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Continuous Selections for Continuous Multifunctions and the Darboux Problem for Third Order Hyperbolic Inclusions. The Refined Case

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Abstract. In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form $u_{xyz} \in F(x, y, z, u)$, where F is a continuous multifunction defined on a compact subset of $\mathbb{R}^{n+3}$ and whose values are non-empty compact and not necessarily convex subsets in $\mathbb{R}^n$. We prove a theorem which establishes the existence of a continuous selection for each of the functions $(x, y, z) \to F(x, y, z, u(x, y, z))$ with respect to a given family of continuous functions $(x, y, z) \to u(x, y, z)$. In two preceding paper [36], [37], we considered F to be a continuous multifunction and respectively F satisfying the Carathéodory type conditions. Now we consider the continuous case refined. Using this result, i.e. the selection theorem, which is stronger than Theorem 4.1 [36] and Schauder's Fixed Point Theorem, it is obtained an existence theorem for an absolutely continuous solution of the Darboux Problem for the specified inclusion.

Keywords: multifunction, upper and lower semicontinuity of multifunctions, the Hausdorff-Pompeiu metric, the characteristic function of a set, selection, hyperbolic inclusion, initial values, absolutely continuous in Carathéodory's sense function, the continuous partition of the unity.

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1 Introduction

In this paper we consider the Darboux Problem for a third order hyperbolic inclusion of the form

$$\frac{\partial^3 u(x, y, z)}{\partial x\, \partial y\, \partial z} \in F(x, y, z, u), \quad (x, y, z) \in D = [0, a] \times [0, b] \times [0, c], \ u \in B \subset \mathbb{R}^n, \tag{1.1}$$

with the initial values

$$\begin{cases} u(x, y, 0) = \varphi(x, y), & (x, y) \in D_1 = [0, a] \times [0, b], \\ u(0, y, z) = \psi(y, z), & (y, z) \in D_2 = [0, b] \times [0, c], \\ u(x, 0, z) = \chi(x, z), & (x, z) \in D_3 = [0, a] \times [0, c], \end{cases} \tag{1.2}$$

where φ, ψ, χ are absolutely continuous in Carathéodory's sense functions [4,§565-§570], $\varphi \in C^*(D_1;\mathbb{R}^n)$, $\psi \in C^*(D_2;\mathbb{R}^n)$, $\chi \in C^*(D_3;\mathbb{R}^n)$ and they satisfy the conditions

$$\begin{cases} u(x,0,0) = \varphi(x,0) = \chi(x,0) = v^1(x), & x \in [0,a], \\ u(0,y,0) = \varphi(0,y) = \psi(y,0) = v^2(y), & y \in [0,b], \\ u(0,0,z) = \psi(0,z) = \chi(0,z) = v^3(z), & z \in [0,c], \\ u(0,0,0) = v^1(0) = v^2(0) = v^3(0) = v^0. \end{cases} \tag{1.3}$$

$F : D \times B \to \operatorname{comp} A$ is a continuous multifunction whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$, A is the closed ball centered at the origin of $\mathbb{R}^n$ with radius M and B is the closed ball centered at the origin of $\mathbb{R}^n$ with radius $r = M_1 + M\,abc$.

We prove a theorem which establishes the existence of a continuous selection for each of the functions $(x,y,z) \to F(x,y,z,u(x,y,z))$ with respect to a given family of continuous functions $(x,y,z) \to u(x,y,z)$. In two preceding paper [36], [37] we considered F as a continuous multifunction and respectively F satisfying the Carathéodory type conditions. Now we consider the continuous case refined. This result is stronger that Theorem 4.1 [36]. The approximation that we obtain, which is basically elementary, consists in systematically using the continuous partition of the unity and also in construction some appropiate approximate selections. The role thereof is somehow analogous to the role played by step functions in approximating continuous functions with real values defined on a compact interval, and to the role played by the approximate solutions in construction the solutions of the ordinary or generalized differential equations, that is differential inclusions as well. Using this result, i.e. the selection theorem and Schauder's Fixed Point Theorem it is obtained an existence theorem of an absolutely continuous solution for Darboux Problem for the specified inclusion.

In [32] we considered the Darboux Problem (1.1) + (1.2) where $F : D \times \Omega \to 2^{\mathbb{R}^n}$ is a multifunction with compact convex and non-empty values and $\Omega \subset \mathbb{R}^n$ is an open subset. Under suitable assumptions, we proved an existence theorem for a local solution of the Darboux Problem (1.1) + (1.2) using the Kakutani Ky-Fan Fixed Point Theorem, and that the set of its solutions is compact in Banach space $C(D_0;\mathbb{R}^n)$, $D_0 = [0,x_0] \times [0,y_0] \times [0,z_0] \subseteq D$; moreover, as a function of the initial values this set defines an upper semi-continuous multifunction.

In [33] we proved a theorem of prolongation for the solutions of the considered problem (1.1) + (1.2) and also an existence theorem for a saturated solution.

In [34] we proved a characterization theorem for the solutions of Darboux Problem (1.1)+ (1.2) using the Aumann integral [2] defined for multifunctions.

In [35], using the notion of uniform convergence on compact sets defined by Arrigo Cellina [7], [8] for a sequence of single-valued functions $f_k : \Lambda \to \mathbb{R}^n$ such that $f_k \to F$, where F is a multifunction, we considered a sequence of approximating univalued equations of the form $u_{xyz} = f_k(x,y,z,u)$ and we proved that they have a unique solution, using Schauder's Fixed Point Theorem. Using a characterization theorem for the solutions of the Darboux Problem (1.1) + (1.2) for the specified inclusion [34], we proved that the sequence of solutions to the approximating univalued equations uniformly converges, on compact sets, to a solution of the Darboux Problem (1.1) + (1.2) for the considered inclusion; hence we obtained a global solution of this problem as the uniform limit of the sequence of solutions for the approximating equations.

In [36] we considered the Darboux Problem (1.1) + (1.2), where $F : D \times \Omega \to \operatorname{comp} A$ is a continuous multifunction whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$. We proved a theorem which establishes the existence of a continuous selection for each of the functions $(x, y, z) \to F(x, y, z, u(x, y, z))$ with respect to a given family of continuous functions $(x, y, z) \to u(x, y, z)$. Using this result and Schauder's Fixed Point Theorem it is obtained an existence theorem of an absolutely continuous solution for Darboux Problem (1.1) + (1.2).

In [37] we considered the Darboux Problem (1.1) + (1.2), where $F : D \times B \to \operatorname{comp} A$ satisfy the Carathéodory type conditions and whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$. We proved an existence theorem of a continuous selection similar to [36] and another existence theorem of an absolutely continuous solution for Darboux Problem (1.1) + (1.2).

This paper has been suggested by [1], [27], [28], [31], [36], [37] and it provides an extension of the results in those articles.

2 Preliminaries

The definitions and Theorems 2.1 – 2.6 in this section are recalled from [1]-[26].

Definition 2.1. Let X and Y be two non-empty sets. A *multifunction* $\Phi : X \to 2^Y$ is a function from X into the family of all non-empty subsets of Y.

To each $x \in X$, a subset $\Phi(x)$ of Y is associated by the multifunction Φ. The set $\bigcup\limits_{x\in X} \Phi(x)$ is the *range* of Φ. $\Phi(X) = \left\{ \bigcup\limits_{x\in X} \Phi(x) \mid x \in X \right\}$.

Definition 2.2. Let us consider $\Phi : X \to 2^Y$.

a) If $A \subset X$, the *image* of A by Φ is $\Phi(A) = \bigcup\limits_{x\in A} \Phi(x)$;

b) If $B \subset Y$, the *counterimage* of B by Φ is

$$\Phi^-(B) = \{x \in X \mid \Phi(x) \cap B \neq \emptyset\};$$

c) The *graph* of Φ, denoted $\operatorname{graph}\Phi$, is the set

$$\operatorname{graph}\Phi = \{(x, y) \in X \times Y \mid y \in \Phi(x)\}.$$

Definition 2.3. Let us now take $\Phi : X \to 2^Y$. An element $x \in X$ with the property that $x \in \Phi(x)$ is called a *fixed point* of the multifunction Φ.

