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Collisional Models for Strongly Magnetized Plasmas. The Gyrokinetic Fokker-Planck Equation

Mihai BOSTAN

Abstract. The subject matter of this paper concerns the derivation of asymptotic models in collisional plasma physics, under the action of strong magnetic fields, motivated by the magnetic fusion context. The limit procedure reduces to averaging with respect to the fast giration motion of particles around the magnetic lines. We investigate the Fokker-Planck collision operator and we compute its gyroaverage. It is shown that the averaged collision operator still satisfies the usual physical conservations (particle, momentum, energy) and ensures relaxation towards local Maxwellian distributions. This formalism applies for inhomogeneous magnetic fields in three dimensional setting.

Keywords: Fokker-Planck equation, Average operator, Gyrokinetic theory.

Mathematics Subject Classification (2000): 35Q75, 78A35, 82D10.

1 Introduction

One of the main topics in plasma physics, motivated by the magnetic fusion, concerns the evolution of charged particles in a tokamak, subject to very large magnetic fields. We appeal to kinetic models of Fokker-Planck type, for several species of particles $s \in \mathcal{S}$ (electrons and ions). We denote by $f_s = f_s(t,x,v)$ the distribution function of the species s, depending on time t, position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The quantity $f_s(t,x,p)\,\mathrm{d}x\mathrm{d}v$ represents the particle number of species s at time t, inside the infinitesimal volume $\mathrm{d}x\mathrm{d}v$ around (x,v). The time evolution of the distribution f_s, when taking into account the collisions between all species, is described by the Fokker-Planck equation

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{\mathcal{F}_s}{m_s} \cdot \nabla_v f_s = \sum_{s'} Q_{ss'}(f_s, f_{s'}), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \qquad (1)$$

where m_s denotes the mass of the species s, $\mathcal{F}_s$ is the Lorentz force acting on the particles of species s and $Q_{ss'}(f_s, f_{s'})$ is the Fokker-Planck collision operator between particles of species s and s' cf. [14]

$$Q_{ss'}(f_s, f_{s'})(v) = \frac{1}{m_s}\mathrm{div}_v \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 S(v-v_1)\left(\frac{1}{m_s} f_{s'}(v_1)(\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v)(\nabla_v f_{s'})(v_1)\right) \mathrm{d}v_1. \tag{2}$$

The equation (1) states that the rate of change of f_s along the particle trajectories in the phase space (x, v) are due to the collisions between particles. Here $\mu_{ss'} = m_s m_{s'}/(m_s + m_{s'})$ is the reduced mass of the pair $\{m_s, m_{s'}\}$, $\sigma_{ss'} = \sigma_{s's} > 0$ stands for the scattering cross section and $S(w) = I - \frac{w \otimes w}{|w|^2}$ is the projection onto the plane orthogonal to $w \in \mathbb{R}^3 \setminus \{0\}$ (the notation $v \otimes w$, with $v, w \in \mathbb{R}^3$ stands for the matrix $(v_i w_j)_{1 \le i,j \le 3}$). Obviously we have for any species s, s'

$$\int_{\mathbb{R}^3} Q_{ss'}(f_s, f_{s'})(v)\,\mathrm{d}v = 0.$$

Taking into account that $\mathrm{div}_v \mathcal{F}_s = 0$, one gets after integration with respect to (x, v)

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s(t,x,v)\,\mathrm{d}v\mathrm{d}x = 0, \quad t \in \mathbb{R}_+ \tag{3}$$

saying that the particle number is conserved for any species. It is well known that the momentum and energy are conserved as well when no force is applied. These statements are straightforward consequences of the following standard results.

Lemma 1.1 *Assume that $(f_s)_s$ are smooth enough with sufficient rapid decay as $|v| \to +\infty$. Then we have*

$$P := \sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s v Q_{ss'}(f_s, f_{s'})(v)\,\mathrm{d}v = 0 \tag{4}$$

$$W := \sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s \frac{|v|^2}{2} Q_{ss'}(f_s, f_{s'})(v)\,\mathrm{d}v = 0. \tag{5}$$

Proof. Notice that

$$m_s Q_{ss'}(f_s, f_{s'})(v) = \mathrm{div}_v \int_{\mathbb{R}^3} A_{ss'}(v, v_1)\,\mathrm{d}v_1$$

with

$$A_{ss'}(v, v_1) = \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 S(v-v_1)\left(\frac{1}{m_s} f_{s'}(v_1)(\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v)(\nabla_v f_{s'})(v_1)\right).$$

Taking into account that $\mu_{ss'} = \mu_{s's}$, $\sigma_{ss'} = \sigma_{s's}$, $S(-w) = S(w), w \in \mathbb{R}^3$, it is easily seen that

$$A_{ss'}(v, v_1) + A_{s's}(v_1, v) = 0.$$

Integrating by parts, interchanging the indexes s, s' and the variables v, v_1 one gets by the previous equality that $P = -P$. Similar manipulations yield, by taking into account that $S(v - v_1)(v - v_1) = 0$, that $W = -W$. $\square$

Assuming now that $\mathcal{F}_s = 0$ and multiplying (1) by $m_s v, m_s \frac{|v|^2}{2}$ one gets after integration with respect to v

$$\partial_t \left(\sum_s \int_{\mathbb{R}^3} m_s v f_s \,\mathrm{d}v \right) + \mathrm{div}_x \left(\sum_s \int_{\mathbb{R}^3} m_s v \otimes v f_s \,\mathrm{d}v \right) = P = 0$$

$$\partial_t \left(\sum_s \int_{\mathbb{R}^3} m_s \frac{|v|^2}{2} f_s \,\mathrm{d}v \right) + \mathrm{div}_x \left(\sum_s \int_{\mathbb{R}^3} m_s \frac{|v|^2}{2} v f_s \,\mathrm{d}v \right) = W = 0.$$

Integrating with respect to x implies the momentum and energy conservations

$$\frac{d}{dt} \sum_s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m_s v f_s \,\mathrm{d}v\mathrm{d}x = 0, \quad \frac{d}{dt} \sum_s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} m_s \frac{|v|^2}{2} f_s \,\mathrm{d}v\mathrm{d}x = 0.$$

Another classical property of the Fokker-Planck operator is the relaxation towards a Maxwellian equilibrium. The entropy dissipation rate $\sum_s \sum_{s'} \int_{\mathbb{R}^3} (1 + \ln f_s) Q_{ss'}(f_s, f_{s'})(v) \,\mathrm{d}v$ is given by the following lemma whose proof is left to the reader.

Lemma 1.2 *Under the hypotheses of Lemma 1.1 we have*

$$\begin{aligned} D : &= -\sum_s \sum_{s'} \int_{\mathbb{R}^3} (1 + \ln f_s) Q_{ss'}(f_s, f_{s'})(v) \,\mathrm{d}v \\ &= \frac{1}{2} \sum_s \sum_{s'} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v - v_1|) |v - v_1| f_s(v) f_{s'}(v_1) \\ &\quad \left| (v - v_1) \wedge \left(\frac{\nabla_v \ln f_s}{m_s}(v) - \frac{\nabla_v \ln f_{s'}}{m_{s'}}(v_1) \right) \right|^2 \,\mathrm{d}v_1 \mathrm{d}v \geq 0. \end{aligned}$$

Multiplying now (1) by $1 + \ln f_s$ and observing that $\mathcal{F}_s \cdot \nabla_v f_s \ln f_s = \mathrm{div}_v (f_s \ln f_s \mathcal{F}_s)$, one gets after integration with respect to v

$$\partial_t \left(\sum_s \int_{\mathbb{R}^3} f_s \ln f_s \,\mathrm{d}v \right) + \mathrm{div}_x \left(\sum_s \int_{\mathbb{R}^3} v f_s \ln f_s \,\mathrm{d}v \right) + D(t, x) = 0.$$

