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On Univalence of Certain Integral Operators

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Abstract. In this work some integral operators are studied and the author determines conditions for the univalence of these integral operators.

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1 Introduction

Let $U = \{z \in C : \ |z| < 1\}$ be the unit disk in the complex plane and let A be the class of functions which are analytic in the unit disk normalized with $f(0) = f'(0) - 1 = 0$.

Let S the class of the functions $f \in A$ which are univalent in U.

In this work, we consider the integral operators

$$L_\beta(z) = \left[\beta \int_0^z (h(u))^{\beta - 1} \, du\right]^{\frac{1}{\beta}} \tag{1.1}$$

and

$$H_{\beta,\gamma}(z) = \left[\beta \int_0^z u^{\beta-1} \left(e^{g(u)}\right)^{\gamma} du\right]^{\frac{1}{\beta}} \tag{1.2}$$

and we obtain sufficient conditions for the univalence of these integral operators.

2 Preliminary Results

To discuss our integral operators, we need the following theorem and lemma.

Theorem 2.1. [4] *Let α be a complex number and $f \in A$. If*

$$\frac{1 - |z|^{2 \operatorname{Re} \alpha}}{\operatorname{Re} \alpha} \left|\frac{z f''(z)}{f'(z)}\right| \leq 1 \tag{2.1}$$

for all $z \in U$, then, for any complex number β with $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the integral operator

$$F_\beta(z) = \left[\beta \int_0^z u^{\beta - 1} f'(u) du\right]^{\frac{1}{\beta}} \tag{2.2}$$

is in the class S.

Lemma Caratheodory. [2] *Let the function $\varphi(z)$ which are analytic in the open unit disk U with $\varphi(0)=0$. If $\mathrm{Re}\varphi(z)<M$, where $M>0$, for all $z\in U$, then*

$$(1-|z|)\,|\varphi(z)|\le 2M|z| \tag{2.3}$$

for all $z\in U$.

3 Main Results

Theorem 3.1. *Let α,β be complex numbers, $\mathrm{Re}\alpha>0$ and $h\in A$, $h(z)=z+a_2z^2+\dots$ If*

$$\mathrm{Re}\,\beta\ge\mathrm{Re}\,\alpha \tag{p_1}$$

and

$$\mathrm{Re}\left[e^{i\theta}\left(\frac{zh'(z)}{h(z)}-1\right)\right]\le\frac{\mathrm{Re}\,\alpha}{4\,|\beta-1|}\ \textit{for}\ \mathrm{Re}\,\alpha\in(0,1) \tag{p_2}$$

or

$$\mathrm{Re}\left[e^{i\theta}\left(\frac{zh'(z)}{h(z)}-1\right)\right]\le\frac{1}{4\,|\beta-1|}\ \textit{for}\ \mathrm{Re}\,\alpha\in[1,\infty) \tag{p_3}$$

for all $\theta\in[0,\,2\pi]$, then the function

$$L_\beta(z)=\left[\beta\int_0^z(h(u))^{\beta-1}\,du\right]^{\frac{1}{\beta}} \tag{3.1}$$

is in the class S.

Proof. We observe that

$$L_\beta(z)=\left[\beta\int_0^z u^{\beta-1}\left(\frac{h(u)}{u}\right)^{\beta-1}du\right]^{\frac{1}{\beta}}. \tag{3.2}$$

We note

$$p(z)=\int_0^z\left(\frac{h(u)}{u}\right)^{\beta-1}du \tag{3.3}$$

which is a regular function in U.

We have

$$\frac{zp''(z)}{p'(z)}=(\beta-1)\left(\frac{zh'(z)}{h(z)}-1\right). \tag{3.4}$$

We take $\varphi(z)=e^{i\theta}\left(\frac{zh'(z)}{h(z)}-1\right)$, $\theta\in[0,\,2\pi]$, and we see that $\varphi(0)=0$.

We consider the cases:

k_1) Let $\mathrm{Re}\,\alpha\in(0,1)$. The function $q:(0,1)\to\Re$, $q(x)=1-a^{2x}$ $(0<a<1)$ is a increasing function and for $a=|z|$, $z\in U$, we obtain

$$1-|z|^{2\,\mathrm{Re}\,\alpha}\le 1-|z|^2 \tag{3.5}$$

for all $z\in U$.

From (p_2) and Lemma Caratheodory we obtain

$$|\varphi(z)|\le\frac{|z|\,\mathrm{Re}\,\alpha}{2\,|\beta-1|(1-|z|)} \tag{3.6}$$

for all $z \in U$.

We have

$$\frac{1-|z|^{2\,\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| = \frac{1-|z|^{2\,\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}|\beta-1|\left|\frac{zh'(z)}{h(z)}-1\right| \le \tag{3.7}$$

$$\le \frac{1-|z|^2}{\operatorname{Re}\alpha}|\beta-1|\frac{|z|\operatorname{Re}\alpha}{2\,|\beta-1|\,(1-|z|)} \le 1.$$

for all $z \in U$.

k_2) Let $\operatorname{Re}\alpha \in [1,\infty)$. The function $t : [1,\infty) \to \Re,\ t(x) = \frac{1-a^{2x}}{x}$ $(0 < a < 1)$ is a decreasing function, and if we take $a = |z|,\ z \in U$, then

$$\frac{1-|z|^{2\,\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le 1-|z|^2 \tag{3.8}$$

for all $z \in U$.