Definition 2.4. A univalued function $\varphi : X \to Y$ is said to be a *selection* of $\Phi : X \to 2^Y$ if $\varphi(x) \in \Phi(x)$ for all $x \in X$.

Definition 2.5. Let X and Y be two topological spaces. The multifunction $\Phi : X \to 2^Y$ is *upper semi-continuous* if, for any closed $B \subseteq Y$, $\Phi^-(B)$ is closed in X.

Definition 2.6. If X and Y are two topological spaces, the multifunction $\Phi : X \to 2^Y$ is *lower semi-continuous* if, for every open subset $\Omega \subseteq Y$, the set $\Phi^-(\Omega)$ is open in X.

Definition 2.7. The multifunction $\Phi : X \to 2^Y$ is *continuous* if it is upper semi-continuous and lower semi-continuous.

Definition 2.8. If $(X, \mathcal{F})$ is a measurable space and Y is a topological space, the multifunction $\Phi : X \to 2^Y$ is *measurable* (*weakly measurable*), if $\Phi^- (B) \in \mathcal{F}$ for every closed (open) subset $B \subseteq Y$, $\mathcal{F}$ being the σ-algebra of the measurable sets of X, i.e. $\Phi^- (B)$ is measurable.

Theorem 2.1 [23]. *Let X and Y be two metric spaces, Y being compact, and $\Phi : X \to 2^Y$ a multifunction with the property that $\Phi (x)$ is a closed subset of Y for any $x \in X$. The following assertions are equivalent:*

i) the multifunction Φ is upper semi-continuous;

ii) the graph of Φ is a closed subset of $X \times Y$;

iii) any would be the sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, from $x_n \to x$, $y_n \in \Phi (x_n)$ and $y_n \to y$ it follows that $y \in \Phi (x)$.

Theorem 2.2 [15]. *When X is first countable and Y is a metric space, $\Phi : X \to 2^Y$ is lower semi-continuous if and only if, for every $\tilde{x} \in X$, every sequence $\{x_n\}$ in X converging to $\tilde{x}$ and every $\tilde{y} \in \Phi (\tilde{x})$, there exists a sequence $\{y_n\}$ in Y converging to $\tilde{y}$, such that $y_n \in \Phi (x_n)$ for all $n \in \mathbb{N}$.*

Definition 2.9. [4], [9], [13] The function $u : \triangle \to \mathbb{R}^n, \triangle \subset \mathbb{R}^2$, is *absolutely continuous in Carathéodory's sense* [4, §565 - §570] if and only if it is continuous on $\triangle$, absolutely continuous in x (for any y), absolutely continuous in y (for any x), $u_x (x, y)$ is (possibly after a suitable definition on a two-dimensional set of zero measure) absolutely continuous in y (for any x) and u_{xy} is Lebesgue-integrable on $\triangle$.

Theorem 2.3. [4], [9], [26] *The function $u : \triangle \to \mathbb{R}^n$, $\triangle = [0, a] \times [0, b] \subset \mathbb{R}^2$, is absolutely continuous in Carathéodory's sense on $\triangle$ if and only if there exist $f \in L^1 (\triangle; \mathbb{R}^n)$, $g \in L^1 ([0, a] ; \mathbb{R}^n)$, $h \in L^1 ([0, b] ; \mathbb{R}^n)$ such that*

$$u (x, y) = \int_0^x \int_0^y f (s, t) \, ds \, dt + \int_0^x g (s) \, ds + \int_0^y h (t) \, dt + u (0, 0) .$$

We denote the class of absolutely continuous functions in Carathéodory's sense by $C^* (\triangle; \mathbb{R}^n)$ [13]. In [9], this space is denoted by $AC (\triangle; \mathbb{R}^n)$.

Theorem 2.4. [9] *The space $C^* (\triangle; \mathbb{R}^n)$ endowed with the norm*

$$\|u(\cdot, \cdot)\| = \int_0^a \int_0^b \|u_{xy}(s, t)\| \, ds \, dt + \int_0^a \|u_x(s, 0)\| \, ds + \int_0^b \|u_y(0, t)\| \, dt + \|u(0, 0)\| ,$$

where $\triangle = [0, a] \times [0, b] \subset \mathbb{R}^2$, and $\|\cdot\|$ is the Euclidean norm, is a Banach space.

Definition 2.10. [14] The function $u : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^3$, is absolutely *continuous in Carathéodory's sense* [4, §565 - §570] if and only if $u (x, y, z)$ is continuous on D, absolutely continuous in each variable (for any pair of the other two variables) and similarly for $u_x (x, y, z)$, $u_y (x, y, z)$, $u_z (x, y, z)$, $u_{xy} (x, y, z)$, $u_{yz} (x, y, z)$, $u_{xz} (x, y, z)$, and u_{xyz} is Lebesgue-integrable on D.

Theorem 2.5. (analogous with [9]) *The function* $u : D \to \mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, *is absolutely continuous in Carathéodory's sense on* D *if and only if there exist* $f \in L^1(D;\mathbb{R}^n)$, $g_1 \in L^1(D_1;\mathbb{R}^n)$, $g_2 \in L^1(D_2;\mathbb{R}^n)$, $g_3 \in L^1(D_3;\mathbb{R}^n)$, $h_1 \in L^1([0,a];\mathbb{R}^n)$, $h_2 \in L^1([0,b];\mathbb{R}^n)$, $h_3 \in L^1([0,c];\mathbb{R}^n)$, *such that*

$$\begin{aligned} u(x,y,z) = &\int_0^x \int_0^y \int_0^z f(r,s,t)\,dr\,ds\,dt + \int_0^x \int_0^y g_1(r,s)\,dr\,ds + \\ &+ \int_0^y \int_0^z g_2(s,t)\,ds\,dt + \int_0^x \int_0^z g_3(r,t)\,dr\,dt + \\ &+ \int_0^x h_1(r)\,dr + \int_0^y h_2(s)\,ds + \int_0^z h_3(t)\,dt + u(0,0,0). \end{aligned}$$

We denote the class of absolutely continuous functions in Carathéodory's sense on D by $C^*(D;\mathbb{R}^n)$ [14].

Theorem 2.6. (analogous with [9]) *The space* $C^*(D;\mathbb{R}^n)$ *endowed with the norm*

$$\begin{aligned} \|u(\cdot,\cdot,\cdot)\| = &\int_0^a \int_0^b \int_0^c \|u_{xyz}(r,s,t)\|\,dr\,ds\,dt + \int_0^a \int_0^b \|u_{xy}(r,s,0)\|\,dr\,ds + \\ &+ \int_0^b \int_0^c \|u_{yz}(0,s,t)\|\,ds\,dt + \int_0^a \int_0^c \|u_{xz}(r,0,t)\|\,dr\,dt + \\ &+ \int_0^a \|u_x(r,0,0)\|\,dr + \int_0^b \|u_y(0,s,0)\|\,ds + \\ &+ \int_0^c \|u_z(0,0,t)\|\,dt + \|u(0,0,0)\|, \end{aligned}$$

where $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, *and* $\|\cdot\|$ *is the Euclidean norm, is a Banach space.*

We denote by $d(x,y)$ the Euclidean distance from x to y, $x,y \in \mathbb{R}^n$, where $\mathbb{R}^n$ is the Euclidean space. $B[x,r]$ is the closed ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$. If $A, B \subset \mathbb{R}^n$, $d(x,A) = \inf\{d(x,y) \mid y \in A\}$, $d^*(A,B) = \sup\{d(y,B) \mid y \in A\}$, $d(x,\emptyset) = \infty$.