Integrating with respect to x yields the dissipation of the entropy H

$$H(t) := \sum_s \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_s(t, x, v) \ln f_s(t, x, v) \,\mathrm{d}v\mathrm{d}x, \quad \frac{dH}{dt} = -\int_{\mathbb{R}^3} D(t, x) \,\mathrm{d}x \leq 0.$$

In particular we have $H(t) + \int_0^t \int_{\mathbb{R}^3} D(\tau, x) \,\mathrm{d}x\mathrm{d}\tau = H(0)$, implying, after standard manipulations, that

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}^3} D(t, x) \,\mathrm{d}x\mathrm{d}t < +\infty.$$

Assuming that the distributions $(f_s)_s$ relax towards some equilibriums $(F_s)_s$ as $t \to +\infty$, we deduce, at least formally, that $\lim_{t\to+\infty} \int_{\mathbb{R}^3} D(t,x)\,\mathrm{d}x = 0$ saying that

$$\frac{1}{2}\sum_s\sum_{s'}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\mu^2_{ss'}\sigma_{ss'}(|v-v_1|)|v-v_1|F_s(v)F_{s'}(v_1)$$
$$\left|(v-v_1)\wedge\left(\frac{\nabla_v \ln F_s}{m_s}(v) - \frac{\nabla_v \ln F_{s'}}{m_{s'}}(v_1)\right)\right|^2 \mathrm{d}v_1\mathrm{d}v\ \mathrm{d}x = 0.$$

For any $s, s' \in \mathcal{S}, x \in \mathbb{R}^3, v, v_1 \in \mathbb{R}^3$ we obtain

$$(v-v_1)\wedge\left(\frac{\nabla_v \ln F_s}{m_s}(v) - \frac{\nabla_v \ln F_{s'}}{m_{s'}}(v_1)\right) = 0. \tag{6}$$

We conclude by appealing to the following easy result

Lemma 1.3 *Assume that $F = F(v) > 0$ is a smooth (C^2) integrable function satisfying*

$$(v-v_1)\wedge(\nabla_v \ln F_s(v) - \nabla_v \ln F_{s'}(v_1)) = 0,\quad v, v_1 \in \mathbb{R}^3. \tag{7}$$

Then F is a Maxwellian.

Proof. For any $w \in \mathbb{R}^3$, $h \in \mathbb{R}$, taking $v_1 = v + hw$ in (7) implies

$$\frac{\nabla_v \ln F(v+hw) - \nabla_v \ln F(v)}{h}\wedge w = 0$$

and therefore

$$(\nabla^2_v \ln F(v)w)\wedge w = 0,\quad v, w \in \mathbb{R}^3.$$

We deduce that for any $v, w \in \mathbb{R}^3$ there is $\lambda(v,w)$ such that $\nabla^2_v \ln F(v)w = \lambda(v,w)w$. In particular for the canonical basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ we have

$$\nabla^2_v F(v)e_i = \lambda_i(v)e_i,\quad i \in \{1,2,3\}.$$

Since $\nabla^2_v \ln F(v)(e_1+e_2) = \lambda(v, e_1+e_2)(e_1+e_2)$ we deduce that $\lambda_1(v) = \lambda_2(v) = \lambda_3(v)$ and finally, for any $v \in \mathbb{R}^3$ there is $\lambda = \lambda(v)$ such that

$$\nabla^2_v \ln F(v) = \lambda(v)I.$$

For any $i \in \{1,2,3\}$ we have $\partial_{v_i}\nabla_v \ln F = \lambda(v)e_i$ and thus $\partial_{v_1}\ln F$ does not depend on v_2, v_3. We deduce that $\lambda(v) = \partial^2_{v_1}\ln F$ does not depend on v_2, v_3 and finally $\lambda(v)$ is a constant function

$$\exists\,\lambda \in \mathbb{R}\ \text{ such that }\ \nabla^2_v \ln F(v) = \lambda I,\quad v \in \mathbb{R}^3.$$

Integrating with respect to v we deduce that $\ln F$ is a quadratic function of v and therefore F is a Maxwellian

$$F(v) = M_{n,u,T}(v) := \frac{n}{(2\pi T/m)^{3/2}}\exp\left(-\frac{m|v-u|^2}{2T}\right),\quad v \in \mathbb{R}^3$$

parametrized by some concentration n, mean velocity u and temperature T (m is the particle mass). Notice that $T > 0$, since F is supposed to be integrable. $\square$

Coming back in (6) with $s = s'$ we deduce that any distribution F_s is a local Maxwellian

$$F_s(x,v) = M_{n_s(x),u_s(x),T_s(x)}(v), \ \ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Taking now $s' \neq s$ one gets

$$\frac{\nabla_v \ln F_s}{m_s}(x,v) - \frac{\nabla_v \ln F_{s'}}{m_{s'}}(x,v_1) = -\left(\frac{v-u_s(x)}{T_s(x)} - \frac{v_1 - u_{s'}(x)}{T_{s'}(x)}\right)$$

and (6) implies that all the distributions $(F_s)_s$ have the same temperature and mean velocity $T_s = T_{s'}, u_s = u_{s'}$. Finally there are $u = u(x)$ and $T = T(x)$ such that

$$F_s(x,v) = M_{n_s(x),u(x),T(x)}(v) = \frac{n_s(x)}{(2\pi T(x)/m_s)^{3/2}} \exp\left(-\frac{m_s|v-u(x)|^2}{2T(x)}\right), \ \ v \in \mathbb{R}^3, s \in \mathcal{S}.$$

The Lorentz force

$$\mathcal{F}_s(t,x,v) = q_s\left(E(t,x) + v \wedge \mathbf{B}(x)\right)$$

corresponds to the electro-magnetic field $(E, \mathbf{B})$, q_s being the charge of the species s. As usual, we prescribe the initial distribution for each species $s \in \mathcal{S}$

$$f_s(0,x,v) = f_s^{\text{in}}(x,v), \ \ (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{8}$$

Generally we close (1) by adding equations for the electro-magnetic field $(E, \mathbf{B})$ (*i.e.,* the Maxwell equations or the Poisson equation). Here we neglect the self-consistent electro-magnetic field, assuming that the magnetic field is stationary, divergence free and that the electric field derives from a given electric potential $E(t) = -\nabla_x \phi(t)$. We investigate the asymptotic behaviour of (1), (8) when the magnetic field becomes large

$$\mathbf{B}^\varepsilon(x) = \frac{\mathbf{B}(x)}{\varepsilon}, \ \ \mathbf{B}(x) = B(x)b(x), \ \ \mathrm{div}_x(Bb) = 0, \ \ 0 < \varepsilon << 1$$

for some scalar positive function $B(x)$ and some field of unitary vectors $b(x)$. We assume that B, b are smooth. Clearly, the dynamics in (1) is dominated by the transport operator

$$\frac{1}{\varepsilon}\frac{q_s B(x)}{m_s}\left(v \wedge b(x)\right) \cdot \nabla_v.$$

Moreover, assuming that $f_s^\varepsilon = f_s + \varepsilon f_s^1 + \varepsilon^2 f_s^2 + ...$ holds true for small $\varepsilon > 0$ and letting $\varepsilon \searrow 0$, it is easily seen that the leading order term f_s belongs to the kernel of $\mathcal{T}_s = \frac{q_s B(x)}{m_s}\left(v \wedge b(x)\right) \cdot \nabla_v$. Indeed, plugging the above ansatz in (1) gives at the lowest order the divergence constraint $\mathcal{T}_s f_s = 0$ and to the next order the evolution equation

$$\partial_t f_s + v \cdot \nabla_x f_s + \frac{q_s}{m_s} E(t,x) \cdot \nabla_v f_s + \mathcal{T}_s f_s^1 = \sum_{s'} Q_{ss'}(f_s, f_{s'}). \tag{9}$$

