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Critical Associated Metrics on Framed φ-Manifolds

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Abstract. Using the induced almost complex and almost contact structures on framed φ-manifolds, defined in [7], we extend some results from the theory of the critical associated metrics on symplectic and on contact manifolds to the case of framed φ-manifolds.

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1 Introduction

Let M be an m-dimensional smooth manifold endowed with a tensor field φ of type (1,1), satisfying the algebraic condition

$$\varphi^3 + \varphi = 0. \tag{1.1}$$

The geometric structure on M defined by φ is called a φ-structure of rank r if the rank r of φ is constant on M and, in this case, M is called a φ-manifold. It follows easily that r is an even number.

If M is a φ-manifold and if there are $m-r$ vector fields ξ_a and $m-r$ differential 1-forms η_a satisfying

$$\varphi^2 = -I + \sum_{a=1}^{m-r} \eta_a \otimes \xi_a, \tag{1.2}$$

$$\eta_a(\xi_b) = \delta_{ab}, \tag{1.3}$$

where $a, b = 1, 2, \ldots, m-r$, M is said to be globally framed or to have a framed φ-structure. In this case M is called a globally framed φ-manifold or, simply, a framed φ-manifold. From (1.2) and (1.3), one obtains, by some algebraic computations

$$\varphi\xi_a = 0, \quad \eta_a \circ \varphi = 0, \quad \varphi^3 + \varphi = 0. \tag{1.4}$$

If $m = 2n+1$ and $\operatorname{rank}\varphi = 2n$ one obtains an almost contact structure on M.

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A framed φ-structure is normal if the tensor field S of type (1,2) defined by

$$S = N_\varphi + \sum_{a=1}^{m-r} d\eta_a \otimes \xi_a, \tag{1.5}$$

vanishes, (see [7]), where

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y], \tag{1.6}$$

with $X, Y \in \chi(M)$, defines the Nijenhuis tensor field, N_φ, of φ.

If g is a (semi-)Riemannian metric on M such that

$$g(\varphi X, \varphi Y) = g(X,Y) - \sum_{a=1}^{m-r} \eta_a(X)\eta_a(Y), \tag{1.7}$$

then we say that $(\varphi, \xi_a, \eta_a, g)$ is a metric framed φ-structure and M is called a metric framed φ-manifold.

The metric g is called an associated (semi-)Riemannian metric.

The fundamental 2-form, Ω, of the considered metric framed φ-manifold, M, is defined just like in the case of the almost Hermitian and almost contact metric manifold, by $\Omega(X,Y) = g(X, \varphi Y)$, for any $X, Y \in \chi(M)$.

The framed φ-manifold M with structure tensors $(\varphi, \xi_a, \eta_a, g)$ is called a $\mathcal{C}$-manifold if it is normal, $d\Omega = 0$ and $d\eta_a = 0$, $a = 1, \dots, m-r$.

If on an almost contact manifold (M, φ, ξ, η) it is defined an associated Riemannian metric g then $(M, \varphi, \xi, \eta, g)$ is called an almost contact metric manifold. If on an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ we have $\Omega = d\eta$, where Ω is the fundamental 2-form on M, then we say that $(M, \varphi, \xi, \eta, g)$ is a contact metric manifold. It is proved that for a $2n+1$-dimensional contact manifold, (M, ξ, η), all associated metrics have the same same volume element, $d\nu = \frac{(-1)^n}{2^n n!}\eta \wedge (d\eta)^n$, (see [5]).

Finally, recall that by a symplectic manifold we mean an even dimensional differentiable manifold, M^{2n}, together with a global 2-form, Ω, which is closed and of maximal rank, that is $d\Omega = 0$ and $\Omega^n \neq 0$. An associated metric on the symplectic manifold, (M, Ω), is a Riemannian metric, g, such that $g(X, JY) = \Omega(X,Y)$, where J is an almost complex structure on M. Note that all associated metrics on a symplectic manifold, (M^{2n}, Ω), have the same volume element, $d\nu = \frac{(-1)^n}{2^n n!}\Omega^n$, (see [5]).