Let (X,d) be a metric space and $\mathcal{P}(X)$ the set of subsets of X. For $A, B \subset X$ we have

Definition 2.11. [9], [23] The function $d_H : \mathcal{P}(X) \to [0,+\infty]$

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\} = \max\left\{\sup_{x\in A} d(x,B), \sup_{x\in B} d(x,A)\right\}$$

is the *Hausdorff-Pompeiu pseudometric*.

The function d_H defines a metric on the space $\mathcal{F}(X)$ of the non-empty and closed subsets of X, called the *Hausdorff-Pompeiu metric*.

3 Continuous selections. The continuous case refined

Let be the multifunction $F : D \times B \to \operatorname{comp} A$ whose values are non-empty compact and not necessarily convex subsets of $\mathbb{R}^n$, $D = [0,a] \times [0,b] \times [0,c] \subset \mathbb{R}^3$, A is the closed ball centered at the origin of $\mathbb{R}^n$ with radius M where M is given by (3.3), and B is the closed ball centered at the origin of $\mathbb{R}^n$ with radius $r = M_1 + Mabc$, where M_1 is given by (3.2).

Proposition 3.1. [1] *A is a compact space for the metric (Euclidean distance) d induced on A by the Euclidean norm of $\mathbb{R}^n$.*

Proposition 3.2. [1] *The set $\operatorname{comp} A$ is a compact metric space for d_H, where d_H is the Hausdorff-Pompeiu metric on $\operatorname{comp} A$ induced by d.*

Let $C(D; \mathbb{R}^n)$ be the Banach space of continuous functions from D into $\mathbb{R}^n$ and $\mathcal{L}^1(D; \mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue-integrable functions on D and taking values in $\mathbb{R}^n$.

Let the following hypotheses be satisfied:

(H_1) $F : D \times B \to \operatorname{comp} A$ is a continuous multifunction;

(H_2) The functions $\varphi \in C^*(D_1; \mathbb{R}^n)$, $\psi \in C^*(D_2; \mathbb{R}^n)$, $\chi \in C^*(D_3; \mathbb{R}^n)$ given by (1.2) are absolutely continuous in Carathéodory's sense functions and satisfy conditions (1.3).

Remark 3.1. The function $\alpha : D \to \mathbb{R}^n$ defined by

$$\begin{aligned} \alpha(x,y,z) &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \psi(0,z) + \psi(0,0) = \\ &= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0, \end{aligned} \tag{3.1}$$

is an absolutely continuous in Carathéodory's sense function on D, $\alpha \in C^*(D; \mathbb{R}^n)$ [4, §565-§570].

Suppose that the following hypothesis holds:

(H_3) The function $\alpha : D \to \mathbb{R}^n$ defined by (3.1) is bounded, that is

$$\|\alpha(x,y,z)\| \le M_1, \quad (x,y,z) \in D. \tag{3.2}$$

Define $\mathcal{K}$ to be the set of absolutely continuous in Carathéodory's sense functions $u : D \to \mathbb{R}^n$ satisfying

$$\left\| \frac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \le M, \quad \text{for a.e.} \quad (x,y,z) \in D, \tag{3.3}$$

and the conditions (1.2).

Proposition 3.3. *The set $\mathcal{K}$ is a non-empty compact and convex subset of $C(D; \mathbb{R}^n)$.*

Proof. The relation $u \in \mathcal{K}$ implies $u \in C(D; \mathbb{R}^n)$. Integrating $\dfrac{\partial^3 u(x,y,z)}{\partial x\, \partial y\, \partial z}$ on D and using the conditions (1.2) we obtain

$$\begin{aligned}
u(x,y,z) &= u(x,y,0) + u(x,0,z) - u(x,0,0) + u(0,y,z) - \\
&- u(0,y,0) - u(0,0,z) + u(0,0,0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t}\, dr\, ds\, dt = \\
&= \varphi(x,y) + \psi(y,z) + \chi(x,z) - \varphi(x,0) - \varphi(0,y) - \\
&- \psi(0,z) + u(0,0,0) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t}\, dr\, ds\, dt = \\
&= \varphi(x,y) + \psi(y,z) + \chi(x,z) - v^1(x) - v^2(y) - v^3(z) + v^0 + \\
&+ \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t}\, dr\, ds\, dt = \\
&= \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t}\, dr\, ds\, dt, \quad (x,y,z) \in D.
\end{aligned} \tag{3.4}$$

The compactness of the set $\mathcal{K}$ results using the theorem of Arzelà-Ascoli. The set $\mathcal{K}$ is equibounded. From (3.2), (3.3) and (3.4) we have

$$\begin{aligned}
\|u(x,y,z)\| &\le \|\alpha(x,y,z)\| + \int_0^x \int_0^y \int_0^z \left\| \frac{\partial^3 u(r,s,t)}{\partial r\, \partial s\, \partial t} \right\| dr\, ds\, dt \le \\
&\le M_1 + \int_0^x \int_0^y \int_0^z M\, dr\, ds\, dt = \\
&= M_1 + Mxyz \le M_1 + Mabc = r, \quad r > 0, \quad (x,y,z) \in D.
\end{aligned}$$

The set $\mathcal{K}$ is equicontinuous. Using the absolute continuity of the integral it follows that

$$\begin{gathered}
\|u(x+h, y+k, z+l) - u(x,y,z)\| < \varepsilon \quad \text{for} \quad h,k,l \in \mathbb{R} \\
\text{with} \quad |h|, |k|, |l| < \delta(\varepsilon), \ \delta(\varepsilon) > 0, \ \varepsilon > 0.
\end{gathered}$$

The set $\mathcal{K}$ is convex. Indeed let $u_1, u_2 \in \mathcal{K}$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ with $0 \le \lambda_1 \le 1$, $0 \le \lambda_2 \le 1$, $\lambda_1 + \lambda_2 = 1$. From (3.3) and (1.2) we have

$$\left\| \frac{\partial^3 u_i(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \le M, \quad (x,y,z) \in D, \quad i = \overline{1,2},$$

and

$$\left\{ \begin{array}{ll}
u_i(x,y,0) = \varphi(x,y), & (x,y) \in D_1 = [0,a] \times [0,b], \\
u_i(0,y,z) = \psi(y,z), & (y,z) \in D_2 = [0,b] \times [0,c], \\
u_i(x,0,z) = \chi(x,z), & (x,z) \in D_3 = [0,a] \times [0,c].
\end{array} \right. \qquad i = \overline{1,2}$$

Using the properties of absolutely continuous in Carathéodory's sense functions (Theorem 2.5), it follows that $\lambda_1 u_1 + \lambda_2 u_2 \in C^*(D; \mathbb{R}^n)$.

The relations (3.3) and (1.2) for the function $\lambda_1 u_1 + \lambda_2 u_2$ hold.

$$\begin{aligned}\left\| \frac{\partial^3 (\lambda_1 u_1 + \lambda_2 u_2)(x,y,z)}{\partial x\, \partial y\, \partial z} \right\| &= \left\| \lambda_1 \frac{\partial^3 u_1 (x,y,z)}{\partial x\, \partial y\, \partial z} + \lambda_2 \frac{\partial^3 u_2 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \leq \\ &\leq \lambda_1 \left\| \frac{\partial^3 u_1 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| + \lambda_2 \left\| \frac{\partial^3 u_2 (x,y,z)}{\partial x\, \partial y\, \partial z} \right\| \leq \\ &\leq \lambda_1 M + \lambda_2 M = (\lambda_1 + \lambda_2) M = M, \quad (x,y,z) \in D,\end{aligned}$$

and

$$\begin{aligned}(\lambda_1 u_1 + \lambda_2 u_2)(x,y,0) &= \lambda_1 u_1 (x,y,0) + \lambda_2 u_2 (x,y,0) = \lambda_1 \varphi (x,y) + \lambda_2 \varphi (x,y) = \\ &= (\lambda_1 + \lambda_2) \varphi (x,y) = \varphi (x,y), \quad (x,y) \in D_1, \\ (\lambda_1 u_1 + \lambda_2 u_2)(0,y,z) &= \lambda_1 u_1 (0,y,z) + \lambda_2 u_2 (0,y,z) = \lambda_1 \psi (y,z) + \lambda_2 \psi (y,z) = \\ &= (\lambda_1 + \lambda_2) \psi (y,z) = \psi (y,z), \quad (y,z) \in D_2, \\ (\lambda_1 u_1 + \lambda_2 u_2)(x,0,z) &= \lambda_1 u_1 (x,0,z) + \lambda_2 u_2 (x,0,z) = \lambda_1 \chi (x,z) + \lambda_2 \chi (x,z) = \\ &= (\lambda_1 + \lambda_2) \chi (x,z) = \chi (x,z), \quad (x,z) \in D_3.\end{aligned}$$

Hence $\lambda_1 u_1 + \lambda_2 u_2 \in \mathcal{K}$ and the set $\mathcal{K}$ is convex.

