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# Brezis-Browder Principles in General Separable Sets

**Mihai TURINICI**

**Abstract.** The version of Brezis-Browder's principle [Adv. Math., 21(1976), 355-364] for general separable sets may be viewed as a logical equivalent of the Zorn-Bourbaki maximality result.

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## 1 Introduction

Let $M$ be some nonempty set; and $(\leq)$, some *quasi-order* (i.e.: reflexive and transitive relation) over it. Further, let $x \vdash \varphi(x)$ stand for a function from $M$ to $R_+ := [0, \infty[$. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w). \tag{1.1}$$

A basic result involving such points is the 1976 Brezis-Browder ordering principle [6]:

**Theorem BB.** *Suppose that*

$$\begin{array}{l} (M, \leq) \textit{ is sequentially inductive:} \\ \textit{each ascending sequence in } M \textit{ has an upper bound} \end{array} \tag{1.2}$$

$$\varphi \textit{ is } (\leq)\textit{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{1.3}$$

*Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$.*

This principle, including the well known Ekeland's [7,8], found some useful applications to convex and nonconvex analysis; we refer to the quoted papers for a survey of these. So, it cannot be surprising that, soon after its formulation, many extensions of Theorem BB were proposed; see, for instance, Altman [2], Turinici [21], Anisiu [3] or Kang and Park [15]. Here, we shall concentrate on the *structural* way of enlargement. This, roughly speaking, consists of $(R_+, \geq)$ being substituted by an ordered structure

$(P, \leq)$ endowed with (cardinal-) countable regularity properties for its chains. Some basic results in this area were obtained in the standard countable case by Gajek and Zagrodny [10]; see also Zhu, Fan and Zhang [25]. A refinement of all these was recently obtained (in the precised setting) by Turinici [23]. It is our aim in the following to show that a further extension of such techniques is possible, beyond the (standard) countable framework; details will be given in Section 4 (the transitive case) and Section 5 (the amorph case). The specific tool for deducing these is a variant of the Zorn-Bourbaki maximality principle for general separable sets, given in Section 3. All preliminary material involving such objects is presented in Section 2. Further discussions about these questions will be performed elsewhere.

## 2 General Separable Sets

**(A)** Let $W$ stand for the class of ordinal numbers, introduced in a "factorial" way; cf. Kuratowski and Mostowski [17, Ch 7, Sect 2]. Precisely, given a partially ordered structure $(P, \leq)$, call it *well ordered* if each (nonempty) part of $P$ admits a first element. Given the couple $(P, \leq)$, $(Q, \leq)$ of such objects, put

$$(P, \leq) \equiv (Q, \leq) \text{ iff there exists a strictly increasing bijection:} P \to Q.$$

This is an equivalence relation; the order type of $(P, \leq)$ (denoted $\text{ord}(P, \leq)$) is just its equivalence class; also referred to as an *ordinal*.

Note that $W$ is not a set, as results from the Burali-Forti paradox; cf. Sierpinski [19, Ch 14, Sect 2]. However, when one restricts to a *Grothendieck universe* $\mathcal{G}$ (taken as in Hasse and Michler [11, Ch 1, Sect 2]) this contradictory character is removed for the class $W(\mathcal{G})$ of all *admissible* (modulo $\mathcal{G}$) ordinals (generated by (non-contradictory) well ordered parts of $\mathcal{G}$). In the following, we drop any reference to $\mathcal{G}$, for simplicity. So, by an *ordinal* in $W$ one actually means a $\mathcal{G}$-admissible ordinal with respect to a "sufficiently large" Grothendieck universe $\mathcal{G}$. Clearly,

$$\xi\text{=admissible ordinal and } \eta \leq \xi \text{ imply } \eta\text{=admissible ordinal.}$$

Hence, in the formulae

$$W(\alpha) = \{\xi \in W; \xi < \alpha\}, \quad W[\alpha, \beta] = \{\xi \in W; \alpha \leq \xi \leq \beta\},$$

the symbol $W$ in the brackets is the "absolute" class of all ordinals.

Now, an enumeration of $W$ is realized via the immediate successor map of a subset $M \subseteq W$

$$\text{suc}(M) = \min\{\xi \in W; M < \xi\} \ \text{ (hence } \text{suc}(\alpha) = \alpha + 1, \ \forall \alpha \in W).$$

(Here, $M < \xi$ means: $\lambda < \xi, \forall \lambda \in M$). It begins with the natural numbers $0, 1, ...$; the set of all these is denoted by $N$. Their immediate successor is $\omega = \text{suc}(N)$ (the first transfinite ordinal); the next in this enumeration is $\omega+1$, and so on. Put $W_0 = W \setminus \{0\} (= \{\xi \in W; \xi > 0\})$. This set is composed of two disjoint classes of ordinals. The former of these, $W_0^1$, collects all *first kind* ordinals $\xi > 0$ [in the sense: $W(\xi)$ admits a last element $\max[W(\xi)] = \xi - 1$]. And the latter of these, $W_0^2$, collects all *second kind* ordinals $\xi > 0$

[in the sense: $W(\xi)$ does not admit a last element; or, equivalently: $\lambda < \xi \Longrightarrow \lambda+1 < \xi$]; this is also referred to as $\xi > 0$ being a *limit* ordinal.

The basic operations with ordinals may be introduced in a synthetic way as follows. Let $\alpha, \beta$ be two ordinals; and $(A, \leq)$, $(B, \leq)$, well ordered structures with $\text{ord}(A, \leq) = \alpha$, $\text{ord}(B, \leq) = \beta$, $A \cap B = \emptyset$. Then *a)* $\alpha + \beta = \text{ord}(A \cup B, \leq)$, where the associated order is given by the concatenation procedure: $x < y$ iff either $[x \in A, y \in B]$ or $[(x, y \in A, x < y)$, $(x, y \in B, x < y)]$; *b)* $\alpha \cdot \beta = \text{ord}(A \times B, \leq)$, where the associated order is defined by the lexicographic procedure: $(x, y) < (u, v)$ iff either $[y < v]$ or $[x < u, y = v]$. The basic properties of these may be found, e.g., in Kuratowski and Mostowski [op. cit., Ch 7, Sect 5].

