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Semi-Invariant Submanifolds of a Generalised 3-Sasakian Structure

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Abstract. We obtain a the necessary and sufficient conditions so that the distributions involved in the study to be integrable. Also we prove that any totally contact umbilical semi-invariant submanifolds of a manifold with a generalized Sasakian 3-structure, the sectional curvature can not be positively.

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Introduction

The notion of CR-submanifold has been introduced by A.Bejancu [2] for the Kähler manifolds and by A.Bejancu-N.Papaghiuc [4] for the Sasakian manifolds (called contact semi-invariant submanifolds). In the case of quaternionic Kähler manifolds has been introduced the concept of CR-submanifolds by Baros-Chen-Urbano [1], by using the decomposition of the tangent bundle of a submanifold and the concept of QR-submanifolds has been introdoced by A.Bejancu [3] by the decomposition of the normal bundle of a submanifold. Later on the notion of CR-submanifolds has been intensively studied from different points of view, several important results have been obtained, some of them being brought together in [2]. Also some important results have been obtained in [3] about the QR-submanifolds of quaternionic Kählerian manifolds and in [6] about the semi-invariant submanifolds of a manifold with a Sasakian 3-structure.

Recently, A.Bejancu-H.R.Faran [5] introduced the notion of the generalised Sasakian 3-structure.

The purpose of this paper is to study the notion of a semi-invartant submanifold on a manifold endowed with a generalised Sasakian 3-structure. More exactly we found the condition so that the distributions involved in the study have to be integrable and also we found the geometry of its leaves. Also we will prove that the sectional curvature of submanifold can not be positive.

1 Preliminaries

Let $\tilde{M}$ be a $(4n+3)$-dimensional differentiable manifold with an almost contact metric 3-structure (f_a, ξ_a, η_a, g), $a \in \{1,2,3\}$, meaning that the next relations are satisfied

$$\begin{gathered} a)\ \ f_a^2 = -I + \eta_a \otimes \xi_a, \qquad b)\ \ \eta_a(\xi_b) = \delta_{ab} \qquad c)\ \ f_a(\xi_b) = -f_b(\xi_a) = \xi_c, \\ d)\ \ \eta_a \circ f_b = -\eta_b \circ f_a = \eta_c, \quad e)\ \ f_a \circ f_b - \eta_b \otimes \xi_a = -f_b \circ f_a + \eta_a \otimes \xi_b = f_c, \\ f)\ \ \eta_a(X) = g(X, \xi_a), \qquad g)\ \ g(f_a X, f_a Y) = g(X,Y) - \eta_a(X)\eta_a(Y), \end{gathered} \tag{1.1}$$

for any cyclic permutation (a,b,c) of $(1,2,3)$, where X and Y are the vector fields tangent to $\tilde{M}$, δ is the Kronecker's delta.

The notion of Sasakian 3-structure has been introduced by Kuo in [8] and was generalised by Bejancu-Faran in [5]. More exactly, the manifold $\tilde{M}$ is a manifold with a generalised Sasakian 3-structure, if there exists the local 1-forms α_{ab} so that $\alpha_{ab} + \alpha_{ba} = 0$ and

$$(\tilde{\nabla}_X f_a)Y = g(X,Y)\xi_a - \eta_a(Y)X + \alpha_{ab}(X) f_b(Y) + \alpha_{ac}(X) f_c(Y), \tag{1.2}$$

for any vector fields X, Y tangent to $\tilde{M}$, $\tilde{\nabla}$ is the Levi-Civita connection on $\tilde{M}$, and (a,b,c) is a cyclic permutation of $(1,2,3)$.

Next, suppose that (a,b,c) is a cyclic permutation of $(1,2,3)$, we call a generalised 3-Sasakian manifold, a manifold endowed with a generalised Sasakian 3-structure. Throughout the paper, all manifolds and maps are supposed differentiable of class C^∞. We denote by $F(M)$ the module of the differentiable functions on $\tilde{M}$ and by $\Gamma(E)$ the module of smooth sections of a vector bundle E over $\tilde{M}$. We use the same notations for any manifolds involved in the study.

By straightforward calculation using (1.1b)-(1.1d) one deduces that

$$\tilde{\nabla}_X \xi_a = -f_a X + \alpha_{ab}(X)\xi_b + \alpha_{ac}(X)\xi_c, \quad \forall X \in \Gamma(T\tilde{M}). \tag{1.3}$$

The curvature tensor K of $\tilde{M}$ is defined by

$$K(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z, \qquad \forall X,Y,Z \in \Gamma(T\tilde{M}), \tag{1.4}$$

and the sectional curvature is given by $K(X,Y) = g(K(X,Y)Y,X)$ for any orthonormal vector fields $X, Y \in \Gamma(T\tilde{M})$.

