

Upper and Lower Solutions Method for Partial Hyperbolic Differential Equations with Caputo's Fractional Derivative

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Abstract. In this paper we investigate the existence of solutions for a class of initial value problem for partial hyperbolic differential equations involving the Caputo fractional derivative by using the lower and upper solutions method combined with Schauder's fixed point theorem.

Keywords: Hyperbolic differential equation, fractional order, upper solution, lower solution, left-sided mixed Riemann-Liouville integral, Caputo fractional-order derivative, fixed point.

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1 Introduction

This paper deals with the existence of solutions to fractional order initial value problems (*IVP* for short), for the system

$$({}^c D_0^r u)(x, y) = f(x, y, u(x, y)); \text{ if } (x, y) \in J := [0, a] \times [0, b], \quad (1)$$

$$u(x, 0) = \varphi(x), \quad u(0, y) = \psi(y); \quad x \in [0, a], \quad y \in [0, b], \quad (2)$$

where $a, b > 0$, ${}^c D_0^r$ is the Caputo's fractional derivative of order $r = (r_1, r_2) \in (0, 1] \times (0, 1]$, $f : J \times \mathbb{R} \rightarrow \mathbb{R}$, is a given function and $\varphi : [0, a] \rightarrow \mathbb{R}$, $\psi : [0, b] \rightarrow \mathbb{R}$ are given absolutely continuous functions with $\varphi(0) = \psi(0)$.

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions was studied in numerous works [17, 27], a similar problem in spaces of continuous functions was studied in [28]. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [9, 11, 12, 14, 21, 22]. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas *et al.* [19], Lakshmikantham *et al.* [20], Miller and Ross [23], Samko *et al.* [26], the papers of Abbas and Benchohra [1, 2], Agarwal *et al.* [3], Belarbi *et al.* [4], Benchohra *et al.* [5, 6, 7], Diethelm [9, 10], Kilbas and Marzan

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[18], Mainardi [21], Podlubny *et al.* [25], Vityuk and Golushkov [29], Zhang [30] and the references therein.

In this paper, we initiate the application of the method of upper and lower solutions for hyperbolic differential equations involving the Caputo fractional derivative. These results based on Schauder's fixed point theorem [13]. The present results extend those considered with integer order derivative [8, 15, 16, 20, 24].

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $C(J, \mathbb{R})$ be the Banach space of continuous functions $u : J \rightarrow \mathbb{R}$ normed by

$$\|u\|_{\infty} = \sup\{|u(x, y)| : (x, y) \in J\}.$$

By $L^1(J, \mathbb{R})$ we denote the space of Lebesgue-integrable functions $f : J \rightarrow \mathbb{R}$ with the norm

$$\|f(x, y)\|_1 = \int_0^a \int_0^b |f(x, y)| dy dx.$$

As usual, by $AC(J, \mathbb{R})$ we denote the space of absolutely continuous functions from J into \mathbb{R} .

Let $r_1, r_2 > 0$ and $r = (r_1, r_2)$. For $f \in L^1(J, \mathbb{R})$, the expression

$$(I_0^r f)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t) dt ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function, is called the left-sided mixed Riemann-Liouville integral of order r .

Definition 2.1 ([19]) For $f \in L^1(J, \mathbb{R})$, the Caputo fractional-order derivative of order $r \in (0, 1] \times (0, 1]$ is defined by the expression $({}^c D_0^r f)(x, y) = (I_0^{1-r} \frac{\partial^2}{\partial x \partial y} f)(x, y)$.

3 Main Results

Let us start by defining what we mean by a solution of the problem (1)-(2).

Definition 3.1 A function $u \in C(J, \mathbb{R})$ with its mixed derivative $\frac{\partial^2 u}{\partial x \partial y}$ exists and is integrable is said to be a solution of (1)-(2) if u satisfies equations (1) and (2) on J .

Definition 3.2 A function $z \in C(J, \mathbb{R})$ is said to be a lower solution of (1)-(2) if z satisfies

$$({}^c D_0^r z)(x, y) \leq f(x, y, z(x, y)), \quad z(x, 0) \leq \varphi(x), \quad z(0, y) \leq \psi(y) \text{ on } J,$$

and $z(0, 0) \leq \varphi(0)$.

The function z is said to be an upper solution of (1)-(2) if the reversed inequalities hold.