In parallel to this, we may (construct and) enumerate the class of all admissible cardinals. Let $P$ and $Q$ be nonempty sets; we put

$$P \preceq Q (P \sim Q) \text{ iff there exists an injection (bijection):} P \to Q.$$

The former is a quasi-order (denoted $\text{card}(P) \leq \text{card}(Q)$); while the latter is an equivalence (written as $\text{card}(P) = \text{card}(Q)$). Denote also

$$P \prec Q \text{ if and only if } P \preceq Q \text{ and } \neg(P \sim Q).$$

This relation is *irreflexive* ($\neg(P \prec P)$, for each $P$) and *transitive*; hence a *strict order* (indicated as $\text{card}(P) < \text{card}(Q)$. Let $\alpha > 0$ be an (admissible) ordinal; we say that it is an (admissible) *cardinal* if

$$W(\xi) \prec W(\alpha), \quad \text{for each } \xi < \alpha.$$

The class of all these will be denoted by $Z$. Now, the enumeration we are looking for is realized via the immediate successor (in $Z$) map

$$\text{SUC}(M) = \min\{\eta \in Z; M < \eta\}, \quad M \subseteq Z.$$

Precisely, this begins with the natural numbers $0, 1, \ldots$. The immediate successor (in $Z$) of all these is $\omega = \text{SUC}(N)$ (the first transfinite cardinal). To describe the remaining ones, we may introduce via transfinite recursion the function $\lambda \vdash \aleph_\lambda$ from $W$ to $Z$ as

$$\begin{aligned}
&\aleph_0 = \omega; \quad \text{and, for each } \lambda > 0,\\
&\aleph_\lambda = \text{SUC}(\aleph_{\lambda-1}), \quad \text{if } \lambda - 1 \text{ exists}\\
&\aleph_\lambda = \text{SUC}\{\aleph_\xi; \xi < \lambda\}, \quad \text{if } \lambda - 1 \text{ does not exist.}
\end{aligned}$$

Note that, in such a case, the order structure of $Z(\omega, \leq) = \{\xi \in Z; \omega \leq \xi\}$ is completely reducible to the one of $W$; further details may be found in Just and Weese [13, Ch 11, Sect 11.2].

Any nonempty part $P$ with $\text{card}(P) < \omega$ ($\text{card}(P) = \omega$) is termed *finite* (*effectively countable*); the union of these ($\text{card}(P) \leq \omega$) is referred to as $P$ is *countable.* When $P = W(\xi)$, all such properties will be transferred to $\xi$. Generally, take some ordinal $\gamma$; and put $\Gamma = \aleph_\gamma$. Any nonempty part $P$ with $\text{card}(P) < \Gamma$ ($\text{card}(P) = \Gamma$) is termed $\Gamma$-*finite* (*effectively* $\Gamma$-*countable*); the union of these ($\text{card}(P) \leq \Gamma$) is referred to as $P$ is $\Gamma$-*countable.* As before, when $P = W(\xi)$, all such properties will be transferred to $\xi$.

Denote by $\Delta$ the immediate successor (in $Z$) of $\Gamma$ [$\Delta = \text{SUC}(\Gamma)$; hence $\Delta = \aleph_{\gamma+1}$, if $\Gamma = \aleph_\gamma$]. By the very definition above, one has

$$\xi \text{ is } \Gamma\text{-countable, for each } \xi < \Delta; \quad \text{but } \Delta \text{ is not } \Gamma\text{-countable.} \tag{2.1}$$

A basic consequence of this is precised in the statement below (to be found, e.g., in Kuratowski and Mostowski [op. cit., Ch 3, Sect 4]):

**Proposition 1.** *The following are valid:*

**i)** *The ordinal* $\Delta$ *cannot be attained via* $\theta$*-net limits (where* $\omega \leq \theta \leq \Gamma$*) of* $\Gamma$*-countable ordinals. In other words: if* $(\alpha_\xi; \xi < \theta)$ *is an ascending* $\theta$*-net (where* $\omega \leq \theta \leq \Gamma$*) of* $\Gamma$*-countable ordinals then*

$$\alpha = \sup_\xi(\alpha_\xi)(= \lim_\xi(\alpha_\xi)) \tag{2.2}$$

*is* $\Gamma$*-countable too.*

**ii)** *Each second kind* $\Gamma$*-countable ordinal is attainable via such nets. In other words: if* $\alpha < \Delta$ *is of second kind then, there exists a strictly ascending* $\theta$*-net (where* $\omega \leq \theta \leq \Gamma$*)* $(\alpha_\xi; \xi < \theta)$ *of* $\Gamma$*-countable ordinals with the property (2.2).*

**(B)** Let $M$ be a nonempty set; and $(\leq)$, some *order* (=antisymmetric quasi-order) on it. By a $(\leq)$-*chain* of $M$ we shall mean any (nonempty) part $A$ of $M$ with $(A, \leq)$ being well ordered (see above). Note that any such object may be written as $A = \{a_\xi; \xi < \lambda\}$, where the net $\xi \vdash a_\xi$ is strictly ascending ($\xi < \eta \Longrightarrow a_\xi < a_\eta$); the uniquely determined ordinal $\lambda$ is just $\mathrm{ord}(A, \leq)$. Let also $\gamma \geq 0$ be arbitrary fixed; and put $\Gamma = \aleph_\gamma$, $\Delta = \mathrm{SUC}(\Gamma)$ (hence $\Delta = \aleph_{\gamma+1}$). Note that, by the very definition above (and (2.3))

$$A \text{ is } \Gamma\text{-countable} \Longleftrightarrow \mathrm{ord}(A, \leq) < \Delta.$$

If, moreover, $\mathrm{ord}(A, \leq) \leq \Gamma$, we say that $A$ is *normally countable.* The following characterization of this concept is useful for us.

**Proposition 2.** *The* $(\leq)$*-chain* $A$ *is normally* $\Gamma$*-countable if and only if*

$$\begin{array}{l} A = \{b(\xi); \xi < \theta\}, \textit{ where } \theta \in W[\omega, \Gamma] \textit{ and} \\ \xi \vdash b(\xi) \textit{ is ascending } (\xi < \eta \Longrightarrow b(\xi) \leq b(\eta)). \end{array} \tag{2.3}$$

**Proof.** There are two steps to be passed.

*i)* Assume that $A$ is normally $\Gamma$-countable. By definition, we have

$$A = \{a_\xi; \xi < \lambda\}, \text{ where } \lambda \leq \Gamma \text{ and } \xi \vdash a_\xi \text{ is strictly ascending.}$$

If $\lambda$ is finite, then (2.3) is accessible via $\theta = \omega$ and $[b(\xi) = a_\xi, \xi < \lambda; b(\xi) = a_{\lambda-1}, \xi \geq \lambda]$. And, if $\lambda$ is infinite, we just take $\theta = \lambda$ and $[b(\xi) = a_\xi, \xi < \theta]$, so as to get (2.3).