Remark 3.2. The membership $u \in \mathcal{K}$ implies $(x,y,z,u(x,y,z)) \in D \times B$ for each $(x,y,z) \in D$. In view of the fact that each $u \in \mathcal{K}$ generates a multifunction $(x,y,z) \to F(x,y,z,u(x,y,z))$ from D into $\operatorname{comp} A$, we shall denote this function by $G(u)$,

$$G(u)(x,y,z) = F(x,y,z,u(x,y,z)), \quad (x,y,z) \in D. \tag{3.5}$$

We are now going to prove that the assertion of Theorem 4.1 [36] can be significantly strenghtened and moreover under the same hypotheses.

We prove the following result, analogous to Lemma 3 [1] and Lemma [31].

Proposition 3.4. (Lemma 3.1) *Let $K : D \to \operatorname{comp} A$ be a continuous multifunction and $v : D \to \mathbb{R}^n$ be a piecewise constant mapping such that $d(v(x,y,z), K(x,y,z)) < \rho$ for every $(x,y,z) \in D$. Then, for every $\varepsilon > 0$ there exists a piecewise constant mapping $w : D \to \mathbb{R}^n$ such that $d(v(x,y,z), w(x,y,z)) < \rho$ and $d(w(x,y,z), K(x,y,z)) < \varepsilon$ for every $(x,y,z) \in D$.*

Proof. Indeed, given $\varepsilon > 0$, we can choose a partition (D_{ijk}), $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq k \leq p$, of $\Delta = [0,a[\times [0,b] \times [0,c[$ consisting of intervals (parallelepipeds) $D_{ijk} = [x_{i-1}, x_i[\times [y_{j-1}, y_j[\times [z_{k-1}, z_k[$ such that $v\Big|_{D_{ijk}} = u_{ijk}$ and $d_H(K(x,y,z), K(x',y',z')) < \varepsilon$ for any (x,y,z), (x',y',z') in D_{ijk}. Then, for each (i,j,k), there exists a point $\zeta_{ijk} \in K(x_{i-1}, y_{j-1}, z_{k-1})$ such that $d(v(x_{i-1}, y_{j-1}, z_{k-1}), \zeta_{ijk}) < \rho$ and $d(\zeta_{ijk}, K(x,y,z)) < \varepsilon$ for every $(x,y,z) \in D_{ijk}$. We define the mapping $w : D \to \mathbb{R}^n$ as follows:

$$\begin{cases} w\Big|_{D_{ijk}} = \zeta_{ijk} \text{ for each } (i,j,k), \\ w(a,y,z) = \lim\limits_{x \to a^-} w(x,y,z), \\ w(x,b,z) = \lim\limits_{y \to b^-} w(x,y,z), \\ w(x,y,c) = \lim\limits_{z \to c^-} w(x,y,z). \end{cases}$$

Obviously, if $(x, y, z) \in \Delta$, then $(x, y, z) \in D_{ijk}$ for an unique D_{ijk}, such that $w(x, y, z) = \zeta_{ijk}$ and $v(x, y, z) = u_{ijk} = v(x_{i-1}, y_{j-1}, z_{k-1})$; consequently

$$\begin{aligned} d(v(x, y, z), w(x, y, z)) &= d(v(x_{i-1}, y_{j-1}, z_{k-1}), \zeta_{ijk}) < \rho \quad \text{and} \\ d(w(x, y, z), K(x, y, z)) &= d(\zeta_{ijk}, K(x, y, z)) < \varepsilon. \end{aligned}$$

By continuity, these inequalities are also true for $x = a$, $y = b$, $z = c$.

Theorem 3.1. *Let $F : D \times B \to \operatorname{comp} A$ be a continuous multifunction (the hypothesis (H_1)). Then, there exists a continuous mapping $g : \mathcal{K} \to \mathcal{L}^1(D; \mathbb{R}^n)$ such that for every $u \in \mathcal{K}$, $g(u)$ is a regular mapping in D and*

$$g(u)(x, y, z) \in G(u)(x, y, z) \quad \textit{for every} \quad (x, y, z) \in D.$$

Proof. We shall construct, for every $n \geq 1$, a continuous mapping $g^n : \mathcal{K} \to \mathcal{L}^1(D; \mathbb{R}^n)$ such that, for every $u \in \mathcal{K}$, $g^n(u)$ is a piecewise constant mapping of D into A which satisfies, at every $(x, y, z) \in D$,

$$d(g^n(u)(x, y, z), G(u)(x, y, z)) < \frac{1}{2^n}, \tag{3.6}$$

$$\left\| g^{n+1}(u)(x, y, z) - g^n(u)(x, y, z) \right\| < \frac{1}{2^{n+1}}. \tag{3.7}$$

It follows that, for every $u \in \mathcal{K}$, the sequence $\{g^n(u)\}$ converges uniformly in D to a mapping $g(u)$ of D into A that is regular in D and satisfies $g(u)(x, y, z) \in G(u)(x, y, z)$ at every $(x, y, z) \in D$. Indeed, since the convergence is uniform in D and each g^n is continuous in $\mathcal{K}$, g will be a continuous mapping of $\mathcal{K}$ into $\mathcal{L}^1(D; \mathbb{R}^n)$ and this will prove the statement.

The construction will be made in two stages. Firstly we choose a decreasing to zero sequence $\{\delta_n\}$ of positive constants such that, for every $n \geq 1$

$$d_H(F(x, y, z, u), F(r, s, t, \tilde{u})) < \frac{1}{2^{n+3}}, \tag{3.8}$$

for any two points (x, y, z, u), $(r, s, t, \tilde{u})$ in $D \times B$ with

$$\|(x, y, z) - (r, s, t)\| < \delta_n, \quad \|u - \tilde{u}\| < \delta_n.$$

This is posible because F is uniformly continuous in D.

We select, for every $n \geq 1$, a finite open covering $(U_l^n)_{1 \leq l \leq N(n)}$ of the compact space $\mathcal{K}$ such that

$$\operatorname{diam} U_l^n < \delta_n, \quad 1 \leq l \leq N(n).$$

Let $(p_l^n)_{1 \leq l \leq N(n)}$ be the continuous patition of the unity subordinate to $(U_l^n)_{1 \leq l \leq N(n)}$. We denote $N(n) = N_1(n) N_2(n) N_3(n)$ and $p_l^n(u) = p_{ijk}^n(u)$ and suppose

$$p_{ijk}^n(u) = q_i^n(u) r_j^n(u) s_k^n(u), \quad i = \overline{1, N_1(n)},\ j = \overline{1, N_2(n)},\ k = \overline{1, N_3(n)}.$$

The functions $p_{ijk}^n : \mathcal{K} \to \mathbb{R}$, $i = \overline{1, N_1(n)}$, $j = \overline{1, N_2(n)}$, $k = \overline{1, N_3(n)}$, satisfy the properties:

a) $0 \leq p_{ijk}^{n}(u) \leq 1$ for $u \in \mathcal{K}$, $i = \overline{1, N_1(n)}$, $j = \overline{1, N_2(n)}$, $k = \overline{1, N_3(n)}$;
b) $p_{ijk}^{n}(u) = 0$ if $u \notin U_l^n = U_{ijk}^n$, $i = \overline{1, N_1(n)}$, $j = \overline{1, N_2(n)}$, $k = \overline{1, N_3(n)}$;
c) $\sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} \sum_{k=1}^{N_3(n)} p_{ijk}^{n}(u) = \sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} \sum_{k=1}^{N_3(n)} q_i^n(u)\, r_j^n(u)\, s_k^n(u) = 1$ for $u \in \mathcal{K}$.

