LIBERTAS MATHEMATICA, vol XXXI (2011)

A Neutral-Convolution Type Functional Equation

C. CORDUNEANU and Yizeng LI

Abstract. We are investigating the existence of almost periodic solutions, in the spaces $AP_r(R,\mathcal{C})$, to a class of neutral functional equations with convolution. The notations and definition of the convolution product are those in the paper [1] of the first author.

1 Introduction

The equations to be considered in this paper are of the form

$$\frac{d}{dt}\,[x(t)+(k*x)(t)] = (fx)(t), \quad t \in R, \tag{1.1}$$

with the linear counterpart

$$\frac{d}{dt}\,[x(t)+(k*x)(t)] = f(t), \quad t \in R, \tag{1.2}$$

where $x \in \mathcal{C}$, $k \in L^1(R,\mathcal{C})$ and $f \in AP_r(R,\mathcal{C})$, $1 \le r \le 2$. Briefly, the convolution product is given by

$$(k*x)(t) \sim \sum_{j=1}^{\infty} \widetilde{x}_j \exp(i\lambda_j t), \tag{1.3}$$

with

$$x(t) \sim \sum_{j=1}^{\infty} x_j \exp(i\lambda_j t), \tag{1.4}$$

and

$$\widetilde{x}_j = x_j \int_R k(s)\exp(-i\lambda_j s)ds, \quad j \ge 1. \tag{1.5}$$

It is understood that (1.4) shows the fact that the Fourier series, in a certain $AP_r(R,\mathcal{C})$, $1 \le r \le 2$, is the (formal) series in the right hand side of (1.4). Also, the right hand side of (1.3) represents the (formal) Fourier series of $k*x$.

In [1] has been shown that, for $k \in L^1(R,\mathcal{C})$,

$$|k*x|_r \le |k|_{L^1}|x|_r, \quad r \in [1,2], \; x \in AP_r(R,\mathcal{C}). \tag{1.6}$$

From (1.6), we see that the convolution product defined by (1.3) enjoys similar properties with the classical concept. In the case $r = 1$, one obtains the same product as in the classical case.

Concerning the formulation of the precise hypotheses, we shall distinguish the linear from the nonlinear case (assuming, in general, that $x \to fx$ is a nonlinear operator acting on $AP_r(R, \mathcal{C})$, with a fixed $r \in [1, 2]$).

2 The linear case (1.2)

We shall consider now equation (1.2), and in order to find an $x \in AP_r(R, \mathcal{C})$, i.e., to construct the series which characterizes a hypothetical solution of (1.2), we shall rewrite (1.2) in the equivalent form

$$x(t) + (k * x)(t) = \int^t f(s)ds + c_0, \quad t \in R, \tag{2.7}$$

the indefinite integral in the right hand side of (2.7) involving the arbitrary constant $c_0 \in \mathcal{C}$. We will discuss later how it can be determined (or not, finding the whole family of solutions to (1.2)).

The fist remark we make is that the function $f \in AP_r(R, \mathcal{C})$ must admit the indefinite integral in the same space. While we don't have a necessary and sufficient condition for the validity of this property, we can supply a *sufficient* condition. Indeed, in [1] it is shown that such a condition requires that f have mean value zero, $M\{f\} = 0$, and its Fourier exponents stay away from zero: $|\lambda_j| \geq m > 0$, $j \geq 1$.

We shall substitute now in (1.1) the x given by (1.4), the convolution product given by (1.3), and on the right hand side the series

$$c_0 - i \sum_{j=1}^{\infty} \lambda_j^{-1} f_j \exp(i\lambda_j t). \tag{2.8}$$

Equating the coefficients of the exponentials, from the first and second side, one obtains (from (2.8)) the equations

$$x_j + x_j \int_R k(s) \exp(-i\lambda_j s) ds = -i\lambda_j^{-1} f_j, \quad j \geq 1. \tag{2.9}$$

From (2.9) we see that the condition

$$1 + \widetilde{k}(i\omega) \neq 0, \quad \omega \in R, \tag{2.10}$$

is securing the solvability of the infinite system (2.9), uniquely, where $\widetilde{k}(i\omega)$ is the Fourier transform of k

$$\widetilde{k}(i\omega) = \int_R k(s) \exp(-i\omega s) ds, \quad \omega \in R. \tag{2.11}$$

Of course, (2.10) is stronger that $\widetilde{k}(i\lambda_j) + 1 \neq 0$, $j \geq 1$, but in the form (2.10) we can use it for any sequence $\{\lambda_j;\ j \geq 1\} \subset R$ of Fourier exponents of the function in the right hand side of the equation (1.2), i.e., for any $f \in AP_r(R, \mathcal{C})$ possessing a primitive in the same space.

