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Existence Results For Functional Semilinear Damped Integrodifferential Equations

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Abstract. In this paper we investigate the existence of mild solutions for first and second order semilinear damped integrodifferential equations in Banach spaces. By using suitable fixed point theorems existence as well as uniqueness results are obtained.

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1 Introduction

In this paper, we shall be concerned with the existence of mild solutions for first and second order functional semilinear damped integrodifferential equations in a Banach space. Firstly, we consider the following first order functional semilinear damped integrodifferential equations of the form:

$$y' - Ay = By + F\left(t, y_t, \int_0^t k(t,s,y_s)ds\right), \quad \text{a.e.} \quad t \in J = [0,b] \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0] \tag{2}$$

where $F : J \times C([-r,0],E) \times E \to E$ is a given function, A is the infinitesimal generator of a family of semigroups $\{T(t) : t \geq 0\}$, B is a bounded linear operator from E into E, $k : J \times J \times C([-r,0],E) \to E$, $\phi \in C([-r,0],E)$ and E a real Banach space with norm $|\cdot|$.

For any continuous function y defined on $[-r,b]$ and any $t \in J$, we denote by y_t the element of $C([-r,0],E)$ defined by $y_t(\theta) = y(t+\theta), \quad \theta \in [-r,0]$. Here $y_t(\cdot)$ represents the history of the state from time $t-r$, up to the present time t.

Later, we study the second order functional damped semilinear integrodifferential equations of the form:

$$y'' - Ay = By' + F\left(t, y_t, \int_0^t k(t,s,y_s)ds\right), \quad \text{a.e.} \quad t \in J = [0,b] \tag{3}$$

$$y_0 = \phi, \quad y'(0) = y_1, \tag{4}$$

where F, B, k, ϕ are as in problem (1)–(2), A is the infinitesimal generator of a family of cosinus operators $\{C(t) : t \geq 0\}$ and $y_1 \in E$.

The study of the dynamical buckling of the hinged extensible beam which is either stretched or compressed by axial force in a Hilbert space, can be modeled by the following hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\alpha + \beta \int_0^L \left|\frac{\partial u}{\partial t}(\xi, t)\right|^2 d\xi\right) \frac{\partial^2 u}{\partial x^2} + g\left(\frac{\partial u}{\partial t}\right) = 0, \qquad (E)$$

where $\alpha, \beta, L > 0$, $u(t, x)$ is the deflection of the point x of the beam at the time t, g is a nondecreasing numerical function, and L is the length of the beam.

Equation (E) has its analogue in $\mathbb{R}^n$ and can be included in a general mathematical model

$$u'' + A^2 u + M(\|A^{1/2}u\|_H^2)Au + g(u') = 0, \qquad (E_1)$$

where A is a linear operator in a Hilbert space H and M and g are real functions. Equation (E) was studied by Patcheu [9] and the equation (E_1) by Matos and Pereira [8]. These equations are special cases of the equation (3).

This paper will be organized as follows. In Section 2 we will recall briefly some basic definitions and preliminary facts which will be used later. In Section 3 we shall establish two theorems for the problem (1)–(2). In Section 4 two theorems are presented for the problem (3)–(4). Our approach in the both sections is based on the Schaefer's fixed point theorem and on the Banach contraction principle. Finally in Section 5 controllability results are presented.

2 Preliminaries

We will briefly recall some basic definitions and preliminary facts that we will use in the sequel. $C([-r, 0], E)$ is the Banach space of all continuous functions from $[-r, 0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

By $C([-r, b], E)$ we denote the Banach space of all continuous functions from $[-r, b]$ into E with the norm

$$\|y\|_\infty = \sup\{|y(t)| : -r \leq t \leq b\}.$$

$B(E)$ is the Banach space of all linear bounded operator from E into E with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [14]). $L^1(J, E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| dt.$$

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if:

(1) $C(0) = I$ (I is the identity operator in E),

(2) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,

(3) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$.