Next, let M be a $(m+3)$-dimensional Riemannian manifold isometrically immersed in $\tilde{M}$, and suppose that the structure vector fields ξ_1, ξ_2, ξ_3 of $\tilde{M}$ are tangent to M, respectively. We denote by TM and $TM^\perp$ the tangent bundle and the normal bundle to M, repectively. We also denote by $\{\xi\}$ the distribution spanned by ξ_1, ξ_2, ξ_3 on M. The induced metric tensor on M will be denoted by the same symbol g.

According to A.Bejancu [3], we say that the submanifold M of a generalised 3-Sasakian manifold $\tilde{M}$ is a semi-invariant submanifold, if there exists a vector subbundle μ of $TM^\perp$ such that

$$f_a(\mu) = \mu; \qquad f_a(\mu^\perp) \subseteq TM, \quad a \in \{1,2,3\},$$

where $\mu^\perp$ is the complementary orthogonal to μ in $TM^\perp$. If in particular $\mu = TM^\perp$ or $\mu = \{0\}$, we say that M is an invariant or an anti-invariant submanifold. It is easy to see that any real hypersurface of $\tilde{M}$ is a semi-invariant submanifold. Suppose that M is a semi-invariant submanifold which is not an invariant submanifold. Next, denote $f_a(\mu_x^\perp)$ by D_{ax}, $x \in M$. Next, we see that D_{1x}, D_{2x}, D_{3x} are mutually orthogonal subspaces of T_xM and have the same dimension s as the dimension of $\mu_x^\perp$. The mapping

$$D^\perp : x \to D_x^\perp = D_{1x} \oplus D_{2x} \oplus D_{3x}, \quad x \in M,$$

is a 3s-dimensional distribution on M ($s = \dim \mu_x^\perp$). By straightforward calculation it is easy to see that

$$a)\ \ f_a(D_{ax}) = \mu_x^\perp; \quad b)\ \ f_a(D_{bx}) = D_{cx},\ x \in M. \tag{1.5}$$

We denote by D the complementary orthogonal distribution to $D^\perp \oplus \{\xi\}$ in TM. It follows that the distribution D is invariant with respect to the action of f_1, f_2, f_3, that is $f_a(D) = D$. So, we obtain the next orthogonal decomposition of $T\tilde{M}$

$$TM = D \oplus D^\perp \oplus \{\xi\},$$

where D, $\{\xi\}$ and $D^\perp$ are the above distributions. We note that $D^\perp$ is not anti-invariant distribution (see (1.5b)).

From the general theory of Riemannian submanifolds, recall the Gauss and Weingarten formulae

$$\begin{aligned} &a)\quad \tilde{\nabla}_X Y = \nabla_X Y + h(X,Y),\\ &b)\quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,\ \ \forall X, Y \in \Gamma(TM),\ \ N \in \Gamma(TM^\perp), \end{aligned} \tag{1.6}$$

where h is the second fundamental form of M, A_N is the shape operator with respect to the normal section N, ∇ and $\nabla^\perp$ are the induced connections by $\tilde{\nabla}$ on TM and $TM^\perp$, respectively. The Codazzi equation is given by

$$\begin{aligned} g(K(X,Y)Z,N) &= g((\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z), N),\\ &\forall X, Y, Z \in \Gamma(TM), \quad N \in \Gamma(TM^\perp). \end{aligned} \tag{1.7}$$

It is known that if $\{e_i\}$ $i = 1, \dots, m$ is an orthonormal basis of $\Gamma(TM)$, then the mean curvature vector field of M, denoted by H, is given by

$$H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i).$$

The submanifold M is called totally contact umbilical if the second fundamental form h of M is expressed as

$$\begin{aligned} h(X,Y) = \sum_{a=1}^{3} (g(f_aX, f_aY)H + \eta_a(X)h(Y,\xi_a) + \eta_a(Y)h(X,\xi_a)),\\ \forall X, Y \in \Gamma(TM). \end{aligned} \tag{1.8}$$

If $H = 0$ and (1.8) hold, then M is called totally contact geodesic submanifold of $\tilde{M}$.