There remains to prove that the function $x(t)$, with the Fourier coefficients

$$x_j = -i[1 + \widetilde{k}(i\lambda_j)]^{-1}\lambda_j^{-1} f_j, \quad j \geq 1, \tag{2.12}$$

is indeed in $AP_r(R, \mathcal{C})$. This assertion will be the consequence of the inequality

$$|[1 + \widetilde{k}(i\omega)]^{-1}\lambda_j^{-1}| \leq C, \quad C > 0, \ \omega \in R, \tag{2.13}$$

which follows from (2.10) and $|\lambda_j| \geq m > 0$, $j \geq 1$, hypotheses that have been accepted. See [1] and [2] for details. Hence, for some $C > 0$,

$$|x_j| \leq C|f_j|, \quad j \geq 1, \tag{2.14}$$

which proves that $x(t)$, as constructed above, belongs to $AP_r(R, \mathcal{C})$.

Before stating the basic result for the linear equation (1.2), we shall make a few more considerations related to the above carried discussion.

Namely, we have chosen the same set of Fourier exponents for both f and x. This fact is justified because the differentiation in the left hand side does not produce new Fourier exponents and $k * x$ has only those exponents that appear in x. Actually, this situation is encountered in many cases, the solutions of an equation having their exponents among those present in the functions involved in equation or in the module generated by the exponents.

To conclude this section regarding the linear case (1.2), we shall formulate the following result.

Theorem 2.1 *Consider the equation* (1.2) *under the following assumptions:*

I. *The kernel* $k \in L^1(R, \mathcal{C})$ *and satisfies* (2.10)*;*

II. $f \in AP_r(R, \mathcal{C})$, $r \in [1, 2]$, *fixed, and its Fourier exponents are such that*

$$|\lambda_j| \geq m > 0, \quad j \geq 1. \tag{2.15}$$

Then, there exists a unique solution $x \in AP_r(R, \mathcal{C})$, *given by* (1.4), *with* x_j, $j \geq 1$, *given by* (2.12).

Remark. The constant c_0 in (2.7), (2.8) must be taken equal to zero, because of the condition (2.10); excepting the case $\widetilde{k}(0) = 0$, not discussed here.

Proof of Theorem 2.1. It has been carried out before the statement. We notice the fact that (2.14) implies $|x_j|^r \leq C^r|f_j|^r$, $j \geq 1$, which has as a direct consequence $x \in AP_r(R, \mathcal{C})$. Details of this kind can be found in [1], [2].

3 On the nonlinear operator fx; examples

The operator $x \to fx$ must be acting on the space $AP_r(R, \mathcal{C})$. If one denotes $y = fx = AP_r(R, \mathcal{C})$, and

$$y \sim \sum_{j=1}^{\infty} y_j \exp(i\lambda_j t), \quad t \in R, \tag{3.16}$$

then $y_j,\ j \geq 1$, depend on $x_j,\ j \geq 1$ (see (1.4)), which can be written concisely as

$$y_j = \varphi_j(x_1, x_2, ..., x_n, ...); \quad j \geq 1, \tag{3.17}$$

in other words, to grasp the generality in its entirety, we should deal with functions depending of an infinity (countable!) of variables.

The case of Volterra type (causal) operators will mean

$$\varphi_j(x_1, x_2, ...) = \varphi_j(x_1, x_2, ..., x_j), \quad j \geq 1. \tag{3.18}$$

A very special case of (3.18) corresponds to the choice

$$\varphi_j(x_1, x_2, ..., x_j) \equiv \varphi_j(x_j), \tag{3.19}$$

when $y = fx$ becomes

$$fx \sim \sum_{j=1}^{\infty} \varphi(x_j) \exp(i\lambda_j t), \tag{3.20}$$

where $\varphi(u)$, $u, \varphi \in \mathcal{C}$, is a function defined on a certain set/disk $|u| \leq A$ in $\mathcal{C}$, such that $|x_j| \leq A$, $j \geq 1$. This type of nonlinear operators on $AP_r(R, \mathcal{C})$ enjoys other simple properties. For instance, if φ is Lipschitz continuous on the disk $|u| \leq A$, it can be easily checked that fx, given by (3.20), is also Lipschitz continuous on $AP_r(R, \mathcal{C})$.