We denote
$$W_l^n = \{u \in U_l^n \mid p_l^n(u) > 0\}, \quad 1 \leq l \leq N(n).$$
Then, for every $n \geq 1$ and every vector index $m = (m_1, m_2, \ldots, m_n)$ such that
$$1 \leq m_\nu \leq N_\nu, \quad \bigcap_{\nu=1}^{n} W_{m_\nu}^{\nu} \neq \emptyset,$$
there exists a piecewise constant mapping $v_m^n : D \to A$ and a point $u_m^n \in \bigcap_{\nu=1}^{n} W_{m_\nu}^{\nu}$ such that, at every $(x, y, z) \in D$,
$$d\left(v_m^n(x, y, z), G(u_m^n)(x, y, z)\right) < \frac{1}{2^{n+1}}. \tag{3.9}$$

This assertion is obviously true for $n = 1$. Suppose that it is true for $n = 1, 2, \ldots, p$. If $m = (m_1, m_2, \ldots, m_n)$ is such that (3.9) holds for $n = p$, we can use Lemma 3.1 and construct, for every integer s such that
$$1 \leq s \leq N(p+1), \quad \bigcap_{\nu=1}^{p} W_{m_\nu}^{\nu} \cap W_s^{p+1} \neq \emptyset,$$
a piecewise constant mapping $v_{(m,s)}^{p+1} : D \to A$ which satisfies, at every $(x, y, z) \in D$,
$$d\left(v_{(m,s)}^{p+1}(x, y, z), G(u_m^n)(x, y, z)\right) < \frac{1}{2^{p+3}}, \tag{3.10}$$
$$\left\| v_{(m,s)}^{p+1}(x, y, z) - v_m^n(x, y, z) \right\| < \frac{1}{2^{p+1}}. \tag{3.11}$$

Remark 3.3. For any n-vector index $m = (m_1, m_2, \ldots, m_n)$ and integer s, we denote (m, s) the $(n+1)$-vector index $(m_1, m_2, \ldots, m_n, s)$.

This, if we fix a point
$$u_{(m,s)}^{p+1} \in \bigcap_{\nu=1}^{p} W_{m_\nu}^{\nu} \cap W_s^{p+1},$$
we deduce that, for every $(x, y, z) \in D$,
$$d\Big(v_{(m,s)}^{p+1}(x,y,z), G\Big(u_{(m,s)}^{p+1}\Big)(x,y,z)\Big) \leq d\Big(v_{(m,s)}^{p+1}(x,y,z), G(u_m^p)(x,y,z)\Big) + d_H(G(u_m^p)(x,y,z)),$$
$$G\left(u_{(m,s)}^{p+1}\right)(x, y, z) < \frac{1}{2^{p+3}} + \frac{1}{2^{p+3}} = \frac{1}{2^{(p+1)+1}}.$$

Hence, the assertion is true for $n = p+1$ and consequently, by induction, for every $n \geq 1$.

We next define, for every $u \in \mathcal{K}$, a sequence of finite partitions of the interval (parallelepiped) Δ as follows: given $u \in \mathcal{K}$, we successively construct, for every $n \geq 1$ and every $3n$-vector index $m = (m_1, m_2, m_3, \ldots, m_n) = m^1 \times m^2 \times m^3$, $m^1 = \left(m_1^1, m_2^1, \ldots, m_n^1\right)$, $m^2 = \left(m_1^2, m_2^2, \ldots, m_n^2\right)$, $m^3 = \left(m_1^3, m_2^3, \ldots, m_n^3\right)$, $1 \leq m_\nu^1 \leq N_1(\nu)$, $1 \leq m_\nu^2 \leq N_2(\nu)$, $1 \leq m_\nu^3 \leq N_3(\nu)$, $1 \leq \nu \leq n$, an interval (parallelepiped) $\Delta_m^n(u) \subset \Delta$, such that

$$\Delta = \bigcup_{\substack{1\leq i\leq N_1(1),\\ 1\leq j\leq N_2(1),\\ 1\leq k\leq N_3(1)}} \Delta_{ijk}^1(u) = \bigcup_{l=1}^{N(n)} \Delta_l^1(u) \tag{3.12}$$

and

$$\Delta_m^n(u) = \bigcup_{1\leq s\leq N(n+1)} \Delta_{(m,s)}^{n+1}(u), \quad n \geq 1. \tag{3.13}$$

Indeed, let

$$\begin{cases} x_0^1(u) = 0 \\ x_i^1(u) = x_{i-1}^1(u) + aq_i^1(u) \displaystyle\sum_{j=1}^{N_2(1)} r_j^1(u) \sum_{k=1}^{N_3(1)} s_k^1(u), \quad i = \overline{1, N_1(1)}, \end{cases}$$

$$\begin{cases} y_0^1(u) = 0 \\ y_j^1(u) = y_{j-1}^1(u) + br_j^1(u) \displaystyle\sum_{i=1}^{N_1(1)} q_i^1(u) \sum_{k=1}^{N_3(1)} s_k^1(u), \quad j = \overline{1, N_2(1)}, \end{cases}$$

$$\begin{cases} z_0^1(u) = 0 \\ z_k^1(u) = z_{k-1}^1(u) + cs_k^1(u) \displaystyle\sum_{i=1}^{N_1(1)} q_i^1(u) \sum_{j=1}^{N_2(1)} r_j^1(u), \quad k = \overline{1, N_3(1)}. \end{cases}$$

For each $u \in \mathcal{K}$ we define the intervals (parallelepipeds)

$$\Delta_{ijk}^1(u) = \left[x_{i-1}^1(u), x_i^1(u)\right[\times \left[y_{j-1}^1(u), y_j^1(u)\right[\times \left[z_{k-1}^1(u), z_k^1(u)\right[$$

for each $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$, $k = \overline{1, N_3(1)}$.

Then, obviously, $\Delta_{ijk}^1(u)$ is non-empty if and only if $u \in W_{ijk}^1$, but (3.12) holds for whatever $u \in \mathcal{K}$ because $\left(p_{ijk}^1\right)$, $1 \leq i \leq N_1(1)$, $1 \leq j \leq N_2(1)$, $1 \leq k \leq N_3(1)$, is a partition of the unity, $p_{ijk}^1(u) = q_i^1(u)\, r_j^1(u)\, s_k^1(u)$.