2 A global invariant on the even dimensional framed φ-manifolds

Theorem 2.1. *Let M be an even dimensional compact manifold, with* $\dim M = 2m+2s$. *Assume that there exists the global 2-form Ω and the global 1-forms $\{\eta_1, \dots, \eta_{2s}\}$, on M, such that the 2-form $\widetilde{\Omega}$, defined by*

$$\widetilde{\Omega} = \Omega + \sum_{a=1}^{s} (\eta_{2a-1} \wedge \eta_{2a}),$$

is closed and of maximal rank. Also assume that M carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, $a = 1, \dots, 2s$, with the fundamental 2-form Ω. Let $\mathcal{A}$ be the set of all associated metrics to the framed φ-structures, (φ, ξ_a, η_a), $a = 1, \dots, 2s$, on M, with the fundamental 2-form Ω, (note that the 1-forms, $\{\eta_a\}_{a=1}^{2s}$, and the vector fields $\{\xi_a\}_{a=1}^{2s}$ are the same for all this structures). Then

$$I = \int_M (R + R^* + T) d\nu$$

is a constant on the set $\mathcal{A}$, where R and R^* are the scalar and the $*$-scalar curvature of a metric $g \in \mathcal{A}$, and the local expression of T is

$$T = -\frac{1}{2} R^j_{lhk} [2\Omega^{hk} + \sum_{a=1}^{s} (\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a})] \sum_{b=1}^{s} (\eta_{2bj}\xi^l_{2b-1} - \eta_{2b-1j}\xi^l_{2b}),$$

where R^j_{lhk} are the local components of the curvature tensor field of the Levi-Civita connection, ∇, of the metric g, Ω_{ij} are the components of the fundamental 2-form Ω and $\Omega^{hk} = g^{hi}\Omega_{ij}g^{jk}$, $g^{ij}g_{jk} = \delta^i_k$, and $\xi_a = \xi^i_a \frac{\partial}{\partial x^i}$, $\eta_a = \eta_{ai}dx^i$, for any $a = 1, \dots, 2s$, with respect the local coordinates on M.

Proof. First, from the hypothesis, it follows that $(M, \widetilde{\Omega})$ is a symplectic manifold.

Let

$$J = \varphi + \sum_{a=1}^{s} (\eta_{2a} \otimes \xi_{2a-1} - \eta_{2a-1} \otimes \xi_{2a}),$$

be the almost complex structure on M, induced by a framed φ-structure, defined in [7].

It is easy to see that an associated metric on the framed φ-manifold, $(M, \varphi, \xi_a, \eta_a)$, is an associated metric for the almost complex manifold (M, J) too.

After a straightforward computation, one obtains that the fundamental 2-form on (M, J, g) is given by

$$\widetilde{\Omega} = \Omega + \sum_{a=1}^{s} (\eta_{2a-1} \wedge \eta_{2a}),$$

where $g \in \mathcal{A}$ is an associated metric. Since the all associated metrics on a symplectic or a contact manifold have the same volume element, (see [5]), it follows that I is well defined. The same argument is available in the case of all functionals on the spaces of associated metrics, which we use in this paper.

It is proved in [8] that the integral

$$\widetilde{I} = \int_M (\widetilde{R} + \widetilde{R}^*) d\nu$$

is a constant on the set $\widetilde{\mathcal{A}}$, of all associated metrics on the symplectic manifold $(M, \widetilde{\Omega})$, where $\widetilde{R}$ and $\widetilde{R}^*$ denotes the scalar and the $*$-scalar curvature of a metric $g \in \widetilde{\mathcal{A}}$.

As we have seen, $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$ and then $\widetilde{I}$ is a constant on $\mathcal{A}$.

Let $g \in \mathcal{A}$ be an associated metric on $(M, \varphi, \xi_a, \eta_a)$. Then, the local coordinate expressions of the Ricci tensor field and the scalar curvature are

$$R_{ij} = R^h_{ihj} = g^{hk} R_{hikj}, \quad R = g^{ij} R_{ij},$$

where g_{ij} are the local components of g and $g_{ih}g^{hj} = \delta_{ij}$.