Let $f \in L^1(J, \mathbb{R})$ and consider the following problem

$$\begin{cases} ({}^c D_0^r u)(x, y) = f(x, y); & (x, y) \in J, \\ u(x, 0) = \varphi(x); & x \in [0, a], \\ u(0, y) = \psi(y); & y \in [0, b], \\ \varphi(0) = \psi(0), \end{cases} \quad (3)$$

For the existence of solutions for the problem (1) – (2), we need the following lemma:

Lemma 3.3 ([1, 2]) *A function $u \in AC(J, \mathbb{R})$ is a solution of problem (3) if and only if $u(x, y)$ satisfies*

$$u(x, y) = \mu(x, y) + (I_0^r f)(x, y); \quad (x, y) \in J,$$

where

$$\mu(x, y) = \varphi(x) + \psi(y) - \varphi(0).$$

Corollary 3.4 *The function $u \in AC(J, \mathbb{R})$ is a solution of problem (1)-(2) if and only if u satisfies the equation*

$$u(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, u(s, t)) dt ds,$$

for all $(x, y) \in J$.

Further, we present conditions for the existence of a solution of our problem.

Theorem 3.5 *Assume that the following hypotheses*

(H1) *The function $f : J \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous,*

(H2) *There exist v and $w \in C(J, \mathbb{R})$, lower and upper solutions for the problem (1)-(2) such that $v \leq w$,*

hold. Then the problem (1)-(2) has at least one solution u such that

$$v(x, y) \leq u(x, y) \leq w(x, y) \quad \text{for all } (x, y) \in J.$$

Proof. Transform the problem (1)-(2) into a fixed point problem. Consider the following modified problem,

$$({}^c D_0^r u)(x, y) = g(x, y, u(x, y)), \quad \text{if } (x, y) \in J, \quad (4)$$

$$u(x, 0) = \varphi(x), \quad u(0, y) = \psi(y); \quad x \in [0, a], \quad y \in [0, b], \quad (5)$$

where

$$g(x, y, u(x, y)) = f(x, y, h(x, y, u(x, y))),$$

$$h(x, y, u(x, y)) = \max\{v(x, y), \min\{u(x, y), w(x, y)\}\},$$

for each $(x, y) \in J$.

A solution to (4)-(5) is a fixed point of the operator $N : C(J, \mathbb{R}) \rightarrow C(J, \mathbb{R})$ defined by,

$$N(u)(x, y) = \mu(x, y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} g(s, t, u(s, t)) dt ds.$$

Notice that g is a continuous function, and from (H2) there exists $M > 0$ such that

$$|g(x, y, u)| \leq M, \text{ for each } (x, y) \in J, \text{ and } u \in \mathbb{R}. \quad (6)$$

Set

$$\eta = \|\mu\|_\infty + \frac{Ma^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)},$$

and

$$D = \{u \in C(J, \mathbb{R}) : \|u\|_\infty \leq \eta\}.$$

Clearly D is a closed convex subset of $C(J, \mathbb{R})$ and that N maps D into D . We shall show that N satisfies the assumptions of Schauder's fixed point theorem. The proof will be given in several steps.

Step 1: N is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \rightarrow u$ in D . Then

$$\begin{aligned} & |N(u_n)(x, y) - N(u)(x, y)| \\ & \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y |x-s|^{r_1-1} |y-t|^{r_2-1} |g(s, t, u_n(s, t)) - g(s, t, u(s, t))| dt ds \\ & \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b |x-s|^{r_1-1} |y-t|^{r_2-1} \sup_{(s,t) \in J} |g(s, t, u_n(s, t)) - g(s, t, u(s, t))| dt ds \\ & \leq \frac{\|g(\cdot, \cdot, u_n(\cdot, \cdot)) - g(\cdot, \cdot, u)\|_\infty}{\Gamma(r_1)\Gamma(r_2)} \int_0^a \int_0^b |x-s|^{r_1-1} |y-t|^{r_2-1} dt ds \\ & \leq \frac{a^{r_1}b^{r_2} \|g(\cdot, \cdot, u_n(\cdot, \cdot)) - g(\cdot, \cdot, u(\cdot, \cdot))\|_\infty}{r_1 r_2 \Gamma(r_1)\Gamma(r_2)}. \end{aligned}$$

Since g is a continuous function, we have

$$\|N(u_n) - N(u)\|_\infty \leq \frac{a^{r_1}b^{r_2} \|g(\cdot, \cdot, u_n(\cdot, \cdot)) - g(\cdot, \cdot, u(\cdot, \cdot))\|_\infty}{\Gamma(r_1+1)\Gamma(r_2+1)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Step 2: $N(D)$ is bounded. This is clear since $N(D) \subset D$ and D is bounded.

