,$$

where $\triangle = [0,a] \times [0,b] \subset \mathbb{R}^2$*, and* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

Definition 2.10 [15]. The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is absolutely continuous in Carathéodory's sense [5, §565 - §570] if and only if $u(x,y,z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables) and similarly for $u_x(x,y,z)$, $u_y(x,y,z)$, $u_z(x,y,z)$, $u_{xy}(x,y,z)$, $u_{yz}(x,y,z)$, $u_{xz}(x,y,z)$, and u_{xyz} is Lebesgue-integrable on D.

Theorem 2.5. [11] *The function* $u : D \to \mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, *is absolutely continuous in Carathéodory's sense on* D *if and only if there exist* $f \in L^1(D;\mathbb{R}^n)$, $g_1 \in L^1(D_1;\mathbb{R}^n)$, $g_2 \in L^1(D_2;\mathbb{R}^n)$, $g_3 \in L^1(D_3;\mathbb{R}^n)$, $h_1 \in L^1([0,a];\mathbb{R}^n)$, $h_2 \in L^1([0,b];\mathbb{R}^n)$, $h_3 \in L^1([0,c];\mathbb{R}^n)$, *such that*

$$\begin{aligned} u(x,y,z) = &\int_0^x \int_0^y \int_0^z f(r,s,t)\,dr\,ds\,dt + \int_0^x \int_0^y g_1(r,s)\,dr\,ds + \\ &+ \int_0^y \int_0^z g_2(s,t)\,ds\,dt + \int_0^x \int_0^z g_3(r,t)\,dr\,dt + \\ &+ \int_0^x h_1(r)\,dr + \int_0^y h_2(s)\,ds + \int_0^z h_3(t)\,dt + u(0,0,0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on D by $C^*(D;\mathbb{R}^n)$ [15].

Theorem 2.6 [11] *The space $C^*(D;\mathbb{R}^n)$ endowed with the norm*

$$\begin{aligned}\|u(\cdot,\cdot,\cdot)\| = &\int_0^a\int_0^b\int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a\int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds+\\ &+\int_0^b\int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt + \int_0^a\int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt+\\ &+\int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\|\,ds+\\ &+\int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|,\end{aligned}$$

where $D=[0,a]\times[0,b]\times[0,c]\subset\mathbb{R}^3$, and *$\|\cdot\|$ is the Euclidean norm, is a Banach space.*

The next result is an instance of a general selection theorem due to Kuratowski and Ryll-Nardzewski.

Theorem 2.7 [20], [24]. *Let X be a separable real Banach space and let $H:[0,r]\to 2^X$ be a non-empty and closed valued mapping. If H is weakly measurable then it has at least one measurable selection, i.e. there exists at least one measurable function $h:[0,T]\to X$ such that $h(t)\in H(t)$ a.e. for $t\in(0,T)$, [24], [6, Theorem III 6, p.65].*

Corollary 2.1 [24]. Let X be a separable real Banach space, let C be a non-empty and closed subset in X and let $F:[0,T]\times C\to 2^X$ be a non-empty and closed-value mapping which is lower semi-continuous on $[0,T]\times C$. Then, for each continuous function $u:[0,T]\to C$, the mapping $F\circ u:[0,T]\to 2^X$, $(F\circ u)(t)=F(t,u(t))$ for each $t\in[0,T]$ has at least one measurable selection.

Proof. Obviously $F\circ u$ is lower semi-continuous on $[0,T]$ and hence it is weakly measurable. Thus Theorem 2.7 applies and this completes the proof, [24].

Definition 2.11 [3], [24]. Let K be a non-empty subset in $C([0,T];X)$ and let $\mathcal{F}:K\to 2^{L^1([0,T];X)}$ be a non-empty and closed valued mapping. We say that $\mathcal{F}$ is *decomposable* if for each $u\in K$, each $f,g\in\mathcal{F}(u)$ and each measurable subset E in $[0,T]$ we have

$$f\cdot\chi_E + g\cdot\chi_{[0,T]-E}\in\mathcal{F}(u),$$

where χ_E is the characteristic function of E.