Determining a closure for the dominant term f_s requires to eliminate the first order distribution f_s^1 in (9). For doing that, observe that the kernel and range of $\mathcal{T}_s$ are orthogonal and therefore, it is sufficient to project (9) on the kernel of $\mathcal{T}_s$. Moreover, it can be shown that the orthogonal projection on $\ker \mathcal{T}_s$ coincides with the average operator along the characteristic flow associated with $\mathcal{T}_s$ cf. [4]. Following the method in [5], averaging the left hand side of (9) leads to the transport operator

$$(\partial_t + A_x \cdot \nabla_x + A_v \cdot \nabla_v) f_s := \partial_t f_s + b \otimes b v \cdot \nabla_x f_s + \left(\frac{q_s}{m_s} b \otimes b E + \omega(x,v) \, {}^\perp v \right) \cdot \nabla_v f_s$$

where the frequency $\omega(x,v)$ is given by

$$\omega(x,v) = \frac{|v \wedge b(x)|}{2} \operatorname{div}_x b - (v \cdot b(x)) \left(\partial_x b \, b(x) \cdot \frac{v}{|v \wedge b(x)|} \right), \quad v \wedge b(x) \neq 0$$

and for any (x,v) such that $v \wedge b(x) \neq 0$ the symbol ${}^\perp v$ stands for the velocity orthogonal to v in the plane determined by $b(x)$ and v such that its coordinate along $b(x)$ is positive

$${}^\perp v = |v \wedge b(x)| \, b(x) - (v \cdot b(x)) \, \frac{v - (v \cdot b(x)) b(x)}{|v \wedge b(x)|}.$$

Neglecting for the moment the collisions, we obtain the limit model

$$\partial_t f_s + A_x \cdot \nabla_x f_s + A_v \cdot \nabla_v f_s = 0$$

whose particle trajectories $(X(\tau), V(\tau))$ in the phase space (x,v) are given by

$$\frac{dX}{d\tau} = A_x(X(\tau), V(\tau)) = (b(X) \cdot V(X)) \, b(X), \quad \frac{dV}{d\tau} = A_v(X(\tau), V(\tau)).$$

At the lowest order the particles are advected along the magnetic lines. The plasma is confined by the magnetic field.

For rigorous studies of collisionless models for strongly magnetized plasmas we refer to [2], [3], [4], [5]. The analysis of the Vlasov or Vlasov-Poisson equations with large external magnetic field has been performed in [8], [9]. The nonlinear gyrokinetic theory of the Vlasov-Maxwell equations can be carried out by appealing to Lagrangian and Hamiltonian methods [7], [15], [16]. It is also possible to follow the general method of multiple time scale or averaging perturbation developped in [1]. For the numerical approximation of the gyrokinetic models we refer to [13], [10], [11].

Coming back to (9), in the presence of collisions, we are left with the difficult task of averaging the Fokker-Planck operator, in the right hand side. It is one of the key question for predicting the confinement properties of magnetized plasmas, since the effect of collisions cannot be neglected. We expect that averaging the collision operator will lead to a similar operator, satisfying the usual physical properties : particle, momentum, energy conservations and the relaxation towards a local Maxwellian equilibrium. It turns out that a linearized and gyroaveraged collision operator has been written in [18], but the implementation of this operator seems very hard. We refer to [6] for a general guiding-center bilinear Fokker-Planck collision operator. Another difficulty lies in the relaxation of the distribution function towards a local Maxwellian equilibrium. Most of the available model operators, in particular those which are linearized near a Maxwellian, are missing this property. Very recently a set of model collision operators has been obtained in [12], based on entropy variational principles.

The aim of this paper is twofold. First we compute the effective Fokker-Planck collision operator, averaging with respect to the fast gyromotion of charged particle moving in a nonuniform magnetic field. This approach relies on a central result in ergodic theory *i.e.,* von Neumann's ergodic theorem [17] pp. 57. Moreover, these calculations follow by basic manipulations and do not appeal to any special mathematical tools, as the Lie-transform method, push-forward and pull-back transformations, noncanonical Poisson brackets, action-angle coordinates, etc. Another advantage is that this method provides an explicit formula for the averaged collision operator, expressed in terms of the standard phase space coordinates (x, v), which facilitates its implementation in a simulation code. One deduces the following expression for the averaged Fokker-Planck collision operator cf. Theorem 5.1

$$\begin{aligned}\langle Q_{ss'}(f_s, f_{s'})\rangle (v) \quad &= \quad \frac{1}{m_s}\mathrm{div}_v \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \\ & \quad S_{v,b}(v - v_1) \left(\frac{1}{m_s} f_{s'}(v_1)(\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v)(\nabla_v f_{s'})(v_1) \right) \mathrm{d}v_1\end{aligned}$$

where $S_{v,b}(w) = S(v \wedge b) \circ S(w)$.

Second, we investigate the main properties of the averaged collision operator. We establish the particle, momentum, energy conservations and the relaxation towards a local Maxwellian equilibrium. Therefore this reduced collision operator is well adapted for gyrokinetic simulations.

The outline of the paper is the following. In Section 2 we recall briefly the main properties of the average operator. In Section 3 we investigate the transport operator in the Fokker-Planck equation. Section 4 is devoted to the computation of the averaged collision operator. The main properties of the averaged collision operator are discussed in Section 5.

2 Average operator

The concern of this section is to introduce the main tool of our study, the average operator cf. [4], [5]. For the sake of the completeness we recall here the main results. We work in the $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ framework and define the operator

$$\mathcal{T} u = \mathrm{div}_v \left(\omega_c(x)\; u\; v \wedge b(x)\right), \quad u \in \mathrm{D}(\mathcal{T})$$

$$\mathrm{D}(\mathcal{T}) = \{u(x, v) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \; : \; \mathrm{div}_v \left(\omega_c(x)\; u\; v \wedge b(x)\right) \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)\}$$

where $\omega_c(x) = \frac{qB(x)}{m}$ is the rescaled cyclotronic frequency of a charged particle of mass m and charge q. The notation $\|\cdot\|$ stands for the standard norm of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$. We denote by $(X, V)(s; x, v)$ the characteristics associated to the vector field $(0, \omega_c(x)(v \wedge b(x)))$, that means

$$\frac{\mathrm{d}X}{\mathrm{d}s} = 0, \quad \frac{\mathrm{d}V}{\mathrm{d}s} = \omega_c(X(s))\; V(s) \wedge b(X(s)), \quad (X, V)(0) = (x, v). \tag{10}$$

It is easily seen that x, $|v \wedge b(x)|$, $(v \cdot b(x))$ are left invariant along the characteristic flow (10). Straightforward computations yield the formulae $X(s; x, p) = x$ and

$$V(s; x, p) = \cos(\omega_c(x)s)\; b(x) \wedge (v \wedge b(x)) + \sin(\omega_c(x)s)\; v \wedge b(x) + (v \cdot b(x))\; b(x).$$

The trajectories $(X,V)(s;x,v)$ are $T_c(x) = \frac{2\pi}{\omega_c(x)}$ periodic for any initial condition $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and therefore we introduce the average operator along these trajectories cf. [4]

$$\begin{aligned} \langle u \rangle (x,v) &= \frac{1}{T_c(x)} \int_0^{T_c(x)} u(X(s;x,v), V(s;x,v)) \, \mathrm{d}s \\ &= \frac{1}{2\pi} \int_{S(x)} u(x, |v \wedge b(x)| \, \omega + (v \cdot b(x)) \, b(x)) \, \mathrm{d}\omega \end{aligned}$$

for any function $u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3)$, where $S(x) = \{\omega \in S^2 \; : \; b(x) \cdot \omega = 0\}$. Notice that the kernel of $\mathcal{T}$ is given by the functions in L^2 invariant along the characteristics (10). Therefore we have

$$\ker \mathcal{T} = \{u \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \; : \; \exists \, w \text{ such that } u(x,v) = w(x, |v \wedge b(x)|, (v \cdot b(x)))\}.$$

The proof of the next result can be found in [5].