More generally, if $m = (m_1, m_2, \ldots, m_n)$ is an $3n$-vector index with $1 \leq m_\nu \leq N(\nu)$, $m = m^1 \times m^2 \times m^3$, $m^1 = \left(m_1^1, m_2^1, \ldots, m_n^1\right)$, $m^2 = \left(m_1^2, m_2^2, \ldots, m_n^2\right)$, $m^3 = \left(m_1^3, m_2^3, \ldots, m_n^3\right)$, $1 \leq m_\nu^1 \leq N_1(\nu)$, $1 \leq m_\nu^2 \leq N_2(\nu)$, $1 \leq m_\nu^3 \leq N_3(\nu)$, $1 \leq \nu \leq n$, for each $\Delta_m^n(u)$ has been constructed, let

$$\begin{cases} x_{(m^1,0)}^{n+1}(u) = x_{\left(m_1^1, m_2^1, \ldots, m_n^1 - 1\right)}^n(u), \\ x_{(m^1,s^1)}^{n+1}(u) = x_{(m^1,s^1-1)}^{n+1}(u) + \left(a \displaystyle\prod_{\nu=1}^n q_{m_\nu^1}^\nu(u) \sum_{j=1}^{N_2(n)} r_j(u) \sum_{k=1}^{N_3(n)} s_k(u)\right) q_{s^1}^{n+1}(u), \end{cases}$$

$$\begin{cases} y^{n+1}_{(m^2,0)}(u) = y^n_{\left(m^2_1,m^2_2,\ldots,m^2_n-1\right)}(u), \\ y^{n+1}_{(m^2,s^2)}(u) = y^{n+1}_{(m^2,s^2-1)}(u) + \left(b\prod_{\nu=1}^{n} r^{\nu}_{m^2_\nu}(u)\sum_{i=1}^{N_1(n)} q_i(u)\sum_{k=1}^{N_3(n)} s_k(u)\right) r^{n+1}_{s^2}(u), \end{cases}$$

$$\begin{cases} z^{n+1}_{(m^3,0)}(u) = z^n_{\left(m^3_1,m^3_2,\ldots,m^3_n-1\right)}(u), \\ z^{n+1}_{(m^3,s^3)}(u) = z^{n+1}_{(m^3,s^3-1)}(u) + \left(c\prod_{\nu=1}^{n} s^{\nu}_{m^3_\nu}(u)\sum_{i=1}^{N_1(n)} q_i(u)\sum_{j=1}^{N_2(n)} r_j(u)\right) s^{n+1}_{s^3}(u), \end{cases}$$

and set

$$\begin{aligned} \Delta^{n+1}_{(m,s)}(u) &= \left[x^{n+1}_{(m^1,s^1-1)}(u), x^{n+1}_{(m^1,s^1)}(u)\right[\times \left[y^{n+1}_{(m^2,s^2-1)}(u), y^{n+1}_{(m^2,s^2)}(u)\right[\times \\ &\quad \times \left[z^{n+1}_{(m^3,s^3-1)}(u), z^{n+1}_{(m^3,s^3)}(u)\right[, \end{aligned}$$

where $s = s^1 \times s^2 \times s^3$, for each $s = \overline{1, N(n+1)}$, $s^1 = \overline{1, N_1(n+1)}$, $s^2 = \overline{1, N_2(n+1)}$, $s^3 = \overline{1, N_3(n+1)}$.

Then $\Delta^{n+1}_{(m,s)}(u)$ is non-empty if and only if $u \in \bigcap_{\nu=1}^{n} W^{\nu}_{m_\nu} \cap W^{n+1}_s$, and, in particular, $u \in \bigcap_{\nu=1}^{n} W^{\nu}_{m_\nu}$ implies that (3.13) holds netrivially.

We remark that, in this case, $\operatorname{diam}\Delta^n_m(u) = abc\prod_{\nu=1}^{n} p^{\nu}_{m_\nu}(u) > 0$. However, whatever would be $u \in \mathcal{K}$, we have by construction that

$$\Delta = \bigcup \left\{\Delta^n_m(u) \mid m = (m_1, m_2, \ldots, m_n), 1 \le m_\nu \le N(\nu), 1 \le \nu \le n\right\}. \tag{3.14}$$

We define, for every $n \ge 1$, the required mapping g^n of $\mathcal{K}$ into $\mathcal{L}^1(D; \mathbb{R}^n)$. In view of (3.14) we can do this simply by prescribing, for every $u \in \mathcal{K}$, the restriction of $g^n(u)$ to each of the intervals (parallelepipeds) Δ^n_m. For every $u \in \mathcal{K}$ we define

$$g^1(u)\Big|_{\Delta} = \sum_{\nu=1}^{N(1)} \chi\left(\Delta^1_s(u)\right) v^1_s \tag{3.15}$$

where χ is the characteristic function, an set, for every $n \ge 1$, and every $3n$-vector index $m = (m_1, m_2, \ldots, m_n)$, $m = m^1 \times m^2 \times m^3$, $m^1 = \left(m^1_1, m^1_2, \ldots, m^1_n\right)$, $m^2 = \left(m^2_1, m^2_2, \ldots, m^2_n\right)$, $m^3 = \left(m^3_1, m^3_2, \ldots, m^3_n\right)$, $1 \le m_\nu \le N(\nu)$, $1 \le m^1_\nu \le N_1(\nu)$, $1 \le m^2_\nu \le N_2(\nu)$, $1 \le m^3_\nu \le N_3(\nu)$, $1 \le \nu \le n$,

$$g^{n+1}(u)\Big|_{\Delta} = \sum_{\nu=1}^{N(n+1)} \chi\left(\Delta^{n+1}_{(m,s)}(u)\right) v^{n+1}_{(m,s)}. \tag{3.16}$$

This uniquely defines, for every $n \ge 1$, $g^n(u)$ as a piecewise constant mapping on Δ, and hence we can extended $g^n(u)$ to $D = \overline{\Delta}$, by setting

$$\begin{cases} g^n(u)(a, y, z) = \lim\limits_{x \to a^-} g^n(u)(x, y, z), \\ g^n(u)(x, b, z) = \lim\limits_{y \to b^-} g^n(u)(x, y, z), \\ g^n(u)(x, y, c) = \lim\limits_{z \to c^-} g^n(u)(x, y, z). \end{cases} \tag{3.17}$$

Obviously, for every $n \geq 1$, g^n is mapping of $\mathcal{K}$ into $\mathcal{L}^1(D;\mathbb{R}^n)$, $g^n : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$. This construction implies, similarly with the proof in Proposition 3.4 [36], that each g^n is continuous in $\mathcal{K}$. Thus, only the inequalities (3.6) and (3.7) remain to be verified.

Let $u \in \mathcal{K}$ be given and fix $(x,y,z) \in \Delta$. Then, for every $n \geq 1$, there exists one and only one $3n$-vector index $m = (m_1, m_2, \ldots, m_m)$, $m = m^1 \times m^2 \times m^3$ such that $(x,y,z) \in \Delta_m^n(u)$. This implies that, in particular, $u \in \bigcap_{\nu=1}^{n} W_{m_\nu}^{\nu}$ and consequently, by (3.16),

$$\begin{gathered} d\left(g^n(u)(x,y,z), G(u)(x,y,z)\right) = d\left(v_m^n(x,y,z), G(u)(x,y,z)\right) \leq \\ \leq d\left(v_m^n(x,y,z), G(u_m^n)(x,y,z)\right) + d_H\left(G(u_m^n)(x,y,z), G(u)(x,y,z)\right) < \\ < \frac{1}{2^{n+1}} + \frac{1}{2^{n+3}} < \frac{1}{2^n}. \end{gathered}$$

Moreover, if $(x,y,z) \in \Delta_m^n$ then $(x,y,z) \in \Delta_{(m,s)}^{n+1}$ for one and only one index s, $1 \leq s \leq N(n+1)$, so that $u \in \bigcap_{\nu=1}^{n} W_{m_\nu}^{\nu} \cap W_s^{n+1}$, too. Hence, we deduce from (3.9) and (3.11) that

$$\left\| g^{n+1}(u)(x,y,z) - g^n(u)(x,y,z) \right\| = \left\| v_{(m,s)}^{n+1}(x,y,z) - v_m^n(x,y,z) \right\| < \frac{1}{2^{n+1}}.$$

Thus, the inequalities (3.6) and (3.7) hold at every $(x,y,z) \in D$. Obviously, by (3.17) and by continuity, they remain valid ar $x = a$, $y = b$, $z = c$. This completes the proof.

4 The Darboux Problem for third order hyperbolic inclusions

Definition 4.1. The *Darboux Problem* for the hyperbolic inclusion (1.1) means to determine a *solution* of this inclusion which satisfies the initial conditions (1.2).