In the proof of our existence result it is used the following specific form of a selection theorem due to Fryszkowski [17].

Theorem 2.8 [17], [24]. *Let X be a separable real Banach space, let K be a compact subset in $C([0,T];X)$ and let $\mathcal{F}:K\to 2^{L^1([0,T];X)}$ be a non-empty and closed valued mapping which is lower semi-continuous and decomposable. Then there exists at least one continuous function $f:K\to L^1([0,T];X)$ such that $f(u)\in\mathcal{F}(u)$ for each $u\in K$.*

Theorem 2.9 (Mazur's Lemma, [18], [19], [30], [42]. *The convex envelope of compact set in a Banach space is compact set.*

Theorem 2.10 (Schauder, [18], [30]). *Let X be a Banach space, $f : X \to X$ a continuous mapping and $A \subset X$ a convex compact subset with the property that $f(A) \subseteq A$. Then f has at least a fixed point, i.e. there exists at least an element $x \in A$ with the property that $f(x) = x$.*

The following version of Schauder's Theorem is useful for applications.

The Schauder's Theorem. *Let X be a Banach space and $Y \subset X$ a non-empty bounded convex and closed subset. If $f : Y \to Y$ is a completely continuous mapping, i.e. it is compact and continuous, then f has at least a fixed point, i.e. there exists at least an element $x \in Y$ with the property that $f(x) = x$.*

3 The Darboux Problem for third order hyperbolic inclusions

Definition 3.1. The *Darboux Problem* for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 3.2. A function $u : D \to \mathbb{R}^n$ is called a *solution* of the Darboux Problem (1.1) + (1.2) if it is absolutely continuous in Carathéodory's sense on D, $u \in C^*(D; \mathbb{R}^n)$ [5, §565 - §570], and it satisfies (1.1) for a.e. $(x, y, z) \in D$, and also the initial conditions (1.2) for all $(x, y) \in D_1$, all $(y, z) \in D_2$, all $(x, z) \in D_3$.

Let the following hypotheses be satisfied:

(H_1) $F : D \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ is a non-empty and closed valued multifunction which is lower semi-continuous on $D \times \mathbb{R}^n$.

(H_2) There exists $M > 0$ such that

$$\sup\{\|\zeta\| \mid \zeta \in F(x, y, z, u)\} \leq M$$

for every $(x, y, z, u) \in D \times \mathbb{R}^n$, $(x, y, z) \in D$, $\|u\| \leq C$, $C > 0$.

(H_3) The functions $\varphi \in C^*(D_1; \mathbb{R}^n)$, $\psi \in C^*(D_2; \mathbb{R}^n)$, $\chi \in C^*(D_3, \mathbb{R}^n)$ given by (1.2) are absolutely continuous in Carathéodory's sense functions and satisfy conditions (1.3).

Remark 3.1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned}\alpha(x, y, z) &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - \varphi(x, 0) - \varphi(0, y) - \psi(0, z) + \psi(0, 0) = \\ &= \varphi(x, y) + \psi(y, z) + \chi(x, z) - v^1(x) - v^2(y) - v^3(z) + v^0, \end{aligned} \tag{3.1}$$

is an absolutely continuous in Carathéodory's sense function on D, $\alpha \in C^*(D; \mathbb{R}^n)$ [5, §565-§570].

Suppose that the following hypothesis holds:

(H_4) The function $\alpha : D \to \mathbb{R}^n$ defined by (3.1) is bounded, that is

$$\|\alpha(x,y,z)\| \le M_1, \quad M_1 > 0,\ (x,y,z) \in D. \tag{3.2}$$

Define $\mathcal{K}$ to be the set of absolutely continuous in Carathéodory's sense functions $u : D \to \mathbb{R}^n$, $u \in C^*(D;\mathbb{R}^n)$ satisfying

$$\|u(x,y,z) - \alpha(x,y,z)\| \le r, \quad \text{where } r > 0, \tag{3.3}$$

and

$$\left\| \frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \le M, \quad \text{for a.e. } (x,y,z) \in D, \tag{3.4}$$

and also the conditions (1.2).