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\, ds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [7], Fattorini [6], and to the papers of Travis and Webb [12], [13]. For properties of semigroup theory, we refer the interested reader to the books of Goldstein [7] and Pazy [10].

Definition 2.1. *The map $F : J \times C([-r,0],E) \times E \to E$ is said to be an L^1-Carathéodory if*

(i) *$t \longmapsto F(t,u,z)$ is measurable for each $u \in C([-r,0],E)$ and $z \in E$;*

(ii) *$(u,z) \longmapsto F(t,u,z)$ is continuous for almost all $t \in J$;*

(iii) *For each $\rho > 0$, there exists $\varphi_\rho \in L^1(J,\mathbb{R}_+)$ such that*

$$|F(t,u,z)| \le \varphi_\rho(t)$$

for all $\|u\|, |z| \le \rho$ and for a.e. $t \in J$.

3 First Order Integrodifferential Equations

In this section we are concerned with the existence of solutions for problem (1)–(2). So let us start by defining what we mean by a mild solution of problem (1)–(2).

Definition 3.1. *A function $y \in C([-r,b],E)$ is said a mild solution of (1)–(2) if $y_0 = \phi$ and y satisfies the integral equation*

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)G(s)ds.$$

where $G(t) = F\left(t, y_t, \int_0^t k(t,s,y_s)ds\right)$.

Let us introduce the following hypotheses:

(H1) There exists a function $\alpha \in C(J,\mathbb{R}_+) \cap L^2(J,\mathbb{R}_+)$, such that

$$\left|\int_0^t k(t,s,u)ds\right| \le \alpha(t)\|u\| \quad \text{for each} \quad t \in J \quad \text{and} \quad u \in C([-r,0],E);$$

(H2) There exists a constant $M \ge 1$ such that $\|T(t)\|_{B(E)} \le M$.

(H3) There exist $\beta \in L^2(J, \mathbb{R}_+)$ such that
$$|F(t,u,z)| \le \beta(t)\psi(\|u\| + |z|) \quad \text{for a.e. } t \in J, \quad u \in C([-r,0],E), z \in E,$$
where $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is a continuous, and increasing function with
$$\psi(\alpha(t)\|u\|) \le \alpha(t)\psi(\|u\|) \quad \text{for each } \ t \in J \ \text{ and } \ u \in C([-r,0],E)$$
and
$$\int_0^b m(s)ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)};$$
where $c = M\|\phi\|$ and $m(t) = \max\left\{M\|B\|_{B(E)}, M\beta(t)(1+\alpha(t))\right\}$;

(H4) For each bounded $\mathcal{B} \subseteq C([-r,b],E)$ and $t \in J$ the set
$$\Big\{T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)G(s)ds : y \in \mathcal{B}\Big\},$$
is relatively compact in E.

Theorem 3.2. *Let $F : J \times C([-r,0],E) \times E \to E$ be an L^1-Carathéodory function. Assume that hypotheses (H1)-(H4) hold. Then the IVP (1)–(2) has at least one mild solution.*

Proof. Transform the problem (1)–(2) into a fixed point problem. Consider the operator $N : C([-r,b],E) \longrightarrow C([-r,b],E)$, defined by
$$N(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds & \\ \quad + \int_0^t T(t-s)G(s)ds, & \text{if } t \in J. \end{cases}$$
Clearly the fixed points of N are mild solutions to (1)–(2).

We shall show that N is completely continuous. The proof will be given in several steps.