*ii)* Assume $A$ is as in (2.3); and put $E = \{\min(b^{-1}(x)); x \in A\}$. The map $\xi \vdash b(\xi)$ is a strictly increasing bijection from $(E, \leq)$ to $(A, \leq)$; wherefrom $\mathrm{ord}(E, \leq) = \mathrm{ord}(A, \leq)$. On the other hand,

$$\mathrm{ord}(E, \leq) \leq \mathrm{ord}(W(\theta), \leq) = \theta \leq \Gamma; \text{ because } E \subseteq W(\theta)$$

(see, e.g., Sierpinski [op. cit., Ch 13, Sect 5]). And this, in conjunction with a preceding relation, shows that $A$ is normally $\Gamma$-countable. ■

Let $P, Q$ be nonempty parts with $P \supseteq Q$. We say that $P$ is *majorized* by $Q$ (and write $P \propto Q$) provided

$$Q \text{ is cofinal in } P \ (\forall x \in P, \exists y \in Q \text{ with } x \leq y).$$

The $(\leq)$-chain $S \subseteq M$ is called *upper* $\Gamma$-*countable* in case

$$S \propto T, \text{ for some normally } \Gamma\text{-countable } (\leq)\text{-chain } T \subseteq S. \tag{2.4}$$

Clearly, this happens if $S$ is normally $\Gamma$-countable. As a completion, we have

**Proposition 3.** *The generic relation holds*

$$(\forall (\leq)\text{-chain}) \ \Gamma\text{-countable} \Longrightarrow \text{upper } \Gamma\text{-countable}. \tag{2.5}$$

*Hence, the* $(\leq)$*-chain* $S \subseteq M$ *is upper* $\Gamma$*-countable if and only if*

$$S \propto T, \text{ for some } \Gamma\text{-countable } (\leq)\text{-chain } T \subseteq S. \tag{2.6}$$

**Proof.** Let $S = \{s(\xi); \xi < \lambda\}$ be the representation of this $(\leq)$-chain where $\lambda := \mathrm{ord}\,(S, \leq) < \Delta$. If $\lambda$ is a first kind ordinal, we are done; because $T = \{s(\lambda - 1)\}$ is then cofinal in $S$. Assume now $\lambda$ is a second kind ordinal. By Proposition 1 there exists a strictly ascending $\theta$-net (where $\omega \leq \theta \leq \Gamma$) $(\lambda_\xi; \xi < \theta)$ with $\lambda = \sup_\xi(\lambda_\xi)$. But then, $T = \{s(\lambda_\xi); \xi < \theta\}$ is a normally $\Gamma$-countable $(\leq)$-chain (of $S$) cofinal in $S$; i.e., we are again done. ■

**Remark.** The reciprocal of (2.5) is not in general true; just take any $(\leq)$-chain $S$ of $M$ with $\Delta \leq \mathrm{ord}(S, \leq)=$ first kind ordinal.

**(C)** Let us now return to our initial setting. We say that the order structure $(M, \leq)$ is $\Gamma$-*separable* if

$$\text{any } (\leq)\text{-chain of } M \text{ is upper } \Gamma\text{-countable}. \tag{2.7}$$

For example, this holds (under (2.5)) whenever

$$(M, \leq) \text{ is } \textit{strongly } \Gamma\textit{-separable}\text{: any } (\leq)\text{-chain of } M \text{ is } \Gamma\text{-countable}. \tag{2.8}$$

In fact, the reciprocal holds too; so that, we may formulate

**Proposition 4.** *Under these conventions,*

$$(\forall (M, \leq)\text{=ordered structure}) \ \Gamma\text{-separable} \Longleftrightarrow \text{strongly } \Gamma\text{-separable}. \tag{2.9}$$

**Proof.** Assume that $(M, \leq)$ is $\Gamma$-separable; and let $S = \{s(\xi); \xi < \lambda\}$ be some $(\leq)$-chain of $M$; where $\lambda := \mathrm{ord}(S, \leq)$. If, by absurd, $S$ is not $\Gamma$-countable, we must have $\lambda \geq \Delta$. The initial segment (of $S$) $U = \{s(\xi); \xi < \Delta\}$ is not $\Gamma$-countable too; cf. (2.1). On the other hand, by hypothesis, $U$ is upper $\Gamma$-countable; so, there exists a strictly ascending $\theta$-net (where $\omega \leq \theta \leq \Gamma$) $(\xi_\nu; \nu < \theta)$ of ranks in $W(\Delta)$ with

$$U \propto \{s(\xi_\nu); \nu < \theta\}; \quad \text{hence } \Delta = \lim_\nu(\xi_\nu).$$

This, however, cannot be accepted, in view of Proposition 1. Hence, $S$ is $\Gamma$-countable; and the proof is complete. ■

In the following, we shall give some useful examples of such structures.

*c1)* Let $\mathcal{I}(M) := \{(x, x); x \in M\}$ stand for the identical relation over $M$. By an *almost uniformity* (on $M$) we shall mean any family $\mathcal{U}$ of parts in $M \times M$ with

$$\mathcal{I}(M) \subseteq U, \text{ for each } U \in \mathcal{U} \quad (\text{i.e.: } \mathcal{I}(M) \subseteq \cap\mathcal{U}).$$

Suppose that we fixed such an object. Call the (ascending) net $(a_\xi;\xi<\lambda)$, *$\mathcal{U}$-Cauchy*, when

$$\forall U\in\mathcal{U},\exists\mu=\mu(U),\text{ such that }\mu\le\xi\le\eta\Longrightarrow(a_\xi,a_\eta)\in U.$$

Likewise, call the (ascending) sequence $(b_n;n<\omega)$, *$\mathcal{U}$-asymptotic*, in case

$$\forall U\in\mathcal{U},\exists k=k(U),\text{ such that }n\ge k\Longrightarrow(b_n,b_{n+1})\in U.$$

It is not hard to see that the global conditions below are equivalent

$$\text{each ascending net is }\mathcal{U}\text{-Cauchy}\tag{2.10}$$

$$\text{each ascending sequence is }\mathcal{U}\text{-asymptotic.}\tag{2.11}$$

By definition, either of these will be referred to as $\mathcal{U}$ is (strongly) *regular.*

**Lemma 1.** *Assume the almost uniformity $\mathcal{U}$ is (strongly) regular and*

$$\begin{aligned}&\mathcal{U}\textit{ is }\Gamma\textit{-pseudometrizable: there exists a }\Gamma\textit{-countable subfamily}\\&\mathcal{V}\subseteq\mathcal{U},\textit{ cofinal in }(\mathcal{U},\supseteq)\ (\forall U\in\mathcal{U},\exists V\in\mathcal{V}:\ U\supseteq V)\end{aligned}\tag{2.12}$$

$$\mathcal{U}\textit{ is sufficient}\quad(\cap\mathcal{U}=\mathcal{I}(M)).\tag{2.13}$$

*Then, $(M,\le)$ is (strongly) $\Gamma$-separable.*

**Proof.** Without loss, one may assume $\mathcal{U}$ itself is $\Gamma$-countable; i.e., written as a $\theta$-net $(U_\xi;\xi<\theta)$, where $\theta<\Delta$. (Otherwise, we simply replace $\mathcal{U}$ by $\mathcal{V}$). The case of $\theta$ being finite is clear; so, it remains to discuss the alternative $\theta\ge\omega$. Let $S$ be some $(\le)$-chain in $M$. If there exists a last element $s=\max(S)$, we are done; so, without restriction, one may assume that

$$\text{for each }x\in S\text{ there exists }y\in S\text{ with }x<y.\tag{2.14}$$

This, and the (strong) regularity of $\mathcal{U}$, yields (cf. Turinici [22])

$$\begin{aligned}&\forall x\in S,\forall U\in\mathcal{U},\text{ there exists }y=y(x,U)\in S(x,<)\\&\text{such that, for each }p,q\in S:\ y\le p\le q\Longrightarrow(p,q)\in U.\end{aligned}\tag{2.15}$$

[Here, for each $P\subseteq M$ and each relation $(*)$ over $M$ we put

$$P(a,*)=\{x\in P;a*x\},\ a\in P\quad\text{(the }a\text{-}\textit{section}\text{ of }(*)\text{ in }P)].$$

Fix $a\in S$. By (2.15), there exists (in $S$) $a_0>a$ with $[p,q\in S,\ a_0\le p\le q\Longrightarrow(p,q)\in U_0]$. Further, again by this relation, there exists (in $S$) $a_1>a_0$ with $[p,q\in S,\ a_1\le p\le q\Longrightarrow(p,q)\in U_1]$; and so on. Generally, assume that, for $\mu<\theta$ we constructed a net $(a_\xi;\xi<\mu)$ in $S$ with the property that, for each $\lambda<\mu$,

$$\xi<\lambda\quad\text{implies }a_\xi<a_\lambda\tag{$A(\lambda)$}$$

$$p,q\in S,\quad a_\lambda\le p\le q\Longrightarrow(p,q)\in U_\lambda.\tag{$B(\lambda)$}$$

Two possibilities may occur.