To summarize the procedure shown above, we can associate to any complex-valued continuous semilinear function $\varphi(u)$, $|u| \leq A < \infty$, an operator acting on $AP_r(R, \mathcal{C})$, without characterizing series is

$$fx \sim \sum_{j=1}^{\infty} \varphi(x_j) \exp(i\lambda_j t), \tag{3.21}$$

for any sequence $\{\lambda_j;\ j \geq 1\} \subset R$, provided

$$\sum_{j=1}^{\infty} |\varphi(x_j)|^r < \infty. \tag{3.22}$$

(3.22) is verified, for instance, when $|\varphi(u)| \leq K|u|$, $K > 0$. The Lipschitz continuity for φ, in a disk $|u| \leq A$, with sufficiently large $A > 0$, will assure the continuity of the operator f from (3.21), on the space $AP_r(R, \mathcal{C})$, $1 \leq r \leq 2$.

4 The nonlinear case, equation (1.1)

Integrating both sides of equation (1.1), one obtains

$$x(t) + (k * x)(t) = c_0 + \int^t (fx)(s)ds, \quad t \in R, \tag{4.23}$$

where $c_0 \in \mathcal{C}$ is a constant.

A first remark is that $fx \subset AP_r(R,\mathcal{C})$ must have the indefinite integral in the same space. As we have seen in Section 2 above, this takes place (*sufficient* condition) in case each fx is deprived of the "free term", and the Fourier exponents satisfy an inequality of the form (2.15): $|\lambda_j| \geq m > 0$, $j \geq 1$.

We shall now establish an auxiliary result, concerning equation (1.2), in view of applying it to the existence proof for equation (1.1).

Lemma 4.2 *Consider equation* (1.1)*, under the assumptions of Theorem* 2.1*. Then, the unique solution, whose existence and uniqueness are stipulated in Theorem* 2.1*, satisfies an estimate of the form*

$$|x|_r \leq K|f|_r, \quad r \in [1,2], \tag{4.24}$$

with $K > 0$ a constant independent of f.

Proof of Lemma 4.2. It has been sketched at the beginning of the proof of Theorem 2.1, up to a point. Only has to be noticed that (2.14) leads to an inequality of the form (4.24).

We can now apply the Banach fixed point theorem, in order to obtain the following result.

Theorem 4.3 *Let us consider equation* (1.1)*, in the space $AP_r(R,\mathcal{C})$, $r \in [1,2]$, under the following assumptions:*

I. *The same as in Theorem* 2.1*.*

II. *The map $x \to fx$ is acting on $AP_r(R,\mathcal{C})$, for fixed $r \in [1,2]$, and satisfies a Lipschitz type condition*

$$|fx - fy|_r \leq \rho|x-y|_r, \tag{4.25}$$

with $\rho > 0$ sufficiently small. Then there exists a unique solution of (1.1)*, in $AP_r(R,\mathcal{C})$.*

Proof of Theorem 4.3. Starting from equation (1.1), and taking into account the result of Theorem 2.1, we define on $AP_r(R,\mathcal{C})$, for a fixed $r \in [1,2]$, an operator T whose fixed point will be the solution of (1.1). Namely, the operator $y \to Ty = x$, is defined by means of the equation

$$\frac{d}{dt}\,[x(t) + (k * x)(t)] = (fy)(t), \quad t \in R, \tag{4.26}$$

for $y \in AP_r(R,\mathcal{C})$, and taking the unique solution $x \in AP_r(R,\mathcal{C})$ of (4.26). Actually, we restrict our considerations to the subspace of $AP_r(R,\mathcal{C})$, such that $M\{f\} = 0$ and the Fourier exponents satisfy (2.15), with $m > 0$ the same for all elements. Applying (4.24), we obtain from (4.26), taking the pairs $y_1 \to x_1 = Ty_1$, $y_2 \to x_2 = Ty_2$,

$$|x_1 - x_2|_r = |Ty_1 - Ty_2|_r \leq K\rho|y_1 - y_2|_r, \tag{4.27}$$

which shows the contraction of T when $\rho < K^{-1}$. Hence, Theorem 4.3 is proven.

Generalizations to the case when x takes values in $\mathcal{C}^n$, or even in a Hilbert space, are possible.

References

[1] C. Corduneanu, A scale of almost periodic function spaces, *Differential and Integral Equations*, **24** (2011), 1–27.

[2] C. Corduneanu, $AP_r(R, \mathcal{C})$-almost periodic solutions to functional differential equations with deviated argument, to appear in *Functional Differential Equations*.

[3] M. Mahdavi, AP_r-almost periodic solutions to the equation $\dot{x}(t) = Ax(t)+(k*x)(t)+f(t)$ (to appear in *African Diaspora Mathematical Journal*).

[4] C. Corduneanu, Yizeng Li and M. Mahdavi, *Special Topics in Functional Equations* (in preparation).