Step 1: *N is continuous.*

Let y_n be a sequence in $C([-r,b],E)$ such that $y_n \to y$. We shall prove that $N(y_n) \to N(y)$. For each $t \in J$ we have
$$N(y_n)(t) = T(t)\phi(0) + \int_0^t T(t-s)B(y_n(s))ds + \int_0^t T(t-s)G_n(s)ds,$$
where $G_n(t) = F\left(t, y_{n,t}, \int_0^t k(t,s,y_{n,s})ds\right)$. Then
$$|N(y_n)(t) - N(y)(t)| \le \int_0^t |T(t-s)||B(y_n(s)) - B(y(s))|ds + \int_0^t |T(t-s)||G_n(s) - G(s)|ds$$
$$\le bM\|B\|_{B(E)}\|y_n - y\|_\infty + M\int_0^b |G_n(s) - G(s)|ds.$$
Since G is continuous, B is bounded and F is an L^1-Carathéodory function we have by the Lebesgue dominated convergence theorem
$$\|N(y_n) - N(y)\|_\infty \le bM\|B\|_{B(E)}\|y_n - y\|_\infty + M\int_0^b |G_n(s) - G(s)|ds \to 0,$$

as $n \to \infty$. Thus N is continuous.

Step 2: *N maps bounded sets into bounded sets in $C([-r,b],E)$.*

Indeed, it is enough to show that for any $q>0$, there exists a positive constant ℓ such that for each $y \in \mathcal{B}_q = \{y \in C([-r,b],E) : \|y\|_\infty \le q\}$ one has $\|N(y)\|_\infty \le \ell$. Let $y \in \mathcal{B}_q$. Then for each $t \in J$ we have

$$N(y)(t) = T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)G(s)ds.$$

By (H1)–(H3) we have for each $t \in J$

$$\begin{aligned}|N(y)(t)| &\le M\|\phi\| + M\int_0^b |B(y(s))|ds + M\int_0^b \varphi_q(s)ds \\ &\le M\|\phi\| + Mbq\|B\|_{B(E)} + M\|\varphi_q\|_{L^1} := \ell.\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of $C([-r,b],E)$.*

Let $\tau_1, \tau_2 \in J,\ \tau_1 < \tau_2$ and $\mathcal{B}_q$ be a bounded set of $C([-r,b],E)$ as in Step 2. Let $y \in \mathcal{B}_q$. Then for each $t \in J$ we have

$$N(y)(t) = T(t)\phi(0) + \int_0^t T(t-s)(By(s))ds + \int_0^t T(t-s)G(s)ds.$$

Then

$$\begin{aligned}|N(y)(\tau_2) - N(y)(\tau_1)| \le{}& |T(\tau_2) - T(\tau_1)|\|\phi\| + \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||By(s)|ds \\ &+ \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)||By(s)|ds + \int_0^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)|\varphi_q(s)ds \\ &+ \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)|\varphi_q(s)ds.\end{aligned}$$

The right-hand side tends to zero as $\tau_2 - \tau_1 \to 0$. As a consequence of Steps 1 to 3 and (H4) together with the Arzela- Ascoli theorem we can conclude that $N : C([-r,b],E) \longrightarrow C([-r,b],E)$ is a completely continuous operator.

Step 4: *Now it remains to show that the set*

$$\mathcal{M} := \{y \in C([-r,b],E) : y = \lambda N(y), \quad \textit{for some} \ \ 0 < \lambda < 1\}$$

is bounded.

Let $y \in \mathcal{M}$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$y(t) = \lambda\left[T(t)\phi(0) + \int_0^t T(t-s)By(s)ds + \int_0^t T(t-s)G(s)ds\right].$$

This implies by (H1)–(H3) that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\leq M\|\phi\| + M\|B\|_{B(E)} \int_0^t |y(s)|ds + M \int_0^t \beta(s)\psi(\|y_s\| + \alpha(s)\|y_s\|)ds \\ &\leq M\|\phi\| + M\|B\|_{B(E)} \int_0^t |y(s)|ds + M \int_0^t \beta(s)(1+\alpha(s))\psi(\|y_s\|)ds \\ &\leq M\|\phi\| + \int_0^t m(s)(|y(s)| + \psi(\|y_s\|))ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have

$$\mu(t) = |y(t^*)| \leq M\|\phi\| + \int_0^{t^*} m(s)(|y(s)| + \psi(\|y_s\|)ds \leq M\|\phi\| + \int_0^t m(s)(\mu(s) + \psi(\mu(s))ds.$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds, since $M \geq 1$.