*j)* $\mu$ is a first kind ordinal: $\lambda=\mu-1$ exists. Again by (2.15), there exists (in $S$) $t>a_\lambda$ with $[p,q\in S,t\le p\le q\Longrightarrow(p,q)\in U_\mu]$. It will suffice taking $a_\mu=t$ so that $(A(\mu))$ and $(B(\mu))$ be fulfilled.

*jj)* $\mu$ is a second kind ordinal: $\mu - 1$ does not exist. By the choice of $\theta$, the $(\leq)$-chain (in $S$) $T = \{a_\xi; \xi < \mu\}$ is $\Gamma$-countable. If $T$ is cofinal in $S$, we are done; because (cf. Proposition 3) $S$ is upper $\Gamma$-countable. Otherwise,

$$a_\xi < s, \quad \text{for all } \xi < \mu \text{ and some } s \in S. \tag{$C(\mu)$}$$

Again by (2.15), there exists (in $S$) $t > s$ with $[p, q \in S, t \leq p \leq q \Longrightarrow (p, q) \in U_\mu]$. It will suffice then putting $a_\mu = t$ so that $(A(\mu))$ and $(B(\mu))$ be fulfilled. As a consequence, the process above either stops at a certain stage $\mu < \theta$ (and then, we are done); or else (in the opposite situation) it is continuable over all of $W(\theta)$; i.e., $(A(\lambda))$ and $(B(\lambda))$ hold, for each $\lambda < \theta$. We claim that, in such a case, $T = \{a_\xi; \xi < \theta\}$ is cofinal in $S$; and this, combined with the $\Gamma$-countable property of the same, completes the argument. Assume not; i.e., $(C(\theta))$ is true. By (2.14), there exists $t \in S$ with $t > s$; hence $t \neq s$. On the other hand, by the choice of $(a_\xi; \xi < \theta)$, one has

$$(s, t) \in U_\xi, \quad \text{for all } \xi < \theta\text{; hence } s = t \text{ (by (2.13))}.$$

The obtained facts involving $(s, t)$ are contradictory; hence, $(C(\theta))$ cannot hold; and our claim follows. ■

A basic particular construction of this type may be described along the following lines. By a *pseudometric* over $M$ we mean any map $d : M \times M \to R_+$ with $d(x, x) = 0$, $\forall x \in M$. Let $D = (d_\lambda; \lambda < \alpha)$ be a family of such objects; where $\alpha \geq \omega$. Then $\mathcal{U}(D) = \{U(\lambda, r); \lambda < \alpha, r > 0\}$, where

$$U(\lambda, r) = \{(x, y) \in M \times M; d_\lambda(x, y) < r\}, \quad \lambda < \alpha, r > 0,$$

is an almost uniformity over $M$. The sufficiency condition (2.13) for this object is characterized as

$$D \text{ is sufficient:} \quad [d_\lambda(x, y) = 0, \forall \lambda < \alpha] \Longrightarrow x = y.$$

On the other hand, the subfamily $\mathcal{V} = \{U(\lambda, 2^{-n}); \lambda < \alpha, n < \omega\}$ is cofinal in $(\mathcal{U}, \supseteq)$; and this, in conjunction with

$$\mathrm{card}(W(\alpha) \times W(\omega)) = \mathrm{card}(W(\alpha)) \text{ (cf. Alexandrov [1, Ch 3, Sect 6])}$$

shows that condition (2.12) holds with $\Gamma = \mathrm{card}(W(\alpha))$. A translation of Lemma 1 in terms of $D = (d_\lambda; \lambda < \alpha)$ is immediate; we do not give details.

*c2)* By a *topology* over $M$ we mean any family $\mathcal{T} \supseteq \{\emptyset, M\}$ of parts in $M$, invariant to arbitrary unions and finite intersections. Assume that we fixed such an object; and let "cl" stand for the associated *closure* operator. Any subfamily $\mathcal{B} \subseteq \mathcal{T}$ with the property that each $D \in \mathcal{T}$ is a union of members in $\mathcal{B}$, will be referred to as a *basis* for $\mathcal{T}$. If, in addition, $\mathcal{B}$ is $\Gamma$-countable, then $\mathcal{T}$ will be called *second* $\Gamma$*-countable.* Finally, term the ambient order $(\leq)$, *closed from the left* provided $M(x, \geq)$ is closed, for each $x \in M$.

**Lemma 2.** *Assume that $\mathcal{T}$ is second $\Gamma$-countable and $(\leq)$ is closed from the left. Then, $(M, \leq)$ is (strongly) $\Gamma$-separable.*

**Proof.** Let $\mathcal{B} = \{B_\xi; \xi < \theta\}$ (where $\omega \leq \theta < \Delta$) stand for a $\Gamma$-countable basis of $\mathcal{T}$. Further, take some choice function "Ch" of the nonempty parts in $M$ [$\mathrm{Ch}(X) \in X$, for each $X \subseteq M, X \neq \emptyset$]. Given the arbitrary fixed $(\leq)$-chain $S$ of $M$, denote $T =$

$\{\mathrm{Ch}(B \cap S); B \in \mathcal{B}\}$ (hence $T \subseteq S$). For the moment, $T$ is $\Gamma$-countable (because $T \preceq \mathcal{B}$). In addition, we claim that $\mathrm{cl}(T) \supseteq S$ [wherefrom, $T$ is dense in $S$]. In fact, let $s$ be some point of $S$; and $U$ stand for an open neighborhood of it. By the definition of $\mathcal{B}$, $U$ may be written as a union of members in this family; so

$$U \supseteq B \ni s \text{ (hence } U \ni \mathrm{Ch}(B \cap S)), \quad \text{for some } B \in \mathcal{B};$$

and our claim follows. If $T$ is cofinal in $S$, we are done (cf. Proposition 3). Otherwise, there must be some $s \in S$ with $T \subseteq M(s, \geq)$; wherefrom

$$S \subseteq \mathrm{cl}(T) \subseteq \mathrm{cl}(M(s, \geq)) = M(s, \geq);$$

i.e., $\{s\}$ is cofinal in $S$. The proof is thereby complete. ■

It remains to establish under which conditions is $\mathcal{T}$, second $\Gamma$-countable. An appropriate answer is to be given in a uniform context:

there exists an (up-directed) family $D = (d_\lambda; \lambda < \alpha)$ of
semimetrics (over $M$) whose associated topology is just $\mathcal{T}$.