From (p_3) and Lemma Caratheodory we get

$$|\varphi(z)| \le \frac{|z|}{2\,|\beta-1|(1-|z|)} \tag{3.9}$$

From (9), (13) and (14) we obtain

$$\frac{1-|z|^{2\,\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zp''(z)}{p'(z)}\right| \le 1 \tag{3.10}$$

for all $z \in U$.

Because $p'(z) = \left(\frac{h(z)}{z}\right)^{\beta-1}$ and $\operatorname{Re}\beta \ge \operatorname{Re}\alpha$, from Theorem 2.1, it results that the function $L_\beta(z)$ is regular and univalent in U.

Theorem 3.2. *Let* $\alpha,\ \beta,\ \gamma$ *be complex numbers,* $\operatorname{Re}\alpha > 0$ *and* $g \in A,\ g(z) = z+a_2z^2+\ldots$ *If*

$$\operatorname{Re}\beta \ge \operatorname{Re}\alpha \tag{j_1}$$

and

$$\operatorname{Re}\left[e^{i\theta}\ zg'(z)\right] \le \frac{\operatorname{Re}\alpha}{4\,|\gamma|}\ \text{for}\ \operatorname{Re}\alpha \in (0,1) \tag{j_2}$$

or

$$\operatorname{Re}\left[e^{i\theta}\ zg'(z)\right] \le \frac{1}{4\,|\gamma|}\ \text{for}\ \operatorname{Re}\alpha \in [1,\infty) \tag{j_3}$$

for all $\theta \in [0,\ 2\pi]$, *then the function*

$$H_{\beta,\,\gamma}(z) = \left[\beta\int_0^z u^{\beta-1}\left(e^{g(u)}\right)^\gamma du\right]^{\frac{1}{\beta}} \tag{3.11}$$

is in the class S.

Proof. We consider the function

$$f(z) = \int_0^z \left(e^{g(u)}\right)^\gamma du \tag{3.12}$$

which is a regular function in U.

We have

$$\frac{zf''(z)}{f'(z)} = \gamma z g'(z) \tag{3.13}$$

We note $\varphi(z) = e^{i\theta} z g'(z)$ and we see that $\varphi(0) = 0$.

We have the cases:

i_1) $0 < \operatorname{Re}\alpha < 1$. The function $q : (0,1) \to \Re$, $q(x) = 1 - a^{2x}$ $(0 < a < 1)$ is a increasing function and for $a = |z|$, $z \in U$, we obtain

$$1 - |z|^{2\operatorname{Re}\alpha} \le 1 - |z|^2 \tag{3.14}$$

for all $z \in U$.

From Lemma Caratheodory and (j_2) we obtain

$$|\varphi(z)| \le \frac{|z| \operatorname{Re}\alpha}{2|\gamma|(1-|z|)} \tag{3.15}$$

for all $z \in U$.

We have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| = \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} |\gamma| \, |zg'(z)| \le \tag{3.16}$$

$$\le \frac{1-|z|^2}{\operatorname{Re}\alpha} |\gamma| \frac{|z| \operatorname{Re}\alpha}{2|\gamma|(1-|z|)} \le 1.$$

for all $z \in U$.

i_2) $\operatorname{Re}\alpha \ge 1$. The function $t : [1,\infty) \to \Re$, $t(x) = \frac{1-a^{2x}}{x}$ $(0 < a < 1)$ is a decreasing function, and if we take $a = |z|$, $z \in U$, then

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \le 1 - |z|^2 \tag{3.17}$$

for all $z \in U$.

From Lemma Caratheodory and (j_3) we obtain

$$|\varphi(z)| \le \frac{|z|}{2|\gamma|(1-|z|)} \tag{3.18}$$

for all $z \in U$.

We have

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| = \frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} |\gamma| \, |zg'(z)| \le \tag{3.19}$$

$$\le \left(1-|z|^2\right) |\gamma| \frac{|z| \operatorname{Re}\alpha}{2|\gamma|(1-|z|)} \le 1.$$

for all $z \in U$.

Because $f'(z) = \left(e^{g(z)}\right)^{\gamma}$, from Theorem 2.1 we conclude that the function $H_{\beta,\gamma}(z)$ is in the class S.

Corollary 3.1. *If* $0 < \operatorname{Re}\alpha \le 1$, $g \in A$ *and* $\operatorname{Re}\left[e^{i\theta} z g'(z)\right] \le \frac{\operatorname{Re}\alpha}{4}$ *for all* $\theta \in [0, 2\pi]$, *then the function*

$$G(z) = \int_0^z e^{g(u)} du \tag{3.20}$$

is in the class S.

Proof. For $0 < \mathrm{Re}\,\alpha \leq 1,\ \beta = \gamma = 1,\ g \in A$ and $\mathrm{Re}\left[e^{i\theta} z g'(z)\right] \leq \frac{\mathrm{Re}\,\alpha}{4}$, from Theorem 3.2 we obtain $H_{1,1}(z) = G(z)$ and hence, it results that the function $G(z)$ is in the class S.

Remark. Let be inverse boundary problem in the unit disk U. In [1] we obtain that the solution of the problem is as the form

$$G(z) = \int_0^z e^{g(u)} du \tag{3.21}$$

where $g(z)$ is a regular known function in U. The solution of the inverse boundary problem is necessary to be univalent. The Corolarry 3.1. can be applied to demonstrate the univalence of the solution of the inverse boundary problem.

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