Let us denote the right-hand side of the above inequality as $v(t)$. Then we have

$$\mu(t) \leq v(t) \text{ for all } \quad t \in [0, b],$$

and

$$v(0) = M\|\phi\|, \qquad v'(t) = m(t)(\mu(t) + \psi(\mu(t)), \quad t \in J.$$

Using the increasing character of ψ we get

$$v'(t) \leq m(t)(v(t) + \psi(v(t)) \text{ a.e.} \quad t \in [0, b].$$

Then for each $t \in [0, b]$ we have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \leq \int_0^b m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

This inequality implies that there exists a constant L such that $v(t) \leq L, \ t \in J$, and hence $\mu(t) \leq L, \ t \in J$. Since for every $t \in J, \|y_t\| \leq \mu(t)$, we have

$$\|y\|_\infty := \sup\{|y(t)| : -r \leq t \leq b\} \leq L,$$

where L depends only on b and on the functions α, β and ψ. This shows that $\mathcal{M}$ is bounded.

Set $X := C([-r, b], E)$. As a consequence of Schaefer's theorem ([11] p. 29) we deduce that N has a fixed point which is a mild solution of (1)–(2).

Now, we present a uniqueness result for the problem (1)–(3) Our considerations are based on the Banach fixed point theorem. Let us introduce the following hypotheses:

(A1) $|F(t, u, v) - F(t, \overline{u}, \overline{v})| \leq d[\|u - \overline{u}\| + |v - \overline{v}|]$, for each $t \in J$ and $u, \overline{u} \in C([-r, 0], E)$, $v, \overline{v} \in E$, where d is a nonnegative constant.

(A2) $|k(t, s, u) - k(t, s, \overline{u})| \leq K_1\|u - \overline{u}\|$, where $u, \overline{u} \in C([-r, 0], E)$ and K_1 is a nonnegative constant.

Theorem 3.3. *Suppose that hypotheses (H2), (A1), (A2) are satisfied. If*

$$Mb\|B\|_{B(E)} + Mbd(1 + bK_1) < 1,$$

then the IVP (1)–(2) has a unique mild solution.

Proof. Transform the problem (1)–(2) into a fixed point problem. Let the operator $N : C([-r, 0], E) \to C([-r, 0], E)$ defined as in Theorem 3.2. We shall show that N is a contraction. Indeed, consider $y, \overline{y} \in C([-r, b], E)$. Then we have for each $t \in J$

$$\begin{aligned}|N(y)(t) - N(\overline{y})(t)| &\leq \int_0^t M|B(y(s)) - B(\overline{y}(s))|ds + Md\int_0^t [\|y_s - \overline{y}_s\| + bK_1\|y_s - \overline{y}_s\|]ds \\ &\leq M\|B\|_{B(E)} \int_0^t |y(s) - \overline{y}(s)|ds + Md(1 + bK_1) \int_0^t \|y_s - \overline{y}_s\|ds \\ &\leq Mb\|B\|_{B(E)}\|y - \overline{y}\|_\infty + Mbd(1 + bK_1)\|y - \overline{y}\|_\infty \\ &= \big(Mb\|B\|_{B(E)} + Mbd(1 + bK_1)\big)\, \|y - \overline{y}\|_\infty.\end{aligned}$$

Then

$$\|N(y) - N(\overline{y})\|_\infty \leq \big(Mb\|B\|_{B(E)} + Mbd(1 + bK_1)\big)\, \|y - \overline{y}\|_\infty,$$

showing that N is a contraction and hence it has a unique fixed point which is a mild solution to (1)–(2).