Then, e.g., the condition below yields the desired property for $\mathcal{T}$ [where $\Gamma = \mathrm{card}(W(\alpha))$]:

$$M \text{ has a } \Gamma\text{-countable dense subset.} \tag{2.16}$$

The proof is very similar to the one in Bourbaki [5, Ch 9, Sect 2.8]; see also Alexandrov [op. cit., Ch 4, Sect 4]. Finally, note that, when $\Gamma = \omega$, the developments above reduce to the ones in Turinici [23]. Some related aspects may be found in Zhu, Fan and Zhang [25].

## 3 Zorn-Bourbaki Principles

**(A)** Let $M$ be a nonempty set; and $(\leq)$, some *order* (antisymmetric quasi-order) on it. Call the point $z \in M$, $(\leq)$-*maximal* in case

$$w \in M, z \leq w \Longrightarrow z = w; \text{ i.e.: } z < x \text{ is false, for each } x \in M. \tag{3.1}$$

(Here, $(<)$ is the *strict order* attached to $(\leq)$). Sufficient conditions for the existence of such elements may be obtained as follows. Call the (nonempty) part $A$ of $M$, a *linear* $(\leq)$-*chain* provided $(A, \leq)$ is linearly ordered [$\forall x, y \in A$: either $x \leq y$ or $y \leq x$]; and a *(natural)* $(\leq)$-*chain*, when $(A, \leq)$ is well ordered [cf. Section 2].

**Theorem ZB.** *Suppose that one of the conditions below holds*

$$\textit{each linear } (\leq)\textit{-chain (of } M\textit{) is bounded above} \tag{3.2}$$

$$\textit{each } (\leq)\textit{-chain (of } M\textit{) is bounded above.} \tag{3.3}$$

*Then, $(\leq)$ is a normal order, in the sense: for each $u \in M$ there exists a $(\leq)$-maximal $v \in M$ with $u \leq v$.*

Some remarks are in order. The first explicit formulation of Theorem ZB in terms of (3.2) was given in 1914 by Hausdorff [12, Ch 6, Sect 1]; a slight different version of it was obtained in 1922 by Kuratowski [16]. Note that the quoted authors regarded Theorem

ZB only as a handy tool in solving various existence problems in the setting of (AC)(= the Axiom of Choice). Finally, again under the lines of (3.2), we must mention the 1935 contribution due to Zorn [26]; who regarded Theorem ZB as an axiom. The version of this result involving (3.3) was stated in Bourbaki [4]; who also established its equivalence with the Well Ordering Principle in Zermelo [24] (equivalent with (AC)). For this reason, it is natural that Theorem ZB be referred to as the Zorn-Bourbaki principle. Note that, in the context of (AC),

$$(3.3) \Longrightarrow (3.2) \quad (\text{hence } (3.3) \Longleftrightarrow (3.2));$$

see also Felgner [9]. Further historical aspects may be found in Moore [18, Ch 4, Sect 4] and the references therein.

**(B)** Now, as results from the developments in Section 2, the verification of (3.3) for (cardinal-) countable chains only will suffice (for its validity) in many concrete cases with a practical relevance. This suggests us considering maximality principles over ordered structures with such regularity properties. So, let $(M, \leq)$ be a (partially) ordered set. Further, take some $\gamma \geq 0$ and put $\Gamma = \aleph_\gamma$, $\Delta = \mathrm{SUC}(\Gamma)$ (hence $\Delta = \aleph_{\gamma+1}$). Assume firstly that

$$\begin{array}{l}(M, \leq) \text{ is sequentially } \Gamma\text{-inductive: each normally } \Gamma\text{-countable} \\ (\leq)\text{-chain of } M \text{ is bounded from above (modulo } (\leq)).\end{array} \tag{3.4}$$

Note that, by Proposition 2, this notion is identical with the one of (1.2) when $\Gamma = \omega$. Moreover, by Proposition 3, it may be also written as

$$\text{each } \Gamma\text{-countable } (\leq)\text{-chain of } M \text{ is bounded above (modulo } (\leq)). \tag{3.5}$$

Secondly, assume that (cf. Section 2)

$$\begin{array}{l}(M, \leq) \text{ is (strongly) } \Gamma\text{-separable: each } (\leq)\text{-chain } S \subseteq M \\ \text{is majorized by some } \Gamma\text{-countable } (\leq)\text{-chain } T \subseteq S.\end{array} \tag{3.6}$$

Remember that, by Proposition 4, this also reads

$$\text{each } (\leq)\text{-chain of } M \text{ is } \Gamma\text{-countable.} \tag{3.7}$$

**Theorem 1.** *Assume that (3.4)+(3.6) hold. Then, $(\leq)$ is a normal order (in the sense above).*

**Proof.** By the remarks involving (3.5)+(3.7), it is clear that Theorem ZB applies to these data; and, from this, we are done. ■

For the moment, Theorem 1 is deductible from Theorem ZB. The reverse inclusion is also true; just take $\Gamma = \mathrm{card}(M)$ in the above statement. Hence

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem ZB (from a logical viewpoint).} \tag{3.8}$$

In other words: the enlargement of Theorem ZB assured by Theorem 1 is technical in nature.

Now, remember that the regularity conditions in Theorem ZB are logically minimal so that its conclusion be retainable. (See the quoted papers for details). So, it is natural to ask whether this is also true for the conditions in Theorem 1. Two situations may occur.

*i)* Assume that in Theorem 1 condition (3.4) does not hold. By definition, there exists a normally $\Gamma$-countable $(\leq)$-chain $K$ of $M$ which is not bounded above (in $M$). As a consequence, $(K, \leq)$ is not sequentially $\Gamma$-inductive; but it is (strongly) $\Gamma$-separable. This, added to $(K, \leq)$ having no $(\leq)$-maximal elements, proves the logical minimality of (3.4).

*ii)* Assume that, in Theorem 1, condition (3.6) does not hold. By definition, there must be a $(\leq)$-chain $L \subseteq M$ with (cf. Proposition 3)

$$L \propto Q \text{ is false, for each } \Gamma\text{-countable } (\leq)\text{-chain } Q \subseteq L.$$

As a consequence, the structure $(L, \leq)$ is sequentially $\Gamma$-inductive; but not (strongly) $\Gamma$-separable. This, added to $(L, \leq)$ having no $(\leq)$-maximal elements, proves the logical minimality of (3.6).

Summing up, we proved

**Proposition 5.** *Either of the regularity conditions (3.4) and (3.6) in Theorem 1 is logically minimal for the conclusions given there to hold.*

In particular, when $\Gamma = \omega$, Theorem 1 (and Proposition 5 as well) is just the statement in Turinici [23] proved via similar techniques.