Proposition 2.1 *The average operator is linear continuous. Moreover it coincides with the orthogonal projection on the kernel of $\mathcal{T}$ i.e.,*

$$\langle u \rangle \in \ker \mathcal{T} \; : \; \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \varphi \, \mathrm{d}v \mathrm{d}x = 0, \;\; \forall \, \varphi \in \ker \mathcal{T}.$$

The above result allows us to characterize the closure of the range of $\mathcal{T}$. Indeed, since $\langle \cdot \rangle = \mathrm{Proj}_{\ker \mathcal{T}}$ and $\mathcal{T}^\star = -\mathcal{T}$ we have

$$\ker \langle \cdot \rangle = (\ker \mathcal{T})^\perp = (\ker \mathcal{T}^\star)^\perp = \overline{\mathrm{Range}\, \mathcal{T}}.$$

Moreover we have the orthogonal decomposition of $L^2(\mathbb{R}^3 \times \mathbb{R}^3)$ into invariant functions along the characteristics (10) and zero average functions *i.e.,*

$$u = \langle u \rangle + (u - \langle u \rangle), \;\; \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (u - \langle u \rangle) \, \langle u \rangle \; \mathrm{d}v \mathrm{d}x = 0.$$

If the magnetic field remains away from 0, the range of $\mathcal{T}$ is closed, leading to the equality $\mathrm{Range}\, \mathcal{T} = \ker \langle \cdot \rangle$, which gives a solvability condition for $\mathcal{T} u = v$. We have the Poincaré inequality cf. [5]

Proposition 2.2 *We assume that $\inf_{x \in \mathbb{R}^3} B(x) > 0$. Then $\mathcal{T}$ restricted to $\ker \langle \cdot \rangle$ is one to one map onto $\ker \langle \cdot \rangle$. Its inverse belongs to $\mathcal{L}(\ker \langle \cdot \rangle, \ker \langle \cdot \rangle)$ and we have the Poincaré inequality*

$$\|u\| \le \frac{2\pi}{|\omega_0|} \|\mathcal{T} u\|, \;\; \omega_0 = \frac{q}{m} \inf_{x \in \mathbb{R}^3} B(x) \neq 0 \tag{11}$$

for any $u \in \mathrm{D}(\mathcal{T}) \cap \ker \langle \cdot \rangle$.

3 The transport operator average

We intend to find a closure for the dominant distribution f_s after eliminating the first order distribution f_s^1 in (9). The idea is to apply the average operator to (9). Indeed, discarding the species index here, we have under the hypotheses of Proposition 2.2

$$\mathcal{T} f^1 \in \text{Range}\, \mathcal{T} = \ker \langle \cdot \rangle$$

saying that $\left\langle \mathcal{T} f^1 \right\rangle = 0$. In this way, averaging (9) leads to a zeroth-order model whose left hand side (coming by averaging the transport operator) is given by

$$\left\langle \partial_t f + v \cdot \nabla_x f + \frac{q}{m} E \cdot \nabla_v f \right\rangle = ... \tag{12}$$

Recall that the previous average is to be computed under the constraint $\mathcal{T} f = 0$, implying that there is a function $g = g(t, x, r, z)$ depending on time t, and the invariants $x, r = |v \wedge b(x)|, z = (v \cdot b(x))$ such that

$$f(t, x, v) = g(t, x, |v \wedge b(x)|, (v \cdot b(x))). \tag{13}$$

Certainly, the limit model should be completed by the average of the collision terms appearing in the right hand side of (9). This will be done in the next section. Now we concentrate on the derivative averages. It is easily seen that the time derivative and the average operator are commuting since the characteristic system (10) is autonomous. Taking into account that $f \in \ker \mathcal{T}$ we obtain

$$\langle \partial_t f \rangle = \partial_t \langle f \rangle = \partial_t f. \tag{14}$$

For computing the averages of the space and momentum derivatives we apply the chain rule to (13) and we average only the derivatives of the invariants since the derivatives of g depend only on time and the invariants and thus are constant along the characteristic flow (10). By direct computations one gets

$$v \cdot \nabla_x f = v \cdot \nabla_x g - \partial_r g \, \frac{(v \cdot b(x))}{|v \wedge b(x)|} \, (\partial_x b \,:\, v \otimes v) + \partial_z g \, (\partial_x b \,:\, v \otimes v)$$

and

$$\nabla_v f = \frac{\partial_r g}{|v \wedge b(x)|} \, (I - b(x) \otimes b(x)) v + \partial_z g \; b(x).$$

Here the notation $U : V$ stands for the contraction $\sum_{i,j=1}^3 u_{ij} v_{ij}$ of two matrices $U = (u_{ij}), V = (v_{ij}) \in \mathcal{M}_{3\times 3}(\mathbb{R})$. It is easily seen that

$$\langle v \rangle = (v \cdot b(x)) \, b, \;\; \langle v \otimes v \rangle = \frac{|v \wedge b(x)|^2}{2} (I - b(x) \otimes b(x)) + (v \cdot b(x))^2 \, b(x) \otimes b(x).$$

Taking into account that ${}^t\partial_x b \, b = \frac{1}{2} \nabla_x |b|^2 = 0$ we deduce that

$$\langle v \cdot \nabla_x f \rangle = b(x) \otimes b(x) v \cdot \nabla_x g - \frac{(v \cdot b(x)) \, |v \wedge b(x)|}{2} \, \text{div}_x b \, \partial_r g + \frac{|v \wedge b(x)|^2}{2} \, \text{div}_x b \, \partial_z g. \tag{15}$$

and

$$\left\langle \frac{q}{m} E(t) \cdot \nabla_v f \right\rangle = \frac{q}{m} (b(x) \cdot E(t,x))\, \partial_z g. \tag{16}$$

Combining (12), (14), (15), (16) yields the following transport operator in the phase space $(x, r, z) \in \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}$

$$\partial_t g + z\, b(x) \cdot \nabla_x g - \frac{zr}{2} \operatorname{div}_x b\, \partial_r g + \left(\frac{r^2}{2} \operatorname{div}_x b + \frac{q}{m} (b(x) \cdot E(t,x)) \right) \partial_z g = ... \tag{17}$$

It is possible to write the left hand side of the previous equality in terms of the distribution $f = f(t,x,v)$ in the phase space (x, v). For this it is sufficient to express the derivatives of g with respect to the derivatives of f

$$\partial_t g = \partial_t f, \;\; \partial_z g = b(x) \cdot \nabla_v f, \;\; \partial_r g = \frac{v - (v \cdot b) b}{|v \wedge b|} \cdot \nabla_v f$$

$$\nabla_x g = \nabla_x f - ({}^\perp v \cdot \nabla_v f)\, \frac{{}^t\partial_x b\, v}{|v \wedge b|}$$

leading to

$$\partial_t f + b(x) \otimes b(x) v \cdot \nabla_x f + \left(\frac{q}{m} b \otimes b E + \omega(x,v)\, {}^\perp v \right) \cdot \nabla_v f = ... \tag{18}$$

where

$${}^\perp v = |v \wedge b|\, b - (v \cdot b)\, \frac{v - (v \cdot b) b}{|v \wedge b|}$$

and

$$\omega(x,v) = \frac{|v \wedge b(x)|}{2} \operatorname{div}_x b - (v \cdot b(x)) \left(\partial_x b\, b(x) \cdot \frac{v}{|v \wedge b(x)|} \right).$$

4 The collision operator average

We average now the terms $Q_{ss'}(f_s, f_{s'})$ appearing in the right hand side of (9), under the constraints $\mathcal{T}_s f_s = 0$, $\mathcal{T}_{s'} f_{s'} = 0$. By the definitions of $\mathcal{T}_s, \mathcal{T}_{s'}$, notice that $\ker \mathcal{T}_s = \ker \mathcal{T}_{s'}$ and therefore $f_s, f_{s'}$ satisfy the same constraint. The distributions $f_s, f_{s'}$ depend only on $t, x, |v \wedge b(x)|, (v \cdot b(x))$. It is easily seen that the average operator is not depending on the species s and therefore we will omit the index s when averaging *i.e.*, $\langle \cdot \rangle_s = \langle \cdot \rangle = \langle \cdot \rangle_{s'}$.