4 Second Order Integrodifferential Equations

In this section we study the problem (3)–(4). We give first the definition of mild solution of the problem (3)–(4)

Definition 4.1. *A function $y \in C([-r, b], E)$ is said mild solution of (3)–(4) if $y_0 = \phi$, $y'(0) = y_1$ and*

$$y(t) = (C(t) - S(t)B)\phi(0) + S(t)y_1 + \int_0^t C(t - s)By(s)ds + \int_0^t S(t - s)G(s)ds.$$

Theorem 4.2. *Let $F : J \times C([-r, 0], E) \times E \to E$ be an L^1-Carathéodory function. Assume (H1) and the conditions:*

(B1) *There exists a constant $M_1 > 0$ such that $\|C(t)\|_{B(E)} < M_1$ for all $t \in \mathbb{R}$;*

(B2) *There exist $\beta \in L^2(J, \mathbb{R}_+)$ such that*

$$\|F(t, u, z)\| \leq \beta(t)\psi(\|u\| + |z|) \text{ for a.e. } t \in J, \ u \in C([-r, 0], E), z \in E,$$

where $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is a continuous, and increasing function with

$$\psi(\alpha(t)\|u\|) \leq \alpha(t)\psi(\|u\|) \text{ for each } t \in J \text{ and } u \in C([-r, 0], E)$$

and

$$\int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)};$$

where

$$c = M_1(1+b)\|\phi\| + bM_1|y_1| \quad \text{and} \quad \widehat{m}(t) = \max\left\{M_1\|B\|_{B(E)}, M_1 b\beta(t)(1+\alpha(t))\right\};$$

(B3) *For each bounded* $\mathcal{B} \subseteq C([-r,b],E)$ *and* $t \in J$ *the set*

$$\left\{(C(t)-S(t)B)\phi(0) + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)G(s)ds : y \in \mathcal{B}\right\}$$

is relatively compact in E,

are satisfied. Then the IVP (3)-(4) has at least one mild solution.

Proof. Transform the problem (3)-(4) into a fixed point problem. Consider the operator $\overline{N} : C([-r,b],E) \to C([-r,b],E)$ defined by:

$$\overline{N}(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ (C(t)-S(t)B)\phi(0) + S(t)y_1 & \\ +\int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)G(s)ds, & \text{if } t \in J. \end{cases}$$

As in Theorem 3.2 we can show that $\overline{N}$ is completely continuous. Now we prove only that the set

$$\overline{\mathcal{M}} := \{y \in C([-r,b],E) : y = \lambda\overline{N}(y), \quad \text{for some } \ 0 < \lambda < 1\}$$

is bounded. Let $y \in \overline{\mathcal{M}}$. Then we have:

$$y(t) = \lambda\Big[(C(t)-S(t))\phi(0) + S(t)y_1 + \int_0^t C(t-s)By(s)ds + \int_0^t S(t-s)G(s)ds\Big], \quad t \in [0,b].$$

This implies by (H1) and (B1)-(B2) that, for each $t \in J$, we have

$$\begin{aligned} |y(t)| &\le (M_1 + bM_1)\|\phi\| + bM_1|y_1| + M_1\int_0^t |By(s)|ds \\ &\quad + M_1 b\int_0^t \beta(s)\psi(\|y_s\| + \alpha(s)\|y_s\|)ds \\ &\le (M_1 + bM_1)\|\phi\| + bM_1|y_1| + M_1\|B\|_{B(E)}\int_0^t |y(s)|ds \\ &\quad + M_1 b\int_0^t \beta(s)(1+\alpha(s))\psi(\|y_s\|)ds \\ &\le (M_1 + bM_1)\|\phi\| + bM_1|y_1| + \int_0^t \widehat{m}(s)(|y(s)| + \psi(\|y_s\|))ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r,t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have

$$\begin{aligned} \mu(t) = |y(t^*)| &\le (M_1 + bM_1)\|\phi\| + bM_1|y_1| + \int_0^{t^*} \widehat{m}(s)(|y(s)| + \psi(\|y_s\|))ds \\ &\le (M_1 + bM_1)\|\phi\| + bM_1|y_1| + \int_0^t \widehat{m}(s)(\mu(s) + \psi(\mu(s)))ds. \end{aligned}$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $\overline{v}(t)$, then we have

$$\overline{v}(0) = (M_1 + bM_1)\|\phi\| + bM_1|y_1|, \quad \mu(t) \leq \overline{v}(t), \text{ for a.e. } t \in [0, b]$$

and

$$\overline{v}'(t) = \widehat{m}(t)(\mu(t) + \psi(\mu(t))), \text{ for a.e. } t \in [0, b].$$