We say that the semi-invariant submanifold M manifold $\tilde{M}$ is D-geodesic if we have $h(X,Y) = 0$, $X, Y \in \Gamma(D)$. By straightforward calculation, using (1.3), (1.6), one deduces

Proposition 1.1. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then we have*

$$\begin{aligned} &a)\ h(X,\xi_a)=0; && b)\ \nabla_X\xi_a=-f_aX+\alpha_{ab}(X)\xi_b+\alpha_{ac}(X)\xi_c,\ X\in\Gamma(D);\\ &c)\ h(f_aU,\xi_a)=U; && d)\ \nabla_{f_aU}\xi_a=\alpha_{ab}(f_aU)\xi_b+\alpha_{ac}(f_aU)\xi_c, \\ &e)\ h(f_bU,\xi_a)=0; && f)\ \nabla_{f_bU}\xi_a=f_cU+\alpha_{ab}(f_bU)\xi_b+\alpha_{ac}(f_bU)\xi_c,\ U\in\Gamma(\mu^\perp). \end{aligned} \tag{1.9}$$

2 Integrability of distributions on a semi-invariant submanifolds of a generalised 3-Sasakian manifold

The purpose of this paragraph is to study the integrability of distributions involved in the study and the geometry of its leaves. First we prove

Theorem 2.1. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then the distributions D and $D^\perp$ are not integrable.*

Proof. Let $X\in\Gamma(D)$, $Y=f_aX$ a not null vector field and because ∇ is a Levi-Civita connection, using (1.3) and (1.5) we derive

$$g([X,Y],\xi_a)=g(Y,\tilde{\nabla}_X\xi_a)-g(X,\tilde{\nabla}_Y\xi_a)=-2g(X,X)\neq 0,$$

which prove that the distribution $D\neq\{0\}$ is not integrable. If we take $Z=f_bU$, $W=f_cU$, $U\in\Gamma(\mu^\perp)$, in the same way, we obtain that the distribution $D^\perp$ is not integrable.

Remark 2.1. It is interesting to see that the distributions D_a and $D_a\oplus\{\xi_a\}$ are not integrable if $\tilde{M}$ is a proper generalised 3-Sasakian manifolds, but in the case of a 3-Sasakian manifold ($\alpha_{ab}(X)=0,\ \forall X\in\Gamma(T\tilde{M})$) these distributions are integrable (see [6]).

In the same way as we had proved Theorem 2.1, we can prove

Theorem 2.2. *If M is a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$, Then the distributions $D\oplus\{\xi_a\}$ and $D^\perp\oplus\{\xi_a\}$ are not integrable.*

By using (1.3) one deduces

Proposition 2.1. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$, then the distribution $\{\xi\}$ is involutive.*

Next we have:

Proposition 2.2. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then we have that*

$$\begin{aligned} &a)\ g([X,\xi_a],Y)=g([Y,\xi_a]),X)=0,\quad X\in\Gamma(D),\ Y\in\Gamma(D^\perp)\\ &b)\ g([X,\xi_a],\xi_b)=\alpha_{ab}(X),\ X\in\Gamma(D\oplus D^\perp) \end{aligned} \tag{2.1}$$

Proof. Let $X\in\Gamma(D)$ and $Y=f_bU,\ U\in\Gamma(\mu^\perp)$. By using (1.2), (1.3), (1.6b) and the fact that ∇ is a metric connection, we deduce that

$$g([X,\xi_a],f_bU)=g(\tilde{\nabla}_X\xi_a,f_aU)-g(\tilde{\nabla}_{\xi_a}X,f_aU)=g(\tilde{\nabla}_{\xi_a}f_aX,U)=g(\tilde{\nabla}_{f_aX}\xi_a,U)=0.$$

In the same way we obtain that $g([Y,\xi_a],X) = 0,\ X \in \Gamma(D),\ Y \in \Gamma(D^\perp)$. Next, let $X \in \Gamma(D^\perp \oplus D)$, and using (1.3) we deduce

$$g([X,\xi_a],\xi_b) = g(\tilde{\nabla}_X \xi_a, \xi_b) - g(\tilde{\nabla}_{\xi_a} X, \xi_b) = \alpha_{ab}(X).$$

Theorem 2.3. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then the following three assertions are equivalents*

$$\begin{aligned} &a)\ \text{the distribution } D \oplus \{\xi\} \text{ is integrable,}\\ &b)\ h(X, f_a Y) = h(f_a X, Y),\quad \forall X,Y \in \Gamma(D),\\ &c)\ M \text{ is } D\text{-geodesic.}\end{aligned} \tag{2.2}$$