Definition 4.2. A function $u : D \to \mathbb{R}^n$ is called a *solution* of the Darboux Problem (1.1) + (1.2) if it is absolutely continuous in Carathéodory's sense on D, $u \in C^*(D;\mathbb{R}^n)$ [4,§565-§570], and it satisfies (1.1) for a.e. $(x,y,z) \in D$, and also the initial conditions (1.2) for all $(x,y) \in D_1$, all $(y,z) \in D_2$, all $(x,z) \in D_3$.

As a corollary of Selection Theorem 3.1 we state following existence result.

Theorem 4.1. *Assume that the hypotheses $(H_1) - (H_3)$ are satisfied. Then there exists an absolutely continuous in Carathéodory's sense function $\hat{u} : D \to \mathbb{R}^n$ which is a solution of Darboux Problem $(1.1) + (1.2)$.*

Proof. Using Theorem 3.1, there is a continuous mapping $g : \mathcal{K} \to \mathcal{L}^1(D;\mathbb{R}^n)$ such that, for every $u \in \mathcal{K}$, $g(u)$ is a regulated mapping in D and $g(u)$ is a continuous selection for $G(u)$ given by (3.5).

Let $h(u)$, for each $u \in \mathcal{K}$, be the continuous function, $h(u) : D \to \mathbb{R}^n$, defined by

$$h(u)(x,y,z) = \alpha(x,y,z) + \int_0^x \int_0^y \int_0^z g(u)(r,s,t)\, dr\, ds\, dt, \quad (x,y,z) \in D. \tag{4.1}$$

Using (3.1) we have

$$h(u)(x,y,z)=\varphi(x,y)+\psi(y,z)+\chi(x,z)-\varphi(x,0)-\varphi(0,y)-\psi(0,z)+\psi(0,0)+ \\ +\int_0^x\int_0^y\int_0^z g(u)(r,s,t)\,dr\,ds\,dt,\quad (x,y,z)\in D, \tag{4.2}$$

and which can be rewriten as

$$\begin{aligned} h(u)(x,y,z)=&\int_0^x\int_0^y\int_0^z g(u)(r,s,t)\,dr\,ds\,dt+\int_0^x\int_0^y \frac{\partial^2\varphi(r,s)}{\partial r\,\partial s}dr\,ds+ \\ &+\int_0^y\int_0^z \frac{\partial^2\psi(s,t)}{\partial s\,\partial t}ds\,dt+\int_0^x\int_0^z \frac{\partial^2\chi(r,t)}{\partial r\,\partial t}dr\,dt+ \\ &+\int_0^x \frac{\partial\varphi(r,0)}{\partial r}dr+\int_0^y \frac{\partial\varphi(0,s)}{\partial s}ds+\int_0^z \frac{\partial\psi(0,t)}{\partial t}dt+ \\ &+u(0,0,0),\quad \text{for } (x,y,z)\in D. \end{aligned} \tag{4.3}$$

Indeed, we have

$$\begin{aligned} \int_0^x\int_0^y \frac{\partial^2\varphi(r,s)}{\partial r\,\partial s}dr\,ds=&\int_0^x\left[\int_0^y \frac{\partial^2\varphi(r,s)}{\partial r\,\partial s}dr\right]ds=\int_0^x \frac{\partial\varphi(r,s)}{\partial r}\Big|_{s=0}^{s=y}dr= \\ &=\int_0^x\left[\frac{\partial\varphi(r,y)}{\partial r}-\frac{\partial\varphi(r,0)}{\partial r}\right]dr=\varphi(r,y)\Big|_{r=0}^{r=x}-\varphi(r,0)\Big|_{r=0}^{r=x}= \\ &=[\varphi(x,y)-\varphi(0,y)]-[\varphi(x,0)-\varphi(0,0)]= \\ &=\varphi(x,y)-\varphi(0,y)-\varphi(x,0)+\varphi(0,0), \end{aligned} \tag{4.4}$$

$$\begin{aligned} \int_0^y\int_0^z \frac{\partial^2\psi(s,t)}{\partial s\,\partial t}ds\,dt=&\int_0^y\left[\int_0^z \frac{\partial^2\psi(s,t)}{\partial s\,\partial t}dt\right]ds=\int_0^y \frac{\partial\psi(s,t)}{\partial s}\Big|_{t=0}^{t=z}ds= \\ &=\int_0^y\left[\frac{\partial\psi(s,z)}{\partial s}-\frac{\partial\psi(s,0)}{\partial s}\right]ds=\psi(s,z)\Big|_{s=0}^{s=y}-\psi(s,0)\Big|_{s=0}^{s=y}= \\ &=[\psi(y,z)-\psi(0,z)]-[\psi(y,0)-\psi(0,0)]= \\ &=\psi(y,z)-\psi(0,z)-\psi(y,0)+\psi(0,0), \end{aligned} \tag{4.5}$$

$$\begin{aligned} \int_0^x\int_0^z \frac{\partial^2\chi(r,t)}{\partial r\,\partial t}dr\,dt=&\int_0^x\left[\int_0^z \frac{\partial^2\chi(r,t)}{\partial r\,\partial t}dt\right]ds=\int_0^x \frac{\partial\chi(r,t)}{\partial r}\Big|_{t=0}^{t=z}dr= \\ &=\int_0^x\left[\frac{\partial\chi(r,z)}{\partial r}-\frac{\partial\chi(r,0)}{\partial r}\right]dr=\chi(r,z)\Big|_{r=0}^{r=z}-\chi(r,0)\Big|_{r=0}^{r=x}= \\ &=[\chi(x,z)-\chi(0,z)]-[\chi(x,0)-\chi(0,0)]= \\ &=\chi(x,z)-\chi(0,z)-\chi(x,0)+\chi(0,0), \end{aligned} \tag{4.6}$$

$$\int_0^x \frac{\partial\varphi(r,0)}{\partial r}dr=\varphi(r,0)\Big|_{r=0}^{r=x}=\varphi(x,0)-\varphi(0,0), \tag{4.7}$$

$$\int_0^y \frac{\partial\varphi(0,s)}{\partial s}ds = \varphi(0,s)\Big|_{s=0}^{s=y} = \varphi(0,y) - \varphi(0,0), \tag{4.8}$$

$$\int_0^z \frac{\partial\psi(0,t)}{\partial t}dt = \psi(0,t)\Big|_{t=0}^{t=z} = \psi(0,z) - \psi(0,0). \tag{4.9}$$

From (1.3) it result

$$u(0,0,0) = \varphi(0,0) = \psi(0,0) = \chi(0,0). \tag{4.10}$$

Replacing (4.4) – (4.10) in (4.3) leads to (4.2).

Using Theorem 2.5, from (4.3) it follows that $h(u) \in C^*(D;\mathbb{R}^n)$ for each $u \in \mathcal{K}$, i.e. $h(u)$ is an absolutely continuous in Carathéodory's sense function. One obtains $h(u) \in \mathcal{K}$ and $h(\mathcal{K}) \subseteq \mathcal{K}$.

Indeed, from (4.2) it follows that $\dfrac{\partial^3 h(u)(x,y,z)}{\partial x\,\partial y\,\partial z} = g(u)(x,y,z)$, $(x,y,z) \in D$, but

$$g(u)(x,y,z) \in G(u)(x,y,z) = F(x,y,z,u(x,y,z)), \quad (x,y,z) \in D.$$

Hence $\zeta = g(u)(x,y,z)$ is an element of the ball A, and consequently $\|\zeta\| = \|g(u)(x,y,z)\| \leq M$, i.e. $\left\|\dfrac{\partial^3 h(u)(x,y,z)}{\partial x\,\partial y\,\partial z}\right\| \leq M$, and from (4.2) $h(u)$ satisfies (1.2). Using the definition of the set $\mathcal{K}$, this inequality shows that $h(u) \in \mathcal{K}$. From $u \in \mathcal{K}$ implying that $h(u) \in \mathcal{K}$, we conclude that $h(\mathcal{K}) \subseteq \mathcal{K}$.