The $*$-Ricci tensor field and the $*$-scalar curvature on $(M, \varphi, \xi_a, \eta_a, g)$ are given by

$$R^*_{ij} = R_{ihkl}\Omega^{hk}\varphi^l_j = -\frac{1}{2}R_{ilhk}\Omega^{hk}\varphi^l_j, \quad R^* = R^*_{ij}g^{ij} = -\frac{1}{2}R^j_{lhk}\Omega^{hk}\varphi^l_j,$$

where $\Omega_{ij} = g_{ik}\varphi^k_j$ are the components of Ω and we have used g to raise and lower the indices.

Since the metric g is associated on (M, J) we have the following expressions for the Ricci tensor field, the scalar curvature, the $*$-Ricci tensor field and the $*$-scalar curvature, respectively, on (M, J)

$$\widetilde{R}_{ij} = R^h_{ihj} = g^{hk}R_{hikj} = R_{ij}, \quad \widetilde{R} = g^{ij}R_{ij} = R, \tag{2.1}$$

and

$$\widetilde{R}^*_{ij} = R_{ihkl}\widetilde{\Omega}^{hk}J^l_j = -\frac{1}{2}R_{ilhk}\widetilde{\Omega}^{hk}J^l_j, \quad \widetilde{R}^* = \widetilde{R}^*_{ij}g^{ij} = -\frac{1}{2}R^j_{lhk}\widetilde{\Omega}^{hk}J^l_j, \tag{2.2}$$

From this equations, one obtains

$$\widetilde{R}^* = R^* - \frac{1}{2}R^j_{lhk}\Omega^{hk}\sum_{a=1}^{s}(\eta_{2aj}\xi^l_{2a-1} - \eta_{2a-1j}\xi^l_{2a}) - \frac{1}{2}R^j_{lhk}\varphi^l_j\sum_{a=1}^{s}(\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a}) -$$

$$-\frac{1}{2}R^j_{lhk}\sum_{a=1}^{s}(\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a})\sum_{b=1}^{s}(\eta_{2bj}\xi^l_{2b-1} - \eta_{2b-1j}\xi^l_{2b}). \tag{2.3}$$

Using the properties of a metric framed φ-structure $(\varphi, \xi_a, \eta_a, g)$, we have

$$R^j_{lhk}\Omega^{hk}\sum_{a=1}^{s}(\eta_{2aj}\xi^l_{2a-1} - \eta_{2a-1j}\xi^l_{2a}) = R^j_{lhk}\Omega^{hk}g_{ij}\sum_{a=1}^{s}(\xi^i_{2a}\xi^l_{2a-1} - \xi^i_{2a-1}\xi^l_{2a}) = \tag{2.4}$$

$$= R_{ilhk}\Omega^{hk}\sum_{a=1}^{s}(\xi^i_{2a}\xi^l_{2a-1} - \xi^i_{2a-1}\xi^l_{2a}),$$

and

$$R^j_{lhk}\varphi^l_j\sum_{a=1}^{s}(\xi^k_{2a}\xi^h_{2a-1} - \xi^k_{2a-1}\xi^h_{2a}) = R_{ilhk}g^{ij}\varphi^l_j\sum_{a=1}^{s}(\xi^k_{2a}\xi^h_{2a-1} - \xi^k_{2a-1}\xi^h_{2a}) = \tag{2.5}$$

$$= -R_{hkil}\Omega^{il}\sum_{a=1}^{s}(\xi^k_{2a}\xi^h_{2a-1} - \xi^k_{2a-1}\xi^h_{2a}).$$

From (2.3), (2.4) and (2.5) one obtains $\widetilde{R}^* = R^* + T$, and then, using (2.1),

$$\widetilde{I} = \int_M (\widetilde{R} + \widetilde{R}^*)d\nu = \int_M (R + R^* + T)d\nu = I,$$

on the set $\mathcal{A}$. Hence I is a constant on $\mathcal{A}$. □

Remark 2.2. With the assumptions in the previous theorem, as in the symplectic case or in the contact case, (see [5]), it can be proved that the set $\mathcal{A}$ is infinite dimensional, and $\mathcal{A}$ is totaly geodesic in the set $\mathcal{N}$, of the Riemannian metrics on M with the same volume element. Note that the same results are true in the case of the odd dimensional manifolds considered in Theorem 3.2.