Using the nondecreasing character of ψ we get

$$\overline{v}'(t) \leq \widehat{m}(t)(\overline{v}(t) + \psi(\overline{v}(t))), \ t \in [0, b].$$

This implies for each $t \in [0, b]$ that

$$\int_{\overline{v}(0)}^{\overline{v}(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b \widehat{m}(s)ds < \int_{\overline{v}(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}.$$

This inequality implies that there exists a constant b^* such that $\overline{v}(t) \leq b^*$ for each $t \in [0, b]$, and hence $\mu(t) \leq \overline{v}(t) \leq b^*$ for each $t \in J$. Since for every $t \in J$, $\|y_t\| \leq \mu(t)$, we have $\|y\|_\infty \leq b^*$, where b^* depends only on b and on the functions α, β and ψ. This shows that $\overline{\mathcal{M}}$ is bounded. Set $X := C([-r, b], E)$. As a consequence of Schaefer's theorem we deduce that $\overline{N}$ has a fixed point which is a mild solution of (3)–(4).

We present now a uniqueness result for the solutions of the problem (3)–(4) by means of the Banach fixed point principle.

Theorem 4.3. *Suppose that hypotheses (B1), (A1), (A2) are satisfied. If*

$$M_1 b\|B\|_{B(E)} + b^2 M_1 d(1 + bK_1) < 1,$$

then the IVP (3)–(4) has a unique mild solution.

Proof. Transform the problem (3)–(4) into a fixed point problem. Consider the operator $\overline{N}$ defined in Theorem 4.2. We shall show that $\overline{N}$ is a contraction. Indeed, consider $y, \overline{y} \in C([-r, b], E)$. Thus for $t \in [0, b]$

$$\begin{aligned}
|\overline{N}(y)(t) - \overline{N}(\overline{y})(t)| &\leq M_1 \int_0^t |B(y(s)) - B(\overline{y}(s))|ds \\
&\quad + M_1 bd \int_0^t [\|y_s - \overline{y}_s\| + bK_1\|y_s - \overline{y}_s\|]ds \\
&\leq M_1\|B\|_{B(E)} \int_0^t |y(s) - \overline{y}(s)|ds + M_1 bd(1 + bK_1) \int_0^t \|y_s - \overline{y}_s\|ds \\
&\leq M_1 b\|B\|_{B(E)}\|y - \overline{y}\|_\infty + M_1 b^2 d(1 + bK_1)\|y - \overline{y}\|_\infty \\
&= \left(M_1 b\|B\|_{B(E)} + M_1 b^2 d(1 + bK_1)\right) \|y - \overline{y}\|_\infty.
\end{aligned}$$

Then

$$\|\overline{N}(y) - \overline{N}(\overline{y})\|_\infty \leq \left(M_1 b\|B\|_{B(E)} + b^2 M_1 d(1 + bK_1)\right) \|y - \overline{y}\|_\infty.$$

Then $\overline{N}$ is a contraction and hence it has a unique fixed point which is a mild solution to (3)–(4).

5 Applications

As applications of our results, we shall give controllability results for first order semilinear integrodifferential equations of the form

$$y'(t) - Ay(t) = By(t) + F\left(t, y(t), \int_0^t k(t,s,y_s)ds\right) + (Z\mathcal{U})(t), \quad t \in J := [0,b], \tag{5}$$

$$y(t) = \phi(t), \quad t \in [-r,0], \tag{6}$$

and

$$y''(t) - Ay(t) = By'(t) + F\left(t, y(t), \int_0^t k(t,s,y_s)ds\right) + (Z\mathcal{U})(t), \quad t \in J := [0,b], \tag{7}$$

$$y_0 = \phi, \quad y'(0) = y_1, \quad t \in [-r,0], \tag{8}$$

where the control function $\mathcal{U}(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions, with U as a Banach space and Z is a bounded linear operator from U to E. For recent controllability results of nonlinear ordinary, functional and neutral functional differential and integrodifferential systems in Banach spaces, by using different tools of fixed point arguments we refer to the paper by Benchohra and Ntouyas [4] and Balachandran et al. [1], [2] and [3].