Proposition 3.1. *The set $\mathcal{K}$ is a non-empty compact subset of $C(D;\mathbb{R}^n)$.*

Proof. The relation $u \in \mathcal{K}$ implies $u \in C(D;\mathbb{R}^n)$. We see that $\dfrac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z}$ exists for a.e. $(x,y,z) \in D$, as $u \in C^*(D;\mathbb{R}^n)$ [5]. Integrating $\dfrac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z}$ on D and using the conditions (1.2) we obtain

$$\begin{aligned}
u(x,y,z) = {} & u(x,y,0) + u(x,0,z) - u(x,0,0) + u(0,y,z) - u(0,y,z) - u(0,0,z) + \\
& + u(0,0,0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t} dr\, ds\, dt = \\
= {} & \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \\
& + u(0,0,0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t} dr\, ds\, dt = \\
= {} & \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0 + \\
& + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t} dr\, ds\, dt, \quad (x,y,z) \in D.
\end{aligned} \tag{3.5}$$

The compactness of the set $\mathcal{K}$results using the Arzelà-Ascoli theorem. The set $\mathcal{K}$ is equibounded. From (3.2), (3.3) and (3.4) we have

$$\begin{aligned}
\|u(x,y,z)\| &\le \|\alpha(x,y,z)\| + \int_0^x \int_0^y \int_0^z \left\| \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t} \right\| dr\, ds\, dt \le \\
&\le M_1 + \int_0^x \int_0^y \int_0^z M\, dr\, ds\, dt = M_1 + Mabc = C, \quad C > 0,\ (x,y,z) \in D.
\end{aligned}$$

The set $\mathcal{K}$ is equicontinuous. Using the absolute continuity of the integral, it follows that

$$\|u(x+h, y+k, z+l) - u(x,y,z)\| < \varepsilon \quad \text{for } h,k,l \in \mathbb{R} \quad \text{with } |h|, |k|, |l| < \delta(\varepsilon).$$

Theorem 3.1. *Let the hypotheses $(H_1) - (H_4)$ be satisfied. Then, the Darboux Problem $(1.1) + (1.2)$ has a local solution on $D_0 = [0, x_0] \times [0, y_0] \times [0, z_0] \subseteq D$.*

Proof. Let us define $\mathcal{F} : \mathcal{K} \to 2^{L^1(D;\mathbb{R}^n)}$ by

$$\mathcal{F}(u) = \left\{ f \in L^1(D;\mathbb{R}^n) \mid f(x,y,z) \in F(x,y,z,u(x,y,z)) \text{ for a.e. } (x,y,z) \in D \right\} \tag{3.6}$$

for each $u \in \mathcal{K}$.

In view of the hypotheses (H_1), (H_2), of (3.3) and of Corollary 2.1, $\mathcal{F}$ has non-empty and closed values. Since F is lower semi-continuous the multifunction $\mathcal{F}$ is lower semi-continuous, too. In addition, $\mathcal{F}$ is decomposable. Then, by Fryszkowski's Theorem 2.8, it follows that there exists a continuous function $f : \mathcal{K} \to L^1(D;\mathbb{R}^n)$ such that

$$f(u)(x,y,z) \in F(x,y,z,u(x,y,z)) \text{ for each } u \in \mathcal{K} \text{ and for a.e. } (x,y,z) \in D. \tag{3.7}$$

Let $h(u) : D \to \mathbb{R}^n$, $u \in \mathcal{K}$, be the continuous function defined by

$$h(u)(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z f(u)(r,s,t)\, dr\, ds\, dt. \tag{3.8}$$