The Fokker-Planck collision operator can be decomposed in gain and loss parts

$$Q_{ss'}(f_s, f_{s'}) = Q^+_{ss'}(f_s, f_{s'}) - Q^-_{ss'}(f_s, f_{s'})$$

where

$$m_s Q^+_{ss'}(f_s, f_{s'})(v) = \operatorname{div}_v \int_{\mathbb{R}^3} A^+_{ss'}(v, v_1)\, \mathrm{d}v_1, \;\; m_s Q^-_{ss'}(f_s, f_{s'})(v) = \operatorname{div}_v \int_{\mathbb{R}^3} A^-_{ss'}(v, v_1)\, \mathrm{d}v_1$$

and

$$A^+_{ss'}(v, v_1) = \mu^2_{ss'} \sigma_{ss'}(|v - v_1|) |v - v_1|^3 S(v - v_1) \frac{1}{m_s} f_{s'}(v_1) \nabla_v f_s(v)$$

$$A^-_{ss'}(v,v_1) = \mu^2_{ss'}\sigma_{ss'}(|v-v_1|)|v-v_1|^3 S(v-v_1)\frac{1}{m_{s'}} f_s(v)\nabla_v f_{s'}(v_1).$$

Both gain and loss operator appearing as a divergence with respect to v, we start by establishing a commuting relation between the operators $\langle\cdot\rangle$ and div_v.

Lemma 4.1 *For any smooth field $A = (A_1, A_2, A_3)(x,v)$ we have*

$$\langle \mathrm{div}_v A\rangle = \mathrm{div}_v \left\{ \left\langle A \cdot \frac{v-(v\cdot b(x))b}{|v\wedge b(x)|}\right\rangle \frac{v-(v\cdot b(x))b}{|v\wedge b(x)|} + \langle A\cdot b\rangle\, b\right\}.$$

Proof. We introduce the notation $e_b(v) = \frac{v-(v\cdot b(x))b}{|v\wedge b(x)|}$, or simply $e(v)$. By Proposition 2.1 we have for any functions $\chi = \chi(x), \varphi(x,v) = \psi(|v\wedge b(x)|, (v\cdot b(x)))$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{div}_v A\ \chi(x)\varphi(x,v)\ \mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \langle \mathrm{div}_v A\rangle\ \chi(x)\varphi(x,v)\ \mathrm{d}v\mathrm{d}x \tag{19}$$

implying that

$$\int_{\mathbb{R}^3} \mathrm{div}_v A\ \varphi(x,v)\ \mathrm{d}v = \int_{\mathbb{R}^3} \langle \mathrm{div}_v A\rangle\ \varphi(x,v)\ \mathrm{d}v,\ \ x\in\mathbb{R}^3. \tag{20}$$

In the sequel we fix $x\in\mathbb{R}^3$ and we compute $\langle \mathrm{div}_v A\rangle(x,\cdot)$ as function of v, using the characterization (20). Integrating by parts, the left hand side of (20) becomes

$$\begin{aligned} -\int_{\mathbb{R}^3} A\cdot\nabla_v\varphi\ \mathrm{d}v &= -\int_{\mathbb{R}^3} A\cdot(\partial_r\psi\ e(v) + \partial_z\psi\ b)\ \mathrm{d}v \\ &= -\int_{\mathbb{R}^3}\partial_r\psi\ \langle A\cdot e(v)\rangle\ \mathrm{d}v - \int_{\mathbb{R}^3}\partial_z\psi\ \langle A\cdot b\rangle\ \mathrm{d}v. \end{aligned} \tag{21}$$

In the last equality we have used the fact that $\partial_r\psi, \partial_z\psi$ depend only on the invariants $r = |v\wedge b|, z = (v\cdot b)$. The next step consists in integrating by parts with respect to the cylindrical coordinates with axis parallel to b. We obtain after computations

$$\begin{aligned} \int_{\mathbb{R}^3}\partial_r\psi\ \langle A\cdot e(v)\rangle\ \mathrm{d}v &= \int_{\mathbb{R}}\int_{\mathbb{R}_+}\partial_r\psi\ \langle A\cdot e(v)\rangle\ 2\pi r\ \mathrm{d}r\mathrm{d}z \\ &= -\int_{\mathbb{R}}\int_{\mathbb{R}_+}\psi(r,z)\left(\partial_r\langle A\cdot e(v)\rangle + \frac{1}{r}\langle A\cdot e(v)\rangle\right) 2\pi r\ \mathrm{d}r\mathrm{d}z \\ &= -\int_{\mathbb{R}^3}\varphi(x,v)\left(\nabla_v\langle A\cdot e(v)\rangle\cdot e(v) + \frac{1}{|v\wedge b|}\langle A\cdot e(v)\rangle\right)\mathrm{d}v \\ &= -\int_{\mathbb{R}^3}\varphi(x,v)\ \mathrm{div}_v\{\langle A\cdot e(v)\rangle\, e(v)\}\ \mathrm{d}v \end{aligned} \tag{22}$$

since $\mathrm{div}_v e(v) = 1/|v \wedge b|$. Similarly one gets

$$\begin{aligned}
\int_{\mathbb{R}^3} \partial_z \psi \left\langle A \cdot b \right\rangle \,\mathrm{d}v &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \partial_z \psi \left\langle A \cdot b \right\rangle \, 2\pi r \,\mathrm{d}r\mathrm{d}z \\
&= -\int_{\mathbb{R}} \int_{\mathbb{R}_+} \psi(r,z) \, \partial_z \left\langle A \cdot b \right\rangle \, 2\pi r \,\mathrm{d}r\mathrm{d}z \\
&= -\int_{\mathbb{R}^3} \varphi(x,v) \, \nabla_v \left\langle A \cdot b \right\rangle \cdot b \,\mathrm{d}v \\
&= -\int_{\mathbb{R}^3} \varphi(x,v) \, \mathrm{div}_v \{ \left\langle A \cdot b \right\rangle b \} \,\mathrm{d}v. \qquad (23)
\end{aligned}$$

Combining (20), (21), (22), (23) yields

$$\int_{\mathbb{R}^3} \varphi \, \mathrm{div}_v \{ \left\langle A \cdot e(v) \right\rangle e(v) + \left\langle A \cdot b \right\rangle b \} \,\mathrm{d}v = \int_{\mathbb{R}^3} \varphi \, \left\langle \mathrm{div}_v A \right\rangle \,\mathrm{d}v$$

for any function φ depending only on $r = |v \wedge b|, z = (v \cdot b)$. Since the functions $\left\langle \mathrm{div}_v A \right\rangle$ and

$$\mathrm{div}_v \{ \left\langle A \cdot e(v) \right\rangle e(v) + \left\langle A \cdot b \right\rangle b \} = \partial_r \left\langle A \cdot e(v) \right\rangle + \frac{1}{r} \left\langle A \cdot e(v) \right\rangle + \partial_z \left\langle A \cdot b \right\rangle$$

satisfy the same property, we deduce that $\left\langle \mathrm{div}_v A \right\rangle = \mathrm{div}_v \{ \left\langle A \cdot e(v) \right\rangle e(v) + \left\langle A \cdot b \right\rangle b \}$. $\square$