# 4 Main Results

With these informations at hand, we may now return to the questions in Section 1. The natural setting for discussing them is the one of *transitive* relations. This, apart from giving us new useful forms of Theorem BB, allows a direct approach to the quasi-order and amorph cases.

**(A)** Let $(M, \nabla)$ and $(P, \nabla)$ be transitive structures. The relation over $M$

$$(x, y \in M) \;\; x < y \text{ iff } x \nabla y \text{ and } \neg(y \nabla x) \tag{4.1}$$

is *irreflexive* ($\neg(x < x), \forall x \in M$) and *transitive*; hence a *strict order*. As a consequence, its completion

$$(x, y \in M) \;\; x \overline{<} y \text{ iff either } x < y \text{ or } x = y \tag{4.2}$$

is an order on $M$. Denote in the same way the strict/standard order on $P$ attached to $(\nabla)$. Further, let $\varphi : M \to P$ be some $(\nabla, \nabla)$-increasing function

$$x \nabla y \Longrightarrow \varphi(x) \nabla \varphi(y) \text{ [equivalently: } \neg(\varphi(x) \nabla \varphi(y)) \Longrightarrow \neg(x \nabla y)]. \tag{4.3}$$

This allows us introducing the relation (in $M$)

$$(x, y \in M) \; x \sqsubset y \text{ iff } x \nabla y \text{ and } \neg(\varphi(y) \nabla \varphi(x)). \tag{4.4}$$

By the remark involving (4.3), one has

$$x \sqsubset y \text{ iff } x < y \text{ and } \varphi(x) < \varphi(y); \tag{4.5}$$

wherefrom, $(\sqsubset)$ is a strict order on $M$. Let $(\sqsubseteq)$ stand for the associated (by (4.2)) order in $M$. Note that, by the very definitions (and remarks) above

$$[(x \sqsubset y, y\nabla z) \text{ or } (x\nabla y, y \sqsubset z)] \text{ imply } x \sqsubset z. \tag{4.6}$$

In fact, assume e.g. that the former of these alternatives holds. As $x \sqsubset y \Longrightarrow x\nabla y$, one gets for the moment $x\nabla z$. If, by absurd, $\varphi(z)\nabla\varphi(x)$ then, combining with $\varphi(y)\nabla\varphi(z)$ (deductible via (4.3) and $y\nabla z$) gives $\varphi(y)\nabla\varphi(x)$; in contradiction with $x \sqsubset y$. Hence, $\neg(\varphi(z)\nabla\varphi(x))$; wherefrom, $x \sqsubset z$. The latter of these alternatives is handled in a similar way; and the claim follows.

Having these precised, call the point $z \in M$, $(\nabla, \nabla; \varphi)$-*maximal*, when

$$\text{for each } w \in M\text{: } z\nabla w \text{ implies } \varphi(w)\nabla\varphi(z). \tag{4.7}$$

Note that, if $(M, \nabla)$ is identical with $(P, \nabla)$, and $(\nabla)$ is an order (on $M$) this concept reduces to the one in Section 3 (when $\varphi$=identity). Hence, maximality results of this type are not without interest for us. The basic step in deducing these is a characterization of our concept in terms of $(\sqsubseteq)$.

**Lemma 3.** *The generic relation is available*

$$(\forall z \in M)\text{: } \quad (\nabla, \nabla; \varphi)\text{-maximal} \Longleftrightarrow (\sqsubseteq)\text{-maximal}. \tag{4.8}$$

**Proof.** It will suffice verifying that

$$z \text{ is not } (\nabla, \nabla; \varphi)\text{-maximal} \Longleftrightarrow z \text{ is not } (\sqsubseteq)\text{-maximal}.$$

The left part of this equivalence means

$$\exists w \in M \text{ such that: } z\nabla w, \neg(\varphi(w)\nabla\varphi(z)); \text{ hence } z \sqsubset w.$$

And, from this, the claim follows. ■

Now, as $(\sqsubseteq)$ is an order, the developments of Section 3 apply to $(M, \sqsubseteq)$; and this yields the following maximality principle to be used further. Take a certain ordinal $\gamma$ and put $\Gamma = \aleph_\gamma$, $\Delta = \mathrm{SUC}(\Gamma)$ (hence $\Delta = \aleph_{\gamma+1}$).

**Theorem 2.** *Assume that*

$$(M, \sqsubseteq) \text{ is sequentially inductive (cf. (3.4))} \tag{4.9}$$

$$(M, \sqsubseteq) \text{ is (strongly) separable (cf. (3.6)).} \tag{4.10}$$

*Then, for each $u \in M$ there exists $v \in M$ with*

$$v \text{ is } (\nabla, \nabla : \varphi)\text{-maximal (cf. (4.7))} \tag{4.11}$$

*in such a way that*

$$u = v \text{ (hence } u \text{ is } (\nabla, \nabla; \varphi)\text{-maximal), whenever } M(u, \nabla) = \emptyset \tag{4.12}$$

$$u\nabla v, \quad \text{whenever } M(u, \nabla) \neq \emptyset. \tag{4.13}$$

**Proof.** By Theorem 1 (applicable, via (4.9)+(4.10)) it follows that, for each $u \in M$ there exists $v \in M$ with

$$u \sqsubseteq v \text{ (i.e.: } u \sqsubset v \text{ or } u = v); \quad \text{and } v \text{ is } (\sqsubseteq)\text{-maximal}. \tag{4.14}$$

The latter of these yields (4.11), if one takes Lemma 3 into account. And the former of these gives the couple of alternatives (4.12)/(4.13). In fact, $M(u, \nabla) = \emptyset$ implies (4.12), in view of $[u \sqsubset v \Longrightarrow u \nabla v]$. Moreover, $M(u, \nabla) \neq \emptyset$ gives (4.13); for, in such a case, (4.14) holds with some $u_1 \in M(u, \nabla)$ in place of $u$. The proof is thereby complete. ■

It remains now to give sufficient conditions (involving our initial data) under which (4.9)+(4.10) be fulfilled. This necessitates further conventions and auxiliary facts. Let the ordinal $\lambda > 0$ be arbitrary fixed. We say that the $\lambda$-net $(b_\xi; \xi < \lambda)$ is *ascending* (modulo $(\nabla)$) when $[\xi < \eta \Longrightarrow b_\xi \nabla b_\eta]$. Given such an object, call $u \in M$ an *upper bound* (modulo $(\nabla)$) of it when

$$b_\xi \nabla u, \ \text{ for all } \xi < \lambda \ \text{ (written as: } (b_\xi; \xi < \lambda) \nabla u).$$

If $u$ is generic in this convention, we say that $(b_\xi; \xi < \lambda)$ is *bounded from above* (modulo $(\nabla)$). Finally, we call $(M, \nabla)$, *sequentially* $\Gamma$*-inductive* provided

$$\begin{array}{l}\text{each ascending (modulo } (\nabla)) \ \theta\text{-net (where } \theta \in W_0^2[\omega, \Gamma]) \\ \text{is bounded from above (modulo } (\nabla)).\end{array}$$