Using (3.1) we have, by (3.8),

$$\begin{aligned} h(u)(x,y,z) = {} & \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \psi(0,0) + \\ & + \int_0^x \int_0^y \int_0^z f(u)(r,s,t)\, dr\, ds\, dt, \quad (x,y,z) \in D, \end{aligned} \tag{3.9}$$

which can be rewritten as

$$\begin{aligned} h(u)(x,y,z) = {} & \int_0^x \int_0^y \int_0^z f(u)(r,s,t)\, dr\, ds\, dt + \int_0^x \int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\, \partial s} dr\, ds + \\ & + \int_0^y \int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\, \partial t} ds\, dt + \int_0^x \int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\, \partial t} dr\, dt + \int_0^x \frac{\partial \varphi(r,0)}{\partial r} dr + \\ & + \int_0^y \frac{\partial \varphi(0,s)}{\partial s} ds + \int_0^z \frac{\partial \psi(0,t)}{\partial t} dt + u(0,0,0), \quad \text{for } (x,y,z) \in D. \end{aligned} \tag{3.10}$$

Indeed, we have

$$\begin{aligned} \int_0^x \int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\, \partial s} dr\, ds & = \int_0^x \left[\int_0^y \frac{\partial^2 \varphi(r,s)}{\partial r\, \partial s} ds \right] dr = \int_0^x \frac{\partial \varphi(r,s)}{\partial r} \bigg|_{s=0}^{s=y} dr = \\ & = \int_0^x \left[\frac{\partial \varphi(r,y)}{\partial r} - \frac{\partial \varphi(r,0)}{\partial r} \right] dr = \varphi(r,y) \Big|_{r=0}^{r=x} - \varphi(r,0) \Big|_{r=0}^{r=x} = \\ & = [\varphi(x,y) - \varphi(0,y)] - [\varphi(x,0) - \varphi(0,0)] = \\ & = \varphi(x,y) - \varphi(0,y) - \varphi(x,0) + \varphi(0,0), \end{aligned} \tag{3.11}$$

$$\begin{aligned} \int_0^y \int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\, \partial t} ds\, dt & = \int_0^y \left[\int_0^z \frac{\partial^2 \psi(s,t)}{\partial s\, \partial t} dt \right] ds = \int_0^y \frac{\partial \psi(s,t)}{\partial s} \bigg|_{t=0}^{t=z} ds = \\ & = \int_0^y \left[\frac{\partial \psi(s,z)}{\partial s} - \frac{\partial \psi(s,0)}{\partial s} \right] ds = \psi(s,z) \Big|_{s=0}^{s=y} - \psi(s,0) \Big|_{s=0}^{s=y} = \\ & = [\psi(y,z) - \psi(0,z)] - [\psi(y,0) - \psi(0,0)] = \\ & = \psi(y,z) - \psi(0,z) - \psi(y,0) + \psi(0,0), \end{aligned} \tag{3.12}$$

$$\int_0^x \int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\, \partial t} dr\, dt = \int_0^x \left[\int_0^z \frac{\partial^2 \chi(r,t)}{\partial r\, \partial t} dt \right] dr = \int_0^x \left. \frac{\partial \chi(r,t)}{\partial r} \right|_{t=0}^{t=z} dr =$$
$$= \int_0^x \left[\frac{\partial \chi(r,z)}{\partial r} - \frac{\partial \chi(r,0)}{\partial r} \right] dr = \chi(r,z) \Big|_{r=0}^{r=x} - \chi(r,0) \Big|_{r=0}^{r=x} =$$
$$= [\chi(x,z) - \chi(0,z)] - [\chi(x,0) - \chi(0,0)] =$$
$$= \chi(x,z) - \chi(0,z) - \chi(x,0) + \chi(0,0), \tag{3.13}$$

$$\int_0^x \frac{\partial \varphi(r,0)}{\partial r} dr = \varphi(r,0) \Big|_{r=0}^{r=x} = \varphi(x,0) - \varphi(0,0), \tag{3.14}$$

$$\int_0^y \frac{\partial \varphi(0,s)}{\partial s} ds = \varphi(0,s) \Big|_{s=0}^{s=y} = \varphi(0,y) - \varphi(0,0), \tag{3.15}$$

$$\int_0^z \frac{\partial \psi(0,t)}{\partial t} dt = \psi(0,t) \Big|_{t=0}^{t=z} = \psi(0,z) - \psi(0,0), \tag{3.16}$$

It follows from (1.3) that

$$u(0,0,0) = \varphi(0,0) = \psi(0,0) = \chi(0,0). \tag{3.17}$$

Replacing (3.11) – (3.17) in (3.10), it follows that (3.9) holds.