Step 3: $N(D)$ is equicontinuous. Let $(x_1, y_1), (x_2, y_2) \in (0, a] \times (0, b]$, $x_1 < x_2$, $y_1 < y_2$ and $u \in D$. Then

$$\begin{aligned}
 & |N(u)(x_2, y_2) - N(u)(x_1, y_1)| \\
 = & |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
 & + \left| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_0^{y_1} [(x_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} - (x_1 - s)^{r_1-1}(y_1 - t)^{r_2-1}] \right. \\
 & \times g(s, t, u(s, t)) dt ds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_{y_1}^{y_2} (x_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} g(s, t, u(s, t)) dt ds \\
 & + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1} \int_{y_1}^{y_2} (x_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} g(s, t, u(s, t)) dt ds \\
 & \left. + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2} \int_0^{y_1} (x_2 - s)^{r_1-1}(y_2 - t)^{r_2-1} g(s, t, u(s, t)) dt ds \right| \\
 \leq & |\mu(x_1, y_1) - \mu(x_2, y_2)| \\
 & + \frac{M}{\Gamma(r_1 + 1)\Gamma(r_2 + 1)} [2y_2^{r_2}(x_2 - x_1)^{r_1} + 2x_2^{r_1}(y_2 - y_1)^{r_2} \\
 & + x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2 - x_1)^{r_1}(y_2 - y_1)^{r_2}].
 \end{aligned}$$

As $x_1 \rightarrow x_2$, $y_1 \rightarrow y_2$ the right-hand side of the above inequality tends to zero. As a consequence of steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : D \rightarrow D$ is continuous and compact. From Schauder's theorem, we deduce that N has a fixed point u which is a solution of the problem (4)-(5).

Step 4: The solution u of (4)-(5) satisfies

$$v(x, y) \leq u(x, y) \leq w(x, y) \text{ for all } (x, y) \in J.$$

We prove that

$$u(x, y) \leq w(x, y) \text{ for all } (x, y) \in J.$$

Assume that $u - w$ attains a positive maximum on J at $(\bar{x}, \bar{y}) \in J$; that is,

$$(u - w)(\bar{x}, \bar{y}) = \max\{u(x, y) - w(x, y) : (x, y) \in J\} > 0.$$

We distinguish the following cases.

Case 1. If $(\bar{x}, \bar{y}) \in (0, a) \times [0, b]$ there exists $(x^*, y^*) \in (0, a) \times [0, b]$ such that

$$u(x^*, y^*) - w(x^*, y^*) \leq 0, \tag{7}$$

and

$$u(x, y) - w(x, y) > 0, \text{ for all } (x, y) \in (x^*, \bar{x}] \times [y^*, b]. \tag{8}$$

By the definition of h one has

$${}^c D^r u(x, y) = f(x, y, w(x, y)) \text{ for all } (x, y) \in [x^*, \bar{x}] \times [y^*, b].$$

An integration on $[x^*, x] \times [y^*, y]$ for each $(x, y) \in [x^*, \bar{x}] \times [y^*, b]$ yields

$$u(x, y) - u(x^*, y^*) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x^*}^x \int_{y^*}^y (x-s)^{r_1-1} (y-t)^{r_2-1} f(s, t, w(s, t)) dt ds. \quad (9)$$

From (9) and using the fact that w is an upper solution to (1)-(2) we get

$$u(x, y) - u(x^*, y^*) \leq w(x, y) - w(x^*, y^*). \quad (10)$$

Thus from (7), (8) and (10) we obtain the contradiction

$$0 < u(x, y) - w(x, y) \leq u(x^*, y^*) - w(x^*, y^*) \leq 0, \text{ for all } (x, y) \in [x^*, \bar{x}] \times [y^*, b].$$

Case 2. If $\bar{x} = 0$, then

$$w(0, \bar{y}) < u(0, \bar{y}) \leq w(0, \bar{y})$$

which is a contradiction. Thus

$$u(x, y) \leq w(x, y) \text{ for all } (x, y) \in J.$$

Analogously, we can prove that

$$u(x, y) \geq v(x, y), \text{ for all } (x, y) \in J.$$

This shows that the problem (4)-(5) has a solution u satisfying $v \leq u \leq w$ which is solution of (1)-(2).

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