Remark 2.3. If on the Riemannian manifold (M, h) with $\dim M = 2m + s$, we have a global 2-form, Ω, such that $\Omega^m \neq 0$, the 1-forms $\{\eta_1, \dots, \eta_s\}$ and the vector fields $\{\xi_1, \dots, \xi_s\}$, such that $\eta_a(\xi_b) = \delta_{ab}$ and $\Omega(X, \xi_a) = 0$, for any $a, b = 1, \dots, s$, $X \in \chi(M)$, then, using the same method as in the case of contact manifolds, (see [5]), it can be proved that M carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, such that the fundamental 2-form of g is Ω.

In [8] the author proves that, if a symplectic manifold, $(M, \widetilde{\Omega})$, carries a Kähler metric g then $\widetilde{R} = \widetilde{R}^*$ and $\widetilde{I} = 2\int_M \widetilde{R} d\nu$. In our case if $(M, \widetilde{\Omega})$, carries a Kähler metric g then $\widetilde{R} = \widetilde{R}^* = R$ and $I = 2\int_M R d\nu$. Since the conditions for the almost complex manifold (M, J, g), with the induced almost complex structure, to be Kähler are the conditions for $\widetilde{\Omega}$ from the previous theorem and in addiction $(M, \varphi, \xi_a, \eta_a, g)$ must be a normal framed φ-manifold, (see [7], [6]), one obtains

Theorem 2.4. *Let M be a compact even dimensional manifold as in Theorem 2.1. If this manifold carries a normal metric framed φ-structure, then*

$$I = 2\int_M R\, d\nu$$

is a constant on the set $\mathcal{A}$, where R is the scalar curvature.

Remark 2.5. It is easy to see that an example of a manifold with the properties in the previous theorems is an even dimensional $\mathcal{C}$-manifold with the condition that $\widetilde{\Omega}$ is of maximal rank, where $\widetilde{\Omega}$ is obtained as in Theorem 2.1.

3 Critical associated metrics on the framed φ-manifolds

In [2] the authors have shown that the associated metric g on a symplectic manifold M is a critical point of the integrals $\int_M R\, d\nu$ and $\int_M (R - R^*) d\nu$ if and only if the Ricci operator, Q, commutes with the almost complex structure, J, on M, (see also [5]).

Using the induced almost complex structure on an even dimensional framed φ-manifold, and the same notations as in Theorem 2.1, one obtains

Theorem 3.1. *Let M be an even dimensional compact manifold, with the same properties as in Theorem 2.1. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$E(g) = \int_M R\, d\nu, \quad F(g) = \int_M (R - R^* - T) d\nu$$

if and only if the Ricci operator, Q, of g, has the properties

$$Q\varphi X = \varphi QX + \sum_{a=1}^{s} \left(\eta_{2a}(QX)\xi_{2a-1} - \eta_{2a-1}(QX)\xi_{2a}\right), \tag{3.1}$$

for $X \in (\operatorname{span}\{\xi_a\}_{a=1}^{2s})^{\perp}$, *and*

$$\begin{cases} \varphi Q\xi_{2a-1} = \varphi^2 Q\xi_{2a}, \\ \eta_{2a-1}(Q\xi_{2a-1}) = \eta_{2a}(Q\xi_{2a}), \\ \eta_{2a}(Q\xi_{2a-1}) = -\eta_{2a-1}(Q\xi_{2a}), \end{cases} \tag{3.2}$$

for any $a = 1, \dots, s$.