By Theorem 2.5 and (3.9) we conclude that $h(u) \in C^*(D; \mathbb{R}^n)$ for each $u \in \mathcal{K}$, i.e. $h(u)$ is absolutely continuous in Carathéodory's sense on D. One obtains $h(u) \in \mathcal{K}$ for each $u \in \mathcal{K}$,hence $h(\mathcal{K}) \subseteq \mathcal{K}$. Indeed, from (3.9) we have

$$\|h(u)(x,y,z) - \alpha(x,y,z)\| = \left\| \int_0^x \int_0^y \int_0^z f(u)(r,s,t)\, dr\, ds\, dt \right\| \leq$$
$$\leq \int_0^x \int_0^y \int_0^z \|f(u)(r,s,t)\|\, dr\, ds\, dt \leq$$
$$\leq Mxyz \leq Mx_0 y_0 z_0 \leq r,$$
$$\text{for } (x,y,z) \in D_0 = [0,x_0] \times [0,y_0] \times [0,z_0] \subseteq D. \tag{3.18}$$

Choose $(x_0, y_0, z_0) \in D$ such that the condition

$$Mx_0 y_0 z_0 \leq r \tag{3.19}$$

holds. By (3.18) and (3.19) we conclude that $h(u)$ satisfies the inequality (3.3).

Moreover, it follows from definition (3.8) that

$$\frac{\partial^3 h(u)(x,y,z)}{\partial x\, \partial y\, \partial z} = f(u)(x,y,z), \quad (x,y,z) \in D,$$

but, from (3.7), we have

$$\zeta = f(u)(x,y,z) \in F(x,y,z,u(x,y,z))$$

for each $u \in \mathcal{K}$ and for a.e. $(x,y,z) \in D$.

From the hypothesis (H_2), it follows that $\|\zeta\| = \|f(u)(x,y,z)\| \leq M$ for every $(x,y,z,u) \in D \times \mathbb{R}^n$, $(x,y,z) \in D$, $\|u\| \leq C$, $C > 0$. Consequently

$$\left\| \frac{\partial^3 h(u)(x,y,z)}{\partial x\,\partial y\,\partial z} \right\| \leq M, \quad (x,y,z,u) \in D \times \mathbb{R}^n, \tag{3.20}$$

i.e. $h(u)$ satisfies (3.4). It follows from (3.8) that $h(u)$ satisfies the conditions (1.2). Using (3.18), (3.19), (3.20), it results from definition of the set $\mathcal{K}$ that $h(u) \in \mathcal{K}$. Since $u \in \mathcal{K}$ implies $h(u) \in \mathcal{K}$, we conclude that $h(\mathcal{K}) \subseteq \mathcal{K}$. Since h is continuous and $\mathcal{K}$ is compact, from Mazur's Theorem 2.9 it follows that the set $\mathcal{K}_0 = \overline{\operatorname{conv} h(\mathcal{K})}$ is compact and convex. Using Schauder's Fixed Point Theorem 2.10, it follows that there exists $\bar{u} \in \mathcal{K}_0$ such that $\bar{u} = h(\bar{u})$, i.e. $h(\bar{u})(x,y,z) = \bar{u}(x,y,z)$, $(x,y,z) \in D_0$. This implies, in view of (3.7) and (3.8), that

$$\frac{\partial^3 \bar{u}(x,y,z)}{\partial x\,\partial y\,\partial z} = f(\bar{u})(x,y,z) \in F(x,y,z,\bar{u}(x,y,z))$$

for a.e. $(x,y,z) \in D_0 \subseteq D$, i.e. $\bar{u}$ satisfies (1.1). Moreover, $\bar{u}$ also satisfies (1.2), consequently $\bar{u}$ is a local solution of the Darboux Problem (1.1) + (1.2). The proof is complete.

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