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A Note on Strong Differential Subordinations Using Sălăgean and Ruscheweyh Operators

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Abstract. In the present paper we establish several strong differential subordinations regardind the new operator SR^m defined by the Hadamard product of the Sălăgean operator S^m and the Ruscheweyh operator R^m, given by $SR^m : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$, $SR^m f(z,\zeta) = (S^m * R^m) f(z,\zeta)$, where $\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),$ $f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U, \zeta \in \overline{U}\}$ is the class of normalized analytic functions.

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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : |z| \le 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z,\zeta) = z + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n+1$, and

$$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U \times \overline{U}),\ f(z) = a + a_n(\zeta) z^n + a_{n+1}(\zeta) z^{n+1} + \ldots,\ z \in U,\ \zeta \in \overline{U}\},$$

for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \ge n$.

Denote by

$$\mathcal{H}_u(U) = \left\{f \in \mathcal{H}^*[a,n,\zeta],\ f(z,\zeta) \text{ univalent in } U, \text{ for all } \zeta \in \overline{U}\right\},$$

$$K^* = \left\{f \in \mathcal{H}^*[a,n,\zeta],\ \operatorname{Re} \frac{zf''(z,\zeta)}{f'(z,\zeta)} + 1 > 0,\ z \in U, \text{ for all } \zeta \in \overline{U}\right\},$$

the class of convex functions and

$$S^* = \left\{f \in \mathcal{H}^*[a,n,\zeta],\ \operatorname{Re} \frac{zf'(z,\zeta)}{f(z,\zeta)} > 0,\ z \in U, \text{ for all } \zeta \in \overline{U}\right\}$$

the class of starlike functions.

Definition 1.1. [2] Let $f(z,\zeta)$, $H(x,\zeta)$ analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\zeta)=H(w(z),\zeta)$ for all $\zeta\in\overline{U}$. In such a case we write $f(z,\zeta)\prec\prec H(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$.

Remark 1.1. [2] (i) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$, and univalent in U, for all $\zeta\in\overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta)=H(0,\zeta)$, for all $\zeta\in\overline{U}$, and $f\left(U\times\overline{U}\right)\subset H\left(U\times\overline{U}\right)$.

(ii) If $H(z,\zeta)\equiv H(z)$ and $f(z,\zeta)\equiv f(z)$, the strong subordination becomes the usual notion of subordination.

Lemma 1.1. [4, p. 71] *Let $h(z,\zeta)$ be a convex function with $h(0,\zeta)=a$ for every $\zeta\in\overline{U}$ and let $\gamma\in\mathbb{C}^*$ be a complex number with $\operatorname{Re}\gamma\geq 0$. If $p\in\mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta)+\frac{1}{\gamma}zp'(z,\zeta)\prec\prec h(z,\zeta),$$

then

$$p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),$$

where $g(z,\zeta)=\dfrac{\gamma}{nz^{\frac{\gamma}{n}}}\int\limits_0^z h(t,\zeta)\,t^{\frac{\gamma}{n}-1}dt$ *is convex and it is the best dominant.*

Lemma 1.2. [3] *Let $g(z,\zeta)$ be a convex function in U, for all $\zeta\in\overline{U}$, and let*

$$h(z,\zeta)=g(z,\zeta)+n\alpha zg'(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

where $\alpha>0$ and n is a positive integer. If

$$p(z,\zeta)=g(0,\zeta)+p_n(\zeta)z^n+p_{n+1}(\zeta)z^{n+1}+\dots,\quad z\in U,\ \zeta\in\overline{U},$$

is holomorphic in U, for all $\zeta\in\overline{U}$, and

$$p(z,\zeta)+\alpha zp'(z,\zeta)\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

then

$$p(z,\zeta)\prec\prec g(z,\zeta)$$

and this result is sharp.