Proof. As we have seen in the previous section, M can be viewed as a symplectic manifold, with the set of all associated metrics, $\widetilde{\mathcal{A}}$, including the set $\mathcal{A}$, of all associated metrics of framed φ-structures, (φ, ξ_a, η_a), on M, with the same fundamental 2-form, Ω.

Since $\widetilde{R} = R$ and $\widetilde{R}^* = R^* + T$, (see Theorem 2.1), it follows, using the specified result in [2], that a metric $g \in \mathcal{A}$ is a critical point for $E(g)$ and $F(g)$ if and only if $QJ = JQ$, where the notations are the same as in the proof of the Theorem 2.1. After a straightforward computations, one obtains that the last condition is equivalent with the equations (3.1) and (3.2). □

In the case of a odd dimensional framed φ-manifold we can use a induced almost contact structure also defined in [7].

Let $(M, \varphi, \xi_a, \eta_a)$ be an odd dimensional framed φ-manifold with $\dim M - \operatorname{rank} \varphi = 2s + 1$. Then $(\widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1})$ is an almost contact structure on M, where $\widetilde{\varphi}$ is a (1,1) tensor field on M, defined by

$$\widetilde{\varphi} = \varphi + \sum_{a=1}^{s} (\eta_{2a} \otimes \xi_{2a-1} - \eta_{2a-1} \otimes \xi_{2a}),$$

(see [7]). If g is an associated metric on $(M, \varphi, \xi_a, \eta_a)$ it is easy to obtain that g is an associated metric on the almost contact manifold $(M, \widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1})$. It is proved in [6] that such a manifold is a contact metric manifold if and only if

$$\widetilde{\Omega} = \Omega + \sum_{i=1}^{s} (\eta_{2i-1} \wedge \eta_{2i}) = d\eta_{2s+1}, \tag{3.3}$$

where $\widetilde{\Omega}$ and Ω are the fundamental 2-forms on the almost contact metric manifold $(M, \widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1})$ and on metric framed φ-manifold $(M, \varphi, \xi_a, \eta_a)$, respectively.

Let M be a compact odd dimensional manifold, with $\dim M = 2m + 2s + 1$, such that there exists the 2-form, Ω, and the 1-forms, $\{\eta_1, \dots, \eta_{2s+1}\}$, with the condition (3.3). If $(\varphi, \xi_a, \eta_a, g)$ is a metric framed φ-structure on M, such that the fundamental 2-form of g is Ω then $(M, \widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1}, g)$ is a contact metric manifold with the fundamental 2-form $\widetilde{\Omega}$. Let us denote by R, R^* the scalar and the $*$-scalar curvature corresponding to g, of the framed φ-manifold M, and by $\widetilde{R}$, $\widetilde{R}^*$ the scalar and the $*$-scalar curvature of the contact metric manifold M. In the same way as in the proof of the Theorem 2.1 one obtains $\widetilde{R} = R$ and $\widetilde{R}^* = R^* + T$, where the local expression of T is

$$T = -\frac{1}{2} R^j_{lhk} \Big[2\Omega^{hk} + \sum_{a=1}^{s} \Big(\xi^h_{2a-1}\xi^k_{2a} - \xi^k_{2a-1}\xi^h_{2a} \Big) \Big] \cdot \sum_{b=1}^{s} \Big(\eta_{2bj}\xi^l_{2b-1} - \eta_{2b-1j}\xi^l_{2b} \Big), \tag{3.4}$$

where R^j_{lhk} are the local components of the curvature tensor field of the Levi-Civita connection, ∇, of the metric g, Ω_{ij} are the components of the fundamental 2-form Ω and

$\Omega^{hk} = g^{hi}\Omega_{ij}g^{jk}$, $g^{ij}g_{jk} = \delta^i_k$, and $\xi_a = \xi^i_a \frac{\partial}{\partial x^i}$, $\eta_a = \eta_{ai}dx^i$, for any $a = 1, \dots, 2s$, with respect of the local coordinates on M.