Now, we can apply Schauder's Fixed Point Theorem and conclude that there exists $\hat{u} \in \mathcal{K}$ such that $\hat{u} = h(\hat{u})$, i.e. $\hat{u}(x,y,z) = h(\hat{u})(x,y,z)$ at every $(x,y,z) \in D$. This implies that $\hat{u}$ satisfies (1.1),

$$\frac{\partial^3 \hat{u}(x,y,z)}{\partial x\,\partial y\,\partial z} = g(\hat{u})(x,y,z) \in F(x,y,z,\hat{u}(x,y,z))$$

for a.e. $(x,y,z) \in D$, and the relations (1.2). Therefore $\hat{u}(x,y,z) = h(\hat{u})(x,y,z)$ is a solution of Darboux Problem (1.1) + (1.2).

References

[1] H.A. Antosiewicz and A. Cellina, *Continuous Selections and Differential Relations*, Journal of Differential Equations, 19 (1975), 386-398.

[2] R.J. Aumann, *Integrals of Set - Valued Functions*, Journal of Mathematical Analysis and Applications, 12 (1956), 1-12.

[3] A.V. Bogatîrev, *Continuous branches of multi-valued mapping with non-convex right-hand side* (Russian), Math.Sb., 120 (162), 3 (1983), 344-353.

[4] C. Carathéodory, *Vorlesungen über Reelle Funktionen*, Chelsea Publishing Company, New York, 1968, 3 Ed.

[5] Ch. Castaing, *Sur les équations différentielles multivoques*, Comptes Rendus Acad. Sci. Paris, 263, No. 2 (1966), Série A, 63-66.

[6] Ch. Castaing, *Quelques problèmes de mésurabilité liés à la théorie de la commande*, Comptes Rendus Acad. Sci. Paris, 262, No. 7 (1966), Série A, 409-411.

[7] A. Cellina, *Approximation of Set-Valued Functions and fixed point theorems*, Ann. Mat. Pura Appl., 82(1969), 17-24.

[8] A. Cellina, *Multivalued differential equations and ordinary differential equations*, SIAM J. Appl. Math. 18, No. 2 (1970), 533-538.

[9] A. Cernea, *Incluziuni diferenţiale hiperbolice şi control optimal*, Editura Academiei Române, Bucureşti, 2001.

[10] A. Cernea, *Aspecte calitative în teoria incluziunilor diferenţiale*, Cartea Universitară, Bucureşti, 2004.

[11] A. Corduneanu, *A note on the Gronwall inequality in two independent variables*, J. Integral Equations, 4(1982), No. 3, 271-276.

[12] K. Deimling, *A Carathéodory theory for systems of integral equations*, Ann. Mat. Pura Appl., (4) 86 (1970), 217-260.

[13] K. Deimling, *Das Picard-Problem für $u_{xy} = f(x, y, u, u_x, u_y)$ unter Carathéodory-Voraussetzungen*, Math. Z., 114(1970), 303-312.

[14] K. Deimling, *Das charakteristiche Anfangswertproblem für $u_{x_1x_2x_3} = f$ unter Carathéodory-Voraussetzungen*, Arch. Math. (Basel), 22(1971), 514-522.

[15] S. Marano, *Generalized Solutions of Partial Differential Inclusions Depending on a Parameter*, Rend. Accad. Naz. Sc. XL, Mem. Mat., 13 (1989), 281-295.

[16] S. Marano, *Classical Solutions of Partial Differential Inclusions in Banach Spaces*, Appl. Anal., 42(1991), no.2, 127-143.

[17] S. Marano, *Controllability of Partial Differential Inclusions Depending on a Parameter and Distributed Parameter Control Process*, Le Matematiche, XLV, Fasc. II (1990), 283-300.

[18] M. Mureşan, *An Introduction to the Set-Valued Analysis*, Cluj University Press, Cluj-Napoca, 1999.

[19] M. Mureşan, *Analiză netedă şi aplicaţii*, Editura Risoprint, Cluj-Napoca, 2001.

[20] A. Petruşel, *Multifuncţii şi aplicaţii*, Presa Universitară Clujeană, Cluj-Napoca, 2002.

[21] A. Petruşel, *Operatorial Inclusions*, House of the Book of Science, Cluj-Napoca, 2002.

[22] A. Petruşel, Ghiocel Moţ, *Multivalued Analysis and Mathematical Economics*, House of the Book of Science, Cluj-Napoca, 2004.

[23] I.A. Rus, *Principii şi aplicaţii ale teoriei punctului fix*, Editura Dacia, Cluj-Napoca, 1979.

[24] I.A. Rus, *Generalized Contractions and Applications*, Cluj University Press, Cluj-Napoca, 2001.

[25] I.A. Rus, *Fixed Point Structure Theory*, Cluj University Press, Cluj-Napoca, 2006.

[26] W. Sosulski, *On neutral partial functional-differential inclusions of hyperbolic type*, Demonstratio Mathematica, XXIII, no. 4 (1990), 893-909.

[27] G. Teodoru, *Continuous selections for multivalued functions*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1986, 273-278.

[28] G. Teodoru, *Continuous selections for multifunctions satisfying the Carathéodory's type conditions. The Darboux Problem associated to a multi-valued equation*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1987, 281-286.

[29] G. Teodoru, *A characterization of the solutions of the Darboux Problem for the equation* $\dfrac{\partial^2 z}{\partial x\,\partial y} \in F\,(x, y, z)$, Analele Ştiinţifice ale Universităţii "Al. I. Cuza" Iaşi, 33(1987), s. I a, Matematică, f. 1, 33-38.

[30] G. Teodoru, *Approximation of the solutions of the Darboux Problem for the equation* $\dfrac{\partial^2 z}{\partial x\,\partial y} \in F\,(x, y, z)$, Analele Ştiinţifice ale Universităţii "Al. I. Cuza" Iaşi, 34(1988), s. I a, Matematică, f. 1, 31-36.

[31] G. Teodoru, *Continuous selections for multifunctions and the Darboux Problem for multi-valued equations*, Itinerant Seminar on Functional Equations, Approximation and Convexity, Cluj-Napoca, 1989, 297-304.

[32] G. Teodoru, *The Darboux Problem for third order hyperbolic inclusions*, Libertas Mathematica, XXIII (2003), 119-127, American Romanian Academy of Arts and Science, Arlington, Texas, U.S.A.

[33] G. Teodoru, *Prolongation of solutions of the Darboux Problem for third order hyperbolic inclusions*, Libertas Mathematica, XXVI (2006), 83-96, American Romanian Academy of Arts and Science, Arlington, Texas, U.S.A.

[34] G. Teodoru, *A characterization of the solutions of the Darboux Problem for third order hyperbolic inclusions*, Buletinul Academiei de Ştiinţe a Republicii Moldova, Matematica, Institutul de Matematică şi Informatică 3(55)(2007) 35-48.

[35] G. Teodoru, *Approximation of the solutions of the Darboux Problem for third order hyperbolic inclusions*, Fixed Point Theory, An International Journal on Fixed Theory, Computation and Applications, House of the Book of Science, Cluj-Napoca, Volume 9, No.1 (2008), 363-381.

[36] G. Teodoru, *Continuous selections for continuous multifunctions. The Darboux Problem for third order hyperbolic inclusions*, Fixed Point Theory, An International Journal on Fixed Point Theory, Computation and Applications, House of the Book of Science, Cluj-Napoca, 11(2010), No.1.

[37] G. Teodoru, *Continuous selections for multifunctions satisfying the Carathéodory's type conditions. The Darboux Problem for third order hyperbolic inclusions*, Libertas Mathematica, XXVIII (2008), 45-59, American Romanian Academy of Arts and Science, Arlington, Texas, U.S.A.