Definition 1.2. (Sălăgean [6]) For $f\in\mathcal{A}^*_{n\zeta}$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, the operator S^m is defined by $S^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,

$$\begin{aligned}S^0f(z,\zeta)&=f(z,\zeta)\\ S^1f(z,\zeta)&=zf'(z,\zeta)\\ &\dots\\ S^{m+1}f(z,\zeta)&=z\left(S^mf(z,\zeta)\right)',\quad z\in U,\ \zeta\in\overline{U}.\end{aligned}$$

Remark 1.2. If $f\in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum\limits_{j=n+1}^{\infty}a_j(\zeta)z^j$, then

$$S^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}j^m a_j(\zeta)z^j,\quad z\in U,\ \zeta\in\overline{U}.$$

Definition 1.3. (Ruscheweyh [5]) For $f\in\mathcal{A}^*_{n\zeta}$, $n\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$, the operator R^m is defined by $R^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,

$$\begin{aligned} R^0 f(z,\zeta)&=f(z,\zeta)\\ R^1 f(z,\zeta)&=zf'(z,\zeta)\\ &\dots\\ (m+1)R^{m+1}f(z,\zeta)&=z(R^m f(z,\zeta))'+mR^m f(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.\end{aligned}$$

Remark 1.3. If $f\in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum\limits_{j=n+1}^{\infty}a_j(\zeta)z^j$, then

$$R^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}a_j(\zeta)z^j,\quad z\in U,\ \zeta\in\overline{U}.$$

2 Main Results

Definition 2.1. [1] Let $m\in\mathbb{N}\cup\{0\}$. Denote by SR^m the operator given by the Hadamard product (the convolution product) of the Sălăgean operator S^m and the Ruscheweyh operator R^m, $SR^m:\mathcal{A}^*_{n\zeta}\to\mathcal{A}^*_{n\zeta}$,

$$SR^m f(z,\zeta)=(S^m * R^m)f(z,\zeta).$$

Remark 2.1. If $f\in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum\limits_{j=n+1}^{\infty}a_j(\zeta)z^j$, then

$$SR^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^m a_j^2(\zeta)z^j,\quad z\in U,\ \zeta\in\overline{U}.$$

Theorem 2.1. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}^*_{n\zeta}$ and the strong differential subordination*

$$\frac{1}{z}SR^{m+1}f(z,\zeta)+\frac{m}{m+1}z(SR^m f(z,\zeta))''\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}\qquad(2.1)$$

holds, then

$$(SR^m f(z,\zeta))'\prec\prec g(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}$$

and this result is sharp.

Proof. With notation

$$p(z,\zeta)=(SR^m f(z,\zeta))'=1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^{m+1}a_j^2(\zeta)z^{j-1}$$

and $p(0,\zeta)=1$, we obtain for

$$f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j,$$
$$p(z,\zeta)+zp'(z,\zeta)=\frac{1}{z}SR^{m+1}f(z,\zeta)+z\frac{m}{m+1}(SR^m f(z,\zeta))''.$$

We have $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.2 we obtain $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $(SR^m f(z,\zeta))'\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ and this result is sharp. □

Theorem 2.2. *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}_{n\zeta}^{*}$ and the strong differential subordination*

$$\frac{1}{z}SR^{m+1}f(z,\zeta)+\frac{m}{m+1}z(SR^m f(z,\zeta))''\prec\prec h(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U} \qquad (2.2)$$

holds, then

$$(SR^m f(z,\zeta))'\prec\prec g(z,\zeta)\prec\prec h(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$ *is convex and it is the best dominant.*

Proof. With notation

$$p(z,\zeta)=(SR^m f(z,\zeta))'=1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^{m+1}a_j^2(\zeta)z^{j-1}$$

and $p(0,\zeta)=1$, we obtain for

$$f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j,$$
$$p(z,\zeta)+zp'(z,\zeta)=\frac{1}{z}SR^{m+1}f(z,\zeta)+z\frac{m}{m+1}(SR^m f(z,\zeta))''.$$

We have $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. Since $p(z,\zeta)\in\mathcal{H}^*[1,n,\zeta]$, using Lemma 1.1, for $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$(SR^m f(z,\zeta))'\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta),\ z\in U,\zeta\in\overline{U},$$

and $g(z,\zeta)$ is convex and it is the best dominant. □

Theorem 2.3. *Let $g(z,\zeta)$ be a convex function, $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$(SR^m f(z,\zeta))' \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.3}$$

then

$$\frac{SR^m f(z,\zeta)}{z} \prec\prec g(z,\zeta), \ z\in U,\ \zeta\in\overline{U},$$

and this result is sharp.