Proof. Let $X, Y \in \Gamma(D),\ U \in \Gamma(\mu)$. By using (1.2), (1.6) and the fact that ∇ is a torsion free connection, we get

$$\begin{aligned} g([X,Y], f_a U) &= g((\tilde{\nabla}_X f_a)Y - (\tilde{\nabla}_Y f_a)X - \tilde{\nabla}_X f_a Y + \tilde{\nabla}_Y f_a X, f_a U)\\ &= g(h(X, f_a Y) - h(Y, f_a X), U),\end{aligned}$$

and using (2.1a) we obtain the equivalence a) $\leftrightarrow$ b). Now, taking into account (1.1e), (1.9b) and the assertion b), we infer that

$$h(X, f_a Y) = h(X, (f_b \circ f_c)Y) = h((f_c \circ f_b)X, Y) = -h(f_a X, Y),$$

and we obtain the assertion c). The proof is complete.

By direct calculation, using (1.1e), (1.2) and (1.6) we obtain:

Proposition 2.3. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then we have*

$$\begin{aligned} &a)\ A_V f_a U = A_U f_a V,\quad \forall U, V \in \Gamma(\mu^\perp),\\ &b)\ g([X,Y],\xi_a) = 0,\quad \forall X, Y \in \Gamma(D \oplus D^\perp).\end{aligned} \tag{2.3}$$

Next, let us consider an ortohnormal field of frames $\{V_1, \dots, V_s\}$ on $\mu^\perp$ in $TM^\perp$. Then on the distribution $D^\perp$ we have the orthonormal field of frames $\{\{E_{11}, \dots, E_{1s}, E_{21}, \dots, E_{2s}, E_{31}, \dots, E_{3s}\}$, where $E_{ai} = f_a V_i,\ i = 1, \dots, s$. We denote by A_i the shape operator with respect to the section V_i. Throughtout the paper $i, j, k, \dots \in \{1, \dots, s\}$ and we use the above orthonormal field of frames.

Lemma 2.1. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then we have*

$$a)\ A_i E_{aj} = A_j E_{ai};\quad b)\ g(\nabla_{E_{ai}} E_{aj}, X) = g(\nabla_{E_{aj}} E_{ai}, X),\ \forall X \in \Gamma(D). \tag{2.4}$$

Proof. The assertion a) follows from (2.3a). Now, let $X \in \Gamma(D)$ and using (1.1g), (1.2) and (1.6b) we obtain

$$g(\nabla_{E_{ai}} E_{aj}, X) = g((\tilde{\nabla}_{E_{ai}} f_a)V_j + f_a \nabla_{E_{ai}} V_j, X) = g(A_j E_{ai}, f_a X),$$

and using assertion a), the assertion b) is proved.

Next we define the differential 1-form B_{aij} on D by

$$B_{aij}(X) = g(\nabla_{E_{ai}} E_{aj}, X) = B_{aji}(X),\ X \in \Gamma(D).$$

Theorem 2.4. *Let M be a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then the following three assertions are equivalents*

a) the distribution $D^{\perp} \oplus \{\xi\}$ is involutive,

b) $B_{aij}(X) = 0$, $X \in \Gamma(D)$,

c) $h(X, Z) \in \Gamma(\mu)$, $X \in \Gamma(D)$, $Z \in \Gamma(D^{\perp})$.

Proof. It is easy to see that $E_{bi} = f_c E_{ai}$ and therefore, by using (1.2) we get

$$g([E_{ai}, E_{aj}], f_c X) = B_{aij}(X) + B_{bij}(X),\ \forall X \in \Gamma(D).$$

Now, from the above relation and using (1.9d), (1.9f) and (2.1a), we obtain that the distribution $D^{\perp} \oplus \{\xi\}$ is integrable iff $B_{aij}(X) = 0$, that is, taking into account (1.9d) and (1.9f), the distribution $D^{\perp} \oplus \{\xi\}$ is parallel. Next by direct calculation we derive

$$B_{aij}(X) = g(\nabla_{E_{ai}} E_{aj}, X) = g(h(f_a X, E_{ai}, V_j),\ X \in \Gamma(D),\ V_j \Gamma(\mu),$$

and the equivalence *b)* $\leftrightarrow$ *c)* is proved.