In order to average the gain/loss collision operators, we apply Lemma 4.1 with $A^{\pm}(v) = \int_{\mathbb{R}^3} A^{\pm}_{ss'}(v, v_1) \,\mathrm{d}v_1$. We split these computations in four steps. The notation $R_{\alpha,b}$, or simply R_α, stands for the rotation of angle α around the axis parallel to b

$$R_\alpha v = \cos\alpha \; b \wedge (v \wedge b) + \sin\alpha \; (v \wedge b) + (v \cdot b) b.$$

The map $v \to R_\alpha v$ is orthogonal for any α and the average operator also writes

$$\left\langle u \right\rangle (x,v) = \frac{1}{2\pi} \int_0^{2\pi} u(x, R_\alpha v) \,\mathrm{d}\alpha, \quad (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3.$$

Lemma 4.2 *Assume that $f_s = f_s(v), f_{s'} = f_{s'}(v)$ depend only on $r = |v \wedge b|$ and $z = (v \cdot b)$. Then, with the notation $d = (v - v_1)/|v - v_1|$, we have*

$$\begin{aligned}
\left\langle A^+ \cdot e(v) \right\rangle &= \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|) |v - v_1|^3 \frac{1}{m_s} f_{s'}(v_1) \\
&\quad \left[(\nabla_v f_s(v) \cdot e(v)) \, |d \wedge e(v)|^2 - (\nabla_v f_s(v) \cdot b)(d \cdot b)(d \cdot e(v)) \right] \,\mathrm{d}v_1.
\end{aligned}$$

Proof. Observe that for any α we have $e(R_\alpha v) = R_\alpha e(v), v \in \mathbb{R}^3$ and $S(R_\alpha w) R_\alpha v = R_\alpha (S(w) v), v, w \in \mathbb{R}^3$. The computations follow by performing orthogonal change of variables and applying Fubini theorem

$$
\begin{aligned}
\langle A^+ \cdot e(v) \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (A^+(R_\alpha v) \cdot e(R_\alpha v)) \, \mathrm{d}\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{R}^3} A^+_{ss'}(R_\alpha v, R_\alpha v_1) \cdot e(R_\alpha v) \, \mathrm{d}v_1 \right) \mathrm{d}\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \\
&\quad \left(S R_\alpha (v - v_1) \frac{1}{m_s} f_{s'}(R_\alpha v_1) \nabla_v f_s(R_\alpha v) \cdot R_\alpha e(v) \right) \mathrm{d}v_1 \, \mathrm{d}\alpha. \qquad (24)
\end{aligned}
$$

Since $f_{s'}(v_1) = g_{s'}(|v_1 \wedge b|, (v_1 \cdot b))$ we deduce that $f_{s'}(R_\alpha v_1) = f_{s'}(v_1)$. Similarly, since $f_s(v) = g_s(|v \wedge b|, (v \cdot b))$ one gets $\nabla_v f_s = \partial_r g_s e(v) + \partial_z g_s b$ implying that

$$\nabla_v f_s(R_\alpha v) = \partial_r g_s R_\alpha e(v) + \partial_z g_s b = R_\alpha(\partial_r g_s e(v) + \partial_z g_s b).$$

Therefore, the calculations in (24) lead to

$$
\begin{aligned}
\langle A^+ \cdot e(v) \rangle &= \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_s} f_{s'}(v_1)(\partial_r g_s e(v) + \partial_z g_s b) \cdot (S(v - v_1) e(v)) \, \mathrm{d}v_1 \\
&= \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_s} f_{s'}(v_1) \\
&\quad \left[(\nabla_v f_s(v) \cdot e(v)) \, |d \wedge e(v)|^2 - (\nabla_v f_s(v) \cdot b)(d \cdot b)(d \cdot e(v)) \right] \mathrm{d}v_1.
\end{aligned}
$$

□

Lemma 4.3 *Under the assumptions in Lemma 4.2 we have*

$$
\begin{aligned}
\langle A^+ \cdot b \rangle &= \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_s} f_{s'}(v_1) \\
&\quad \left[(\nabla_v f_s(v) \cdot b)|d \wedge b|^2 - (\nabla_v f_s(v) \cdot e(v))(d \cdot b)(d \cdot e(v)) \right] \mathrm{d}v_1.
\end{aligned}
$$

Proof. Using the same notations one gets

$$
\begin{aligned}
\langle A^+ \cdot b \rangle &= \frac{1}{2\pi} \int_0^{2\pi} (A^+(R_\alpha v) \cdot b) \, \mathrm{d}\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \left(\int_{\mathbb{R}^3} A^+_{ss'}(R_\alpha v, R_\alpha v_1) \cdot b \, \mathrm{d}v_1 \right) \mathrm{d}\alpha \\
&= \frac{1}{2\pi} \int_0^{2\pi} \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \\
&\quad \left(S R_\alpha (v - v_1) \frac{1}{m_s} f_{s'}(R_\alpha v_1) \nabla_v f_s(R_\alpha v) \cdot b \right) \mathrm{d}v_1 \, \mathrm{d}\alpha \\
&= \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_s} f_{s'}(v_1) \left(S(v - v_1)(\partial_r g_s e(v) + \partial_z g_s b) \cdot b \right) \mathrm{d}v_1
\end{aligned}
$$

(25)

$$
\begin{aligned}
= & \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 \frac{1}{m_s} f_{s'}(v_1) \\
& \left[(\nabla_v f_s(v)\cdot b)|d\wedge b|^2 - (\nabla_v f_s(v)\cdot e(v))(d\cdot b)(d\cdot e(v))\right] \, \mathrm{d}v_1 .
\end{aligned}
$$

□

Combining the expressions obtained in the Lemmas 4.2, 4.3 we deduce

Proposition 4.1 *The average of the gain operator is given by*

$$
m_s \left\langle Q_{ss'}^+(f_s, f_{s'})\right\rangle = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 \frac{1}{m_s} f_{s'}(v_1) S_{v,b}(v-v_1)\nabla_v f_s(v)\, \mathrm{d}v_1 \right\}
$$

where $S_{v,b}(w) = S(v\wedge b)\circ S(w)$.

Proof. Applying Lemma 4.1 with the field A^+ one gets

$$
m_s \left\langle Q_{ss'}^+(f_s, f_{s'})\right\rangle = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 \frac{1}{m_s} f_{s'}(v_1) C^+(v,v_1)\, \mathrm{d}v_1 \right\}
$$

where the field $C^+(v, v_1)$ is given by

$$
\begin{aligned}
C^+(v,v_1) &= [(\nabla_v f_s(v)\cdot e(v))\ |d\wedge e(v)|^2 - (\nabla_v f_s(v)\cdot b)\ (d\cdot b)(d\cdot e(v))]\ e(v) \\
&+ [(\nabla_v f_s(v)\cdot b)\ |d\wedge b|^2 - (\nabla_v f_s(v)\cdot e(v))\ (d\cdot b)(d\cdot e(v))]\ b.
\end{aligned}
$$

Notice that $C^+ = C_1^+ + C_2^+$ where $C_1^+ = (\nabla_v f_s(v)\cdot e(v))\ e(v) + (\nabla_v f_s(v)\cdot b)\ b$ and

$$
C_2^+ = -[(\nabla_v f_s(v)\cdot e(v))\ (d\cdot e(v)) + (\nabla_v f_s(v)\cdot b)\ (d\cdot b)]\ [(d\cdot e(v))\ e(v) + (d\cdot b)\ b].
$$

Since the distribution f_s satisfies the constraint $(v\wedge b)\cdot\nabla_v f_s = 0$, we can write

$$
C_1^+ = \nabla_v f_s(v) - \left(\nabla_v f_s(v)\cdot \frac{v\wedge b}{|v\wedge b|}\right) \frac{v\wedge b}{|v\wedge b|} = \nabla_v f_s(v).
$$