In [3] it is proved that a metric g, associated on a compact contact manifold, (M, φ, ξ, η), is a critical point of $A(g) = \int_M R\, d\nu$ in the set of all associated metrics on M, if and only if the Ricci operator, Q, and φ commute when restricted to $(\mathrm{span}\{\xi\})^\perp$, where R is the scalar curvature of M. Using this result, in the case of odd dimensional framed φ-manifold, one obtains

Theorem 3.2. *Let M be a a compact odd dimensional manifold as above, which carries a metric framed φ-structure, $(\varphi, \xi_a, \eta_a, g)$, $a = 1, \dots, 2s$, with the fundamental 2-form Ω, and let $\mathcal{A}$ be the set of all associated metrics to framed φ-structures, (φ, ξ_a, η_a), on M, with the fundamental 2-form Ω. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$A(g) = \int_M R\, d\nu$$

in $\mathcal{A}$, if and only if

$$Q\varphi X = \varphi Q X + \sum_{a=1}^{s} \left(\eta_{2a}(QX)\xi_{2a-1} - \eta_{2a-1}(QX)\xi_{2a}\right), \tag{3.5}$$

for $X \in (\mathrm{span}\{\xi_a\}_{a=1}^{2s+1})^\perp$, and

$$\begin{cases} \varphi Q\xi_{2a-1} = \varphi^2 Q\xi_{2a}, \\ \eta_{2a-1}(Q\xi_{2a-1}) = \eta_{2a}(Q\xi_{2a}), \\ \eta_{2a}(Q\xi_{2a-1}) = -\eta_{2a-1}(Q\xi_{2a}), \end{cases} \tag{3.6}$$

for any $a = 1, \dots, s$, where Q is the Ricci operator corresponding to the metric g.

Proof. Indeed the conditions (3.5) and (3.6) are equivalent with the condition that the Ricci operator, Q, and the induced tensor field of type (1,1), $\widetilde{\varphi}$, commute when restricted to $(\mathrm{span}\{\xi_{2s+1}\})^\perp$, and, since from the hypothesis one obtains that $(M, \widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1})$ is a contact manifold, it follows that the associated metric, g, is a critical point of $A(g)$ in the set $\widetilde{\mathcal{A}}$, of associated metrics on $(M, \widetilde{\varphi}, \xi_{2s+1}, \eta_{2s+1})$. But $\mathcal{A} \subseteq \widetilde{\mathcal{A}}$. Hence one obtains the desired result. □

In [4] the authors have shown that an associated metric, g, on a compact contact manifold, (M, φ, ξ, η), is a critical point of $I(g) = \int_M (R + R^*) d\nu$ in the set $\mathcal{A}$, of all associated metrics on M, if and only if ξ is a Killing vector field with respect g. From this we have an immediate result, concerning the odd dimensional framed φ-manifolds

Theorem 3.3. *Let M be a a compact odd dimensional manifold as in Theorem 3.2. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$I(g) = \int_M (R + R^* + T) d\nu$$

in $\mathcal{A}$, where T is given by (3.4), if and only if ξ_{2s+1} is a Killing vector field with respect with g.

In [1] the author proves that, if (M, ξ, η) is a compact regular contact manifold and $\mathcal{A}$ is the set of the associated metrics, then $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M Ric(\xi) d\nu$

if and only if ξ is a Killing vector field, where $Ric(\xi)$ denotes the Ricci curvature on the direction ξ. From this and from the results on the beginning of this section one obtains

Theorem 3.4. *Let M be a a compact odd dimensional manifold as in Theorem 3.2. Then a metric $g \in \mathcal{A}$ is a critical point of*

$$L(g) = \int_M Ric(\xi_{2s+1}) d\nu,$$

in $\mathcal{A}$, if and only if ξ_{2s+1} is a Killing vector field with respect with g.

Remark 3.5. By changing indices, it is easy to see that, in the conditions of the previous theorem, a metric $g \in \mathcal{A}$ is a critical point of $L(g) = \int_M Ric(\xi_a) d\nu$, $a = 1, \dots, 2s+1$, if and only if ξ_a is a Killing vector field with respect with g.

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