Definition 5.1. *A function $y \in C([-r,b],E)$ is called a mild solution of (5)–(6) if $y_0 = \phi$ and*

$$y(t) = T(t)\phi(0) + \int_0^t T(t-s)B(y(s))\,ds + \int_0^t T(t-s)[G(s)ds + (Z\mathcal{U})(s)]\,ds.$$

Definition 5.2. *The system (5)–(6) is said to be controllable on the interval $[-r,b]$, if for every continuous initial function ϕ and every $y_0, x_1 \in E$, there exists a control $u \in L^2(J,U)$, such that the mild solution $y(t)$ of (5)–(6) satisfies $y(b) = x_1$.*

Similarly we define the mild solution and the controllability for the problem (7)–(8). We formulate only the controllability result for the problem (5)–(6).

Theorem 5.3. *Let $F : J \times C([-r,0],E) \times E \to E$ be a continuous function. Assume that (H1)-(H3) hold and moreover:*

(C1) *The linear operator $W : L^2(J,U) \to E$, defined by*

$$Wu = \int_0^b T(b-s)Z\mathcal{U}(s)\,ds,$$

has an invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2(J,U)\backslash \ker W$ and there exist positive constants M_1 and M_2 such that $\|Z\| \le M_1$ and $\|\widetilde{W}^{-1}\| \le M_2$.

(C2) *There exist $\beta \in L^2(J,\mathbb{R}_+)$ such that*

$$\|F(t,u,z)\| \le \beta(t)\psi(\|u\| + |z|) \text{ for a.e. } t \in J, \quad u \in C([-r,0],E), z \in E,$$

where $\psi : \mathbb{R}_+ \longrightarrow (0,\infty)$ is a continuous, and increasing function with

$$\psi(\alpha(t)\|u\|) \le \alpha(t)\psi(\|u\|) \text{ for each } t \in J \text{ and } u \in C([-r,0],E)$$

and

$$\int_1^\infty \frac{d\tau}{\tau+\psi(\tau)} = \infty.$$

(C3) *For each bounded* $\mathcal{B} \subseteq C([-r,b],E)$ *and* $t \in J$ *the set*

$$\Big\{T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds + \int_0^t T(t-s)[(Z\mathcal{U})(s)+G(s)]ds, y \in \mathcal{B}\Big\},$$

is relatively compact in E.

Then the problem (5)–(6) is controllable on $[-r,b]$.

Remark 5.4. *For the construction of* $\widetilde{W}^{-1}$ *see* [5].

Using hypothesis (C1), for an arbirtary function $y(\cdot)$, define the control

$$\mathcal{U}_y(t) = \widetilde{W}^{-1}\Big[x_1 - T(b)\phi(0) - \int_0^b T(b-s)[B(y(s))ds - G(s)]ds\Big](t).$$

We shall now show that when using this control, the operator $N_1 : C([-r,b],E) \longrightarrow C([-r,b],E)$ defined by:

$$N_1(y)(t) := \begin{cases} \phi(t), & \text{if } t \in [-r,0] \\ T(t)\phi(0) + \int_0^t T(t-s)B(y(s))ds & \\ \quad + \int_0^t T(t-s)[(Z\mathcal{U}_y)(s)+G(s)]\,ds & \text{if } t \in J, \end{cases}$$

has a fixed point. This fixed point is then a solution of the system (5)-(6).

We shall show that N_1 is completely continuous. The steps for the proofs are parallel to that of Theorems 3.2. So, we omit the details.

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