For the same reason we have

$$
C_2^+ = -(\nabla_v f_s(v)\cdot d)\ [e(v)\otimes e(v) + b\otimes b] d = -(S(v\wedge b)\circ (d\otimes d))\ \nabla_v f_s(v).
$$

Finally, taking into account that $\nabla_v f_s(v) = S(v\wedge b)\nabla_v f_s(v)$ one gets the formula

$$
C^+(v,v_1) = (S(v\wedge b) - S(v\wedge b)\circ(d\otimes d))\ \nabla_v f_s(v) = S_{v,b}(v-v_1)\nabla_v f_s(v)
$$

and our conclusion follows. □

Notice that the averaged gain operator has similar structure as the Fokker-Planck gain operator in (2), the main point here being that the map S should be replaced by $S_{v,b}$. In this way the averaged gain operator depends explicitly on x, through the direction of the magnetic field $b = b(x)$. The collision mechanism is not anymore uniform in space and depends on the magnetic shape. We expect a similar expression for the loss operator. Actually we can prove

Proposition 4.2 *The average of the loss operator is given by*

$$m_s \left\langle Q^-_{ss'}(f_s, f_{s'}) \right\rangle = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_{s'}} f_s(v) S_{v,b}(v - v_1) \nabla_v f_{s'}(v_1) \, \mathrm{d}v_1 \right\}.$$

We need to establish analogous results for $A^-(v)$ as those in the Lemmas 4.2, 4.3. The reader can easily check

Lemma 4.4 *Under the assumptions in Lemma 4.2 we have*

$$\begin{aligned} \left\langle A^- \cdot e(v) \right\rangle \;\; = \;\; & \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_{s'}} f_s(v) \\ & \Big\{ (\nabla_v f_{s'}(v_1) \cdot e(v_1))[\, (e(v) \cdot e(v_1)) - (e(v) \cdot d)\; (e(v_1) \cdot d) \,] \\ & -(\nabla_v f_{s'}(v_1) \cdot b)\; (d \cdot b)(d \cdot e(v)) \Big\} \, \mathrm{d}v_1 \end{aligned}$$

and

$$\begin{aligned} \left\langle A^- \cdot b \right\rangle \;\; = \;\; & \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_{s'}} f_s(v) \\ & \big[(\nabla_v f_{s'}(v_1) \cdot b)(1 - (b \cdot d)^2) - (\nabla_v f_{s'}(v_1) \cdot e(v_1))\; (d \cdot b)(d \cdot e(v_1)) \big] \; \mathrm{d}v_1. \end{aligned}$$

We can establish now the conclusion in Proposition 4.2

Proof. (of Proposition 4.2) As for the gain operator we write

$$m_s \left\langle Q^-_{ss'}(f_s, f_{s'}) \right\rangle = \mathrm{div}_v \left\{ \int_{\mathbb{R}^3} \mu^2_{ss'} \sigma_{ss'}(|v - v_1|)|v - v_1|^3 \frac{1}{m_{s'}} f_s(v) C^-(v, v_1) \, \mathrm{d}v_1 \right\}$$

where the field $C^-(v, v_1)$ is given by

$$\begin{aligned} C^- \;\; = \;\; & \Big\{ (\nabla_v f_{s'}(v_1) \cdot e(v_1))\; [\, (e(v) \cdot e(v_1)) - (e(v) \cdot d)\; (e(v_1) \cdot d) \,] \\ & -(\nabla_v f_{s'}(v_1) \cdot b)\; (d \cdot b)(d \cdot e(v)) \Big\}\; e(v) \\ + \;\; & [(\nabla_v f_{s'}(v_1) \cdot b)\; (1 - (d \cdot b)^2) - (\nabla_v f_{s'}(v_1) \cdot e(v_1))\; (d \cdot b)(d \cdot e(v_1))]\; b \end{aligned}$$

or equivalently $C^- = C^-_1 + C^-_2$ with

$$C^-_1(v, v_1) = (\nabla_v f_{s'}(v_1) \cdot e(v_1))\; (e(v) \cdot e(v_1))\; e(v) + (\nabla_v f_{s'}(v_1) \cdot b)\; b$$

$$C^-_2(v, v_1) = -[(\nabla_v f_{s'}(v_1) \cdot e(v_1))(d \cdot e(v_1)) + (\nabla_v f_{s'}(v_1) \cdot b)(d \cdot b)]\; S(v \wedge b) d.$$

Using the equality

$$(\nabla_v f_{s'}(v_1) \cdot e(v_1))\; e(v_1) + (\nabla_v f_{s'}(v_1) \cdot b)\; b = \nabla_v f_{s'}(v_1) \tag{26}$$

it is easily seen that

$$C^-_1(v, v_1) = S(v \wedge b) \nabla_v f_{s'}(v_1), \quad C^-_2(v, v_1) = -(\nabla_v f_{s'}(v_1) \cdot d)\; S(v \wedge b) d.$$

Finally one gets

$$C^-(v, v_1) = (S(v \wedge b) \circ S(v - v_1)) \nabla_v f_{s'}(v_1)$$

and our conclusion follows. □

5 The limit model

The computations in the previous sections allow us to derive, at least formally, a closed system for the dominant distribution functions $(f_s)_s$. The rigorous justification of the asymptotic limit below is beyond the scope of this study. More details can be found in [4]. The notation $\langle Q_{ss'}\rangle$ stands for the averaged collision operator, *i.e.*, $\langle Q_{ss'}\rangle (f_s, f_{s'}) = \langle Q_{ss'}(f_s, f_{s'})\rangle$.

Theorem 5.1 *For any $\varepsilon > 0$ we assume that $(f_s^\varepsilon)_s$ solve the problem*

$$\partial_t f_s^\varepsilon + v\cdot\nabla_x f_s^\varepsilon + \frac{q_s}{m_s} E\cdot\nabla_v f_s^\varepsilon + \frac{\omega_{cs}(x)}{\varepsilon}(v\wedge b(x))\cdot\nabla_v f_s^\varepsilon = \sum_{s'} Q_{ss'}(f_s, f_{s'}),\ \ (t,x,v)\in\mathbb{R}_+\times\mathbb{R}^3\times\mathbb{R}^3 \tag{27}$$

$$f_s^\varepsilon(0,x,v) = f_s^{\text{in}}(x,v),\ \ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3,\ \ s\in\mathcal{S}$$

where $\omega_{cs}(x) = q_s B(x)/m_s, s \in \mathcal{S}$. Therefore the limit distributions $f_s = \lim_{\varepsilon\searrow 0} f_s^\varepsilon$ solve the problem

$$\partial_t f_s + b\otimes bv\cdot\nabla_x f_s + \left(\frac{q_s}{m_s} b\otimes bE + \omega(x,v)\ {}^\perp v\right)\cdot\nabla_v f_s = \sum_{s'}\langle Q_{ss'}\rangle (f_s, f_{s'}) \tag{28}$$

$$\begin{aligned}\langle Q_{ss'}\rangle (f_s, f_{s'})(v) \quad = \quad & \frac{1}{m_s}\mathrm{div}_v \int_{\mathbb{R}^3} \mu_{ss'}^2 \sigma_{ss'}(|v-v_1|)|v-v_1|^3 \\ & S_{v,b}(v-v_1)\left(\frac{1}{m_s} f_{s'}(v_1)(\nabla_v f_s)(v) - \frac{1}{m_{s'}} f_s(v)(\nabla_v f_{s'})(v_1)\right)\,\mathrm{d}v_1\end{aligned}$$

under the constraints $(v\wedge b(x))\cdot\nabla_v f_s = 0, s\in\mathcal{S}$ and with the initial conditions

$$f_s(0,x,v) = \left\langle f_s^{\text{in}}\right\rangle (x,v),\ \ (x,v)\in\mathbb{R}^3\times\mathbb{R}^3,\ \ s\in\mathcal{S}.$$