**Lemma 4.** *Under these conventions, one has*

$$(M, \nabla) \textit{ is seq. } \Gamma\textit{-inductive} \Longrightarrow (M, \sqsubseteq) \textit{ is seq. } \Gamma\textit{-inductive.} \tag{4.15}$$

**Proof.** Assume that $(M, \nabla)$ is sequentially $\Gamma$-inductive; and let $K$ be some normally $\Gamma$-countable $(\sqsubseteq)$-chain of $M$. By definition, it may be represented as a $\theta$-net (with $\theta \in W_0[\Gamma]$) $K = \{a_\xi; \xi < \theta\}$; where $\xi \vdash a_\xi$ is strictly ascending (modulo $(\sqsubseteq)$). The case $\theta \in W_0^1$ is clear; so, without loss, one may assume $\theta \in W_0^2$ (hence $\theta \geq \omega$). By (4.4) (and the choice of $(a_\xi; \xi < \theta)$)

$$\xi < \eta \Longrightarrow a_\xi \sqsubset a_\eta \Longrightarrow a_\xi \nabla a_\eta. \tag{4.16}$$

This net is therefore ascending (modulo $(\nabla)$); wherefrom (by hypothesis) $(a_\xi; \xi < \theta) \nabla v$, for some $v \in M$. This, along with (4.6), gives (via (4.16)) $(a_\xi; \xi < \theta) \sqsubset v$; hence $(a_\xi; \xi < \theta) \sqsubseteq v$; and the conclusion follows. ■

We are now in position to get an appropriate answer to the posed question.

**Theorem 3.** *Suppose that*

$$(M, \nabla) \textit{ is sequentially } \Gamma\textit{-inductive} \tag{4.17}$$

$$(P, \overline{<}) \textit{ is (strongly) } \Gamma\textit{-separable.} \tag{4.18}$$

*Then, conclusions of Theorem 2 are retainable.*

**Proof.** By Lemma 4, condition (4.9) holds via (4.17). We claim that (4.10) holds too (from (4.18)); and this will complete the argument. Let $S$ be some $(\sqsubseteq)$-chain of $M$; and put $V = \varphi(S)$. Clearly,

$$V \text{ is a } (\overline{<})\text{-chain in } P; \quad \text{(cf. (4.5));}$$

so, in view of (4.18), $V$ is $\Gamma$-countable (in $P$). On the other hand, the same relation (4.5) shows that $\varphi$ is an order isomorphism between $(S, \sqsubseteq)$ and $(V, \overline{<})$; wherefrom, $S$ is countable too; and the claim follows. ■

**(B)** In particular, assume that the (transitive) relation $(\nabla)$ is a quasi-order $(\leq)$ in both $M$ and $P$. By Theorem 3 we then derive (under (4.3))

**Theorem 4.** *Assume (4.18) is true, as well as*

$$(M, \leq) \text{ is sequentially } \Gamma\text{-inductive.} \tag{4.19}$$

*Then, for each $u \in M$, there exists $v \in M$, with*

$$[u \leq v] \text{ and } \quad [v \leq w \Longrightarrow \varphi(w) \leq \varphi(v)]. \tag{4.20}$$

In particular, when $\Gamma = \omega$, this result reduces to the one in Turinici [23]. But, as shown in that paper, the precised statement includes the Brezis-Browder ordering principle [6] (Theorem BB) when $(P, \leq)$ is identical with $(R_+, \geq)$. Hence, so does Theorem 4; and, as such, it may be also referred to in this way. On the other hand, if $(M, \leq)$ is identical with $(P, \leq)$ and $\varphi$=the identity, Theorem 4 is just Theorem 1 (when $(\leq)$ is an order). Summing up,

$$\text{Th } 1 \Longrightarrow \text{Th } 3 \Longrightarrow \text{Th } 4 \Longrightarrow \text{Th } 1; \tag{4.21}$$

hence, all these are mutually equivalent. In particular, this also shows that Theorem 4 includes the "transfinite" version of Theorem BB obtained in Turinici [22]. The question of the reciprocal inclusion being also true remains open; we conjecture that the answer is positive.

**(C)** Let us return to our initial framework. The basic hypothesis used in all these developments is (4.3). So, the question arises of what can be said about such results when (4.3) is no longer available. To this end, put

$$(x, y \in M) \;\; x \bigtriangleup y \text{ iff } x \nabla y \text{ and } \varphi(x) \nabla \varphi(y). \tag{4.22}$$

This is a transitive relation over $M$; and condition (4.3) holds with $(\bigtriangleup, \nabla)$ in place of $(\nabla, \nabla)$. An application of Theorem 3 to these data yields an appropriate answer to the problem we deal with.

**Theorem 5.** *Assume that (4.18) holds, as well as*

$$(M, \bigtriangleup) \text{ is sequentially inductive.} \tag{4.23}$$

*Then, for each $u \in M$, there exists $v \in M$ with*

$$v \text{ is } (\bigtriangleup, \nabla; \varphi)\text{-maximal (cf. (4.7))} \tag{4.24}$$

*in such a way that*

$$u = v \text{ (hence } u \text{ is } (\bigtriangleup, \nabla; \varphi)\text{-maximal), whenever } M(u, \bigtriangleup) = \emptyset \tag{4.25}$$

$$u \bigtriangleup v, \quad \text{whenever } M(u, \bigtriangleup) \neq \emptyset. \tag{4.26}$$

A quasi-order version of this (under the lines of Theorem 4) is immediately obtainable; we do not give details. In particular, when $\Gamma = \omega$, this result is just the one in Turinici [op. cit.]; which, in turn, extends a related statement due to Kada, Suzuki and Takahashi [14]. Further aspects will be discussed in a separate paper.

## 5 Some Amorph Versions

A slight extension of these facts is to be reached when the relation $(\nabla)$ over $M$ is no longer transitive. Further aspects occasioned by the obtained results are then discussed.

**(A)** Let $(\perp)$ stand for an *amorph* relation over $M$. Denote by $(\nabla)$ the transitive relation (over the same) attached to $(\perp)$

$$\begin{array}{l}(x,y\in M)\ x\nabla y \text{ iff } x=u_1\perp ...\perp u_k=y \text{ (in the sense:}\\ u_i\perp u_{i+1}, \forall i\in\{1,...,k-1\}\text{), for some } k\ge 2 \text{ and } u_1,...,u_k\in M.\end{array} \tag{5.1}$$

Take a transitive relation $(\triangle)$ over $P$; as well as a function $\varphi : M\to P$ with

$$\varphi \text{ is } (\perp,\triangle)\text{-increasing: } \ x\perp y\Longrightarrow \varphi(x)\,\triangle\,\varphi(y). \tag{5.2}$$

Note that, under (5.1) above, one gets

$$\varphi \text{ is } (\nabla,\triangle)\text{-increasing (in the sense of (4.3)).}$$

Given $z\in M$, we say that it is $(\perp,\triangle;\varphi)$-*maximal*, if

$$\text{(for each } w\in M\text{):}\quad z\perp w\Longrightarrow \varphi(w)\,\triangle\,\varphi(z). \tag{5.3}$$

Again by (5.1), one gets the generic relation

$$\text{(for each } z\in M\text{): } (\nabla,\triangle;\varphi)\text{-maximal} \Longrightarrow (\perp,\triangle;\varphi)\text{-maximal.} \tag{5.4}$$

So, existence results involving such points are deductible from Theorem 3 above. The only aspect to be clarified is that of expressing (4.17) in terms of $(\perp)$. This will necessitate a lot of new conventions.