Proof. For $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$ we have

$$SR^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^m a_j^2(\zeta)z^j, z\in U,\zeta\in\overline{U}.$$

Consider

$$p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}=\frac{z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^m a_j^2(\zeta)z^j}{z}=1+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^m a_j^2(\zeta)z^{j-1}.$$

We have

$$p(z,\zeta)+zp'(z,\zeta)=(SR^m f(z,\zeta))', \quad z\in U,\ \zeta\in\overline{U}.$$

Then $(SR^m f(z,\zeta))' \prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$ becomes

$$p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1.2 we obtain $p(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $\frac{SR^m f(z,\zeta)}{z}\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. □

Theorem 2.4. *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}^*_{n\zeta}$ and verifies the strong differential subordination*

$$(SR^m f(z,\zeta))' \prec\prec h(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \tag{2.4}$$

then

$$\frac{SR^m f(z,\zeta)}{z}\prec\prec g(z,\zeta)\prec\prec h(z,\zeta), \ z\in U,\ \zeta\in\overline{U},$$

where $g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$ is convex and it is the best dominant.

Proof. For $f\in\mathcal{A}^*_{n\zeta}$, $f(z,\zeta)=z+\sum_{j=n+1}^{\infty}a_j(\zeta)z^j$ we have

$$SR^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C^m_{m+j-1}j^m a_j^2(\zeta)z^j, \quad z\in U,\ \zeta\in\overline{U}.$$

Consider

$$p(z,\zeta)=\frac{SR^m f(z,\zeta)}{z}=\frac{z+\sum\limits_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^j}{z}$$
$$=1+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^{j-1}\in\mathcal{H}^*[1,n,\zeta].$$

We have $p(z,\zeta)+zp'(z,\zeta)=(SR^m f(z,\zeta))'$, $z\in U$, $\zeta\in\overline{U}$.

Then $(SR^m f(z,\zeta))'\prec\prec h(z,\zeta)$, $\quad z\in U$, $\zeta\in\overline{U}$ becomes $p(z,\zeta)+zp'(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. By using Lemma 1.1, for $\gamma=1$, we obtain $p(z,\zeta)\prec\prec g(z,\zeta)\prec\prec h(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e.

$$\frac{SR^m f(z,\zeta)}{z}\prec\prec g(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt\prec\prec h(z,\zeta),\ z\in U,\zeta\in\overline{U},$$

and $g(z,\zeta)$ is convex and it is the best dominant. □

Theorem 2.5. *Let $g(z,\zeta)$ be a convex function such that $g(0,\zeta)=1$ and let h be the function $h(z,\zeta)=g(z,\zeta)+zg'(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$. If $m\in\mathbb{N}\cup\{0\}$, $f\in\mathcal{A}_{n\zeta}^*$ and verifies the strong differential subordination*

$$\left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'\prec\prec h(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},\tag{2.5}$$

then

$$\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\prec\prec g(z,\zeta),\ \ z\in U,\ \zeta\in\overline{U}$$

and this result is sharp.

Proof. For $f\in\mathcal{A}_{n\zeta}^*$, $f(z,\zeta)=z+\sum\limits_{j=n+1}^{\infty}a_j(\zeta)z^j$ we have

$$SR^m f(z,\zeta)=z+\sum_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^j,\quad z\in U,\ \zeta\in\overline{U}.$$

Consider

$$p(z,\zeta)=\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}=\frac{z+\sum\limits_{j=n+1}^{\infty}C_{m+j}^{m+1}j^{m+1}a_j^2(\zeta)z^j}{z+\sum\limits_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^j}$$
$$=\frac{1+\sum\limits_{j=n+1}^{\infty}C_{m+j}^{m+1}j^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum\limits_{j=n+1}^{\infty}C_{m+j-1}^{m}j^m a_j^2(\zeta)z^{j-1}}.$$

We have

$$p'(z,\zeta) = \frac{\left(SR^{m+1}f(z,\zeta)\right)'}{SR^m f(z,\zeta)} - p(z,\zeta) \cdot \frac{\left(SR^m f(z,\zeta)\right)'}{SR^m f(z,\zeta)}.$$

Then

$$p(z,\zeta) + zp'(z,\zeta) = \left(\frac{zSR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)}\right)'.$$

Relation (2.5) becomes

$$p(z,\zeta) + zp'(z,\zeta) \prec\prec h(z,\zeta) = g(z,\zeta) + zg'(z,\zeta), \ z \in U, \ \zeta \in \overline{U}$$

and by using Lemma 1.2 we obtain $p(z,\zeta) \prec\prec g(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.

$$\frac{SR^{m+1}f(z,\zeta)}{SR^m f(z,\zeta)} \prec\prec g(z,\zeta), \ z \in U, \ \zeta \in \overline{U}.$$

□

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