Proof. The limit equation (28) follows immediately by applying the average operator to (9), for any species s, and by replacing the expressions obtained in (18), Proposition 4.1, Proposition 4.2 for the transport operator average and gain/loss operator averages. It remains to justify the initial conditions. Multiplying (27) by the test function $\eta(t)\varphi(x,v)$, with $\eta\in C_c^1(\mathbb{R}_+), \varphi\in C_c^1(\mathbb{R}^3\times\mathbb{R}^3)$, such that the constraint $(v\wedge b(x))\cdot\nabla_v\varphi = 0$ holds true, we obtain easily after integration by parts and taking the limit as $\varepsilon\searrow 0$

$$\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s\varphi\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s\left(v\cdot\nabla_x\varphi + \frac{q_s}{m_s}E\cdot\nabla_v\varphi\right)\mathrm{d}v\mathrm{d}x + \sum_{s'}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} Q_{ss'}(f_s, f_{s'})\varphi\,\mathrm{d}v\mathrm{d}x$$

and

$$\lim_{t\searrow 0}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s(t,x,v)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s^{\text{in}}(x,v)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x.$$

Since the function φ satisfies the constraint $(v\wedge b(x))\cdot\nabla_v\varphi = 0$, we have by the definition of the average operator

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_s^{\text{in}}(x,v)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle f_s^{\text{in}}\right\rangle (x,v)\varphi(x,v)\,\mathrm{d}v\mathrm{d}x$$

and thus the initial condition for f_s (which satisfies the same constraint at any time $t\in\mathbb{R}_+$) must be $f_s(0) = \left\langle f_s^{\text{in}}\right\rangle, s\in\mathcal{S}$. □

In the sequel we inquire about the standard conservations of the model (28). We expect that the limit solutions $(f_s)_s$ satisfy the particle, momentum and energy conservations, when no force is applied. It is easily checked, based on Lemma 1.1, that the averaged collision operators $\langle Q_{ss'} \rangle$ satisfy the same conservations as the collision operators $Q_{ss'}$.

Lemma 5.1 *Under the hypotheses in Lemma 1.1 we have*

$$\int_{\mathbb{R}^3} \langle Q_{ss'} \rangle (f_s, f_{s'})(v) \, \mathrm{d}v = 0 \tag{29}$$

$$\sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s v \langle Q_{ss'} \rangle (f_s, f_{s'})(v) \, \mathrm{d}v = 0 \tag{30}$$

$$\sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s \frac{|v|^2}{2} \langle Q_{ss'} \rangle (f_s, f_{s'})(v) \, \mathrm{d}v = 0. \tag{31}$$

Proof. The first statement is obvious, since $\langle Q_{ss'} \rangle (f_s, f_{s'})$ is a divergence with respect to v. For the second one we use the decomposition $v = v - \langle v \rangle + \langle v \rangle = v - \langle v \rangle + (v \cdot b)\, b$. Therefore

$$\begin{aligned}
\int_{\mathbb{R}^3} v \langle Q_{ss'} \rangle (f_s, f_{s'}) \, \mathrm{d}v &= \int_{\mathbb{R}^3} (v - \langle v \rangle + (v \cdot b)\, b) \langle Q_{ss'}(f_s, f_{s'}) \rangle \, \mathrm{d}v \\
&= \int_{\mathbb{R}^3} (v \cdot b)\, b \langle Q_{ss'}(f_s, f_{s'}) \rangle \, \mathrm{d}v \\
&= \int_{\mathbb{R}^3} (v \cdot b)\, b Q_{ss'}(f_s, f_{s'}) \, \mathrm{d}v
\end{aligned}$$

where in the last equality we have used the variational characterization of the average operator, with the test function $(v \cdot b)\, b$. Using now (4) one gets

$$\begin{aligned}
\sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s v \langle Q_{ss'} \rangle (f_s, f_{s'})(v) \, \mathrm{d}v &= \sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s (v \cdot b)\, b Q_{ss'}(f_s, f_{s'})(v) \, \mathrm{d}v \\
&= b \otimes b \sum_s \sum_{s'} \int_{\mathbb{R}^3} m_s v Q_{ss'}(f_s, f_{s'})(v) \, \mathrm{d}v = 0.
\end{aligned}$$

The last statement comes easily in a similar manner from (5) by using the test function $|v|^2 = |v \wedge b|^2 + (v \cdot b)^2$. □

It remains to justify the relaxation towards a local Maxwellian. As usual we multiply the equation (28) by $1 + \ln f_s$ and we obtain

$$\begin{aligned}
\partial_t (f_s \ln f_s) + b \otimes b\, v \cdot \nabla_x (f_s \ln f_s) &+ \left(\frac{q_s}{m_s} b \otimes b\, E + \omega(x, v)\, {}^\perp v \right) \cdot \nabla_v (f_s \ln f_s) \\
&= \sum_{s'} (1 + \ln f_s) \langle Q_{ss'} \rangle (f_s, f_{s'}).
\end{aligned}$$

In order to perform integration by parts it is worth to write the above equation in conservative form. By direct computations one gets

$$\mathrm{div}_x(b\otimes b\ v)+\mathrm{div}_v\left(\frac{q_s}{m_s}b\otimes b\ E+\omega(x,v)\ {}^\perp v\right)=\frac{(v\cdot b(x))^2}{|v\wedge b(x)|^2}(\partial_x b\ b\cdot(v-(v\cdot b)\ b))$$

and therefore we obtain

$$\begin{aligned}\partial_t(f_s\ln f_s) &+ \mathrm{div}_x(f_s\ln f_s b\otimes b\ v)+\mathrm{div}_v\left\{f_s\ln f_s\left(\frac{q_s}{m_s}b\otimes b\ E+\omega(x,v)\ {}^\perp v\right)\right\}\\ &= f_s\ln f_s\frac{(v\cdot b(x))^2}{|v\wedge b(x)|^2}(\partial_x b\ b\cdot(v-(v\cdot b)\ b))+\sum_{s'}(1+\ln f_s)\left\langle Q_{ss'}\right\rangle(f_s,f_{s'}).\end{aligned}$$

Notice that the extra term in the conservative form gives no contribution when integrating with respect to (x,v) since $f_s\ln f_s$ depends only on the invariants and $v-(v\cdot b)\ b$ is zero average. Therefore we deduce

$$\frac{d}{dt}\sum_s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_s\ln f_s\ \mathrm{d}v\mathrm{d}x-\sum_s\sum_{s'}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+\ln f_s)\left\langle Q_{ss'}\right\rangle(f_s,f_{s'})\ \mathrm{d}v\mathrm{d}x=0$$

and finally we obtain the same entropy dissipation as in Lemma 1.2 since

$$\begin{aligned}&- \sum_s\sum_{s'}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+\ln f_s)\left\langle Q_{ss'}\right\rangle(f_s,f_{s'})\ \mathrm{d}v\mathrm{d}x\\ &= -\sum_s\sum_{s'}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(1+\ln f_s)Q_{ss'}(f_s,f_{s'})\ \mathrm{d}v\mathrm{d}x\\ &= \sum_{s,s'}\frac{\mu_{ss'}^2}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\sigma_{ss'}|v-v_1|f_s(v)f_{s'}(v_1)\\ &\qquad\left|(v-v_1)\wedge\left(\frac{\nabla_v\ln f_s}{m_s}(v)-\frac{\nabla_v\ln f_{s'}}{m_{s'}}(v_1)\right)\right|^2\ \mathrm{d}v_1\mathrm{d}v\ \mathrm{d}x.\end{aligned}$$

The distributions $(f_s)_s$ relax towards local Maxwellians with the same temperature and mean velocity.

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