Let $\alpha>0$ be an ordinal. Remember that $\omega\cdot\alpha=\mathrm{ord}(W(\omega)\times W(\alpha),\le)$; where $(\le)$ stands for the lexicographic order (cf. Section 2). By an $(\omega,\alpha)$-*net*, we shall mean any map $(n,\xi)\vdash b(n,\xi)$ from $W(\omega)\times W(\alpha)$ to $M$. Given such an object, call it *ascending* (modulo $(\perp)$) when

$$b(n,\xi)\perp b(n+1,\xi),\quad \text{for all } n<\omega,\xi<\alpha. \tag{5.5}$$

Any $(\omega,\alpha)$-net $(a(n,\xi);n<\omega,\xi<\alpha)$, with

$$(a(n,\xi);n<\omega) \text{ is a subsequence of } (b(n,\xi);n<\omega)\text{, for each } \xi<\alpha \tag{5.6}$$

will be referred to as a *strong subnet* of $(b(n,\xi);n<\omega,\xi<\alpha)$. Further, call $u\in M$, an *upper bound* (modulo $(\perp)$) of $(b(n,\xi);n<\omega,\xi<\alpha)$, when

$$b(n,\xi)\perp u, \forall n<\omega,\forall\xi<\alpha \text{ (in short: } (b(n,\xi);n<\omega,\xi<\alpha)\perp u\text{)}. \tag{5.7}$$

If this holds only on a strong subnet of $(b(n,\xi);n<\omega,\xi<\alpha)$, we say that $u$ is an *asymptotic upper bound* (modulo $(\perp)$) of this net; written as: $(b(n,\xi);n<\omega,\xi<\alpha)\perp\!\!\perp u$. When $u\in M$ is generic in these conventions, the corresponding property will be referred to as: $(b(n,\xi);n<\omega,\xi<\alpha)$ is *bounded above* (respectively, *asymptotic bounded above*) modulo $(\perp)$. Finally, call the structure $(M,\perp)$, *sequentially* $\Gamma$-*inductive* if

$$\begin{array}{l}\text{each ascending (modulo } (\perp)\text{) } (\omega,\alpha)\text{-net (where } \omega\cdot\alpha\le\Gamma\text{)}\\ \text{is asymptotic bounded above (modulo } (\perp)\text{).}\end{array} \tag{5.8}$$

(Here, the couple of ordinals $[\Gamma = \aleph_\gamma, \Delta = \aleph_{\gamma+1}]$ is the one in Section 4).

The following auxiliary fact is useful for us.

**Lemma 5.** *Under these conventions,*

$$(M, \perp) \text{ is seq. } \Gamma\text{-inductive} \Longrightarrow (M, \nabla) \text{ is seq. } \Gamma\text{-inductive.} \tag{5.9}$$

**Proof.** Assume that $(M, \perp)$ is sequentially $\Gamma$-inductive; and let $(a_\xi; \xi < \theta)$ be some ascending (modulo $(\nabla)$) $\theta$-net (where $\theta \in W_0^2[\omega, \Gamma]$). We have

$$\theta = \omega \cdot \alpha, \quad \text{for some } \alpha > 0 \ \text{ (cf. Sierpinski [19, Ch 14, Sect 11]);}$$

so, this net may be written as a double indexed one $(a(n, \xi); n < \omega, \xi < \alpha)$ (with $\omega \cdot \alpha \leq \Gamma$). In addition, the ascending property for it reads

$$a(n, \xi) \nabla a(n + 1, \xi), \quad \text{for all } n < \omega, \xi < \alpha.$$

So, by (5.1), there exists another double indexed net $(b(n, \xi); n < \omega, \xi < \alpha)$ in such a way that (5.5) holds, as well as (5.6). The working hypothesis yields (via (5.5) and $\omega \cdot \alpha \leq \Gamma$)

$$(b(n, \xi); n < \omega, \xi < \alpha) \perp\perp u, \quad \text{for some } u \in M.$$

And this, in conjunction with (5.6), shows that

$$(a(n, \xi); n < \omega, \xi < \alpha) \nabla u; \quad \text{i.e., } (a_\xi; \xi < \theta) \nabla u.$$

Hence the conclusion. ■

Now, by simply adding this to Theorem 3, one gets

**Theorem 6.** *Assume (4.18) holds, as well as*

$$(M, \perp) \text{ is sequentially } \Gamma\text{-inductive.} \tag{5.10}$$

*Then, for each $u \in M$, there exists $v \in M$ with*

$$v \text{ is } (\perp, \triangle; \varphi)\text{-maximal (cf. (5.3))} \tag{5.11}$$

*in such a way that, either (4.13) is retainable, or else*

$$u = v \text{ (hence } u \text{ is } (\perp, \triangle; \varphi)\text{-maximal) whenever } M(u, \nabla) = \emptyset. \tag{5.12}$$

*(Here, $(\nabla)$ is the transitive relation given by (5.1)).*

In particular, when $(\perp)$ is a transitive relation over $M$, this statement reduces to Theorem 3. Since the opposite inclusion also holds, we get

$$\text{Theorem 6} \Longleftrightarrow \text{Theorem 3 } (\Longleftrightarrow \text{Theorem 1}). \tag{5.13}$$

Hence, this extension is technical in nature.

**(B)** Now, the basic assumption used here is (5.2). So, we may ask of what happens when such a condition is no longer true. To this end, put

$$(x, y \in M) \quad x \top y \text{ iff } x \perp y \text{ and } \varphi(x) \triangle \varphi(y). \tag{5.14}$$

This is an amorph relation over $M$; and condition (5.2) holds with $(\top, \triangle)$ in place of $(\perp, \triangle)$. An application of Theorem 6 to these data gives:

**Theorem 7.** *Assume (4.18) holds, as well as*

$$(M, \top) \text{ is sequentially } \Gamma\text{-inductive.} \tag{5.15}$$

*Then, for each $u \in M$ there exists $v \in M$ with*

$$v \text{ is } (\top, \triangle; \varphi)\text{-maximal (cf. (5.3))} \tag{5.16}$$

*in such a way that, either (4.13) is retainable, or else*

$$u = v \text{ (hence } u \text{ is } (\top, \triangle; \varphi)\text{-maximal) whenever } M(u, \nabla) = \emptyset. \tag{5.17}$$

*(Here, $(\top)$ is the amorph relation of (5.14); and $(\nabla)$, its associated by (5.1) transitive relation).*

A basic particular case of this corresponds to the choice $\Gamma = \omega$. Then, the above results are just the ones in Turinici [23]. But, as precised there, this version of Theorem 7 includes a related maximality principle in Gajek and Zagrodny [10]; hence, so does our statement. Further aspects were delineated in Sonntag and Zalinescu [20].

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