Theorem 2.5. *If M is a semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$, then we have*

a) if the distribution $D \oplus \{\xi\}$ is integrable, then its leaves are totally geodesic immersed in $\tilde{M}$,

b) if the distribution $D^{\perp} \oplus \{\xi\}$ is integrable, then its leaves are totally geodesic immersed in M but not in $\tilde{M}$.

Proof. Suppose that $M^{\star}$ is a leaf of integrable distribution $D \oplus \{\xi\}$ and denote by $h^{\star}$ the second fundamental form of immersion $M^{\star} \to \tilde{M}$. Then, by direct calculation we deduce that:

$$g(h^{\star}(X, Y), E_{ai}) = -g(\tilde{\nabla}_X f_a Y, V_i) = -g(h(X, f_a Y), V_i) = 0,\ \forall X, Y \in \Gamma(D \oplus \{\xi\}),$$

and the assertion a) is proved. Now let M' be a leaf of the integrable distribution $D^{\perp} \oplus \{\xi\}$ and denote by h' the second fundamental form of imersion $M' \to M$. Then we deduce that

$$g(h'(E_{ai}, E_{aj}), X) = B_{aij}(X) = 0,$$

$$g(h'(E_{ai}, E_{bj}), X) = -B_{aij}(f_c X) = 0,\ \forall X \in \Gamma(D),$$

but $h(E_{ai}, \xi_a) = V_i$ and our assertion b) is proved.

3 Totally contact umbilical and totally contact geodesic semi-invariant submanifolds are tangent to the structure vector fields ξ_1, ξ_2, ξ_3

The purpose of this paragraph is to study some properties of totally contact umbilical and totally contact geodesic of semi-invariant submanifolds of a generalised 3-Sasakian manifold. First, from [7] we recall

Theorem 3.1. *If M is a totally contact umbilical semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$, then we have*

if $\dim \mu^{\perp} \neq 1$ then M is totally contact geodesic,

if $\dim \mu^{\perp} = 1$ then M is an intrinsic sphere, that is $\nabla^{\perp}_X H = 0$, $\forall X \in \Gamma(TM)$.

Next from Theorem 2.3 and 2.4 we deduce

Theorem 3.2. *If M is a totally contact umbilical semi-invariant submanifold of a generalised 3-Sasakian structure, then*

a) if the distribution $D \oplus \{\xi\}$ is integrable then $D = \{0\}$, or M is a totally contact geodesic semi-invariant submanifold,

b) the distribution $D^{\perp} \oplus \{\xi\}$ is involutive.

Lemma 3.1. *Let M be a totally contact umbilical semi-invariant submanifold of a generalised 3-Sasakian manifold $\tilde{M}$. Then the curvature tensor field satisfies*

$$g(\tilde{K}(X, E_{ai})X, E_{ai}) + g(\tilde{K}(X, E_{ai})f_a X, V_i) = 0,\ \forall X \in \Gamma(D \oplus \{\xi\} \tag{3.1}$$

Proof. Let $X \in \Gamma(D \oplus \{\xi\})$. By direct calculation, using (1.1g), (1.2) and (3.1) we infer that

$$\begin{aligned} & g(\tilde{\nabla}_X \tilde{\nabla}_{E_{ai}} f_a X, V_i) + g(\tilde{\nabla}_X \tilde{\nabla}_{E_{ai}} X, E_{ai}) = \\ & \quad = g(\tilde{\nabla}_X ((\tilde{\nabla}_{E_{ai}} f_a) X), V_i) + (\tilde{\nabla}_X f_a) \tilde{\nabla}_{E_{ai}} X, V_i) = \\ & \quad = \alpha_{ab}(E_{ai}) g(\tilde{\nabla}_X f_b X, V_i) + \alpha_{ac}(E_{ai}) g(\tilde{\nabla}_X f_c X, V_i) + \\ & \quad\quad + \alpha_{ab}(X) g(f_b \tilde{\nabla}_{E_{ai}} X, V_i) + \alpha_{ac}(X) g(f_c \tilde{\nabla}_{E_{ai}} X, V_i) = \\ & \quad = \alpha_{ab}(X) g(\tilde{\nabla}_{E_{ai}} f_b X, V_i) + \alpha_{ac}(X) g(\tilde{\nabla}_{E_{ai}} f_c X, V_i) = 0. \end{aligned}$$

In the same way one deduces

$$g(\tilde{\nabla}_{E_{ai}} \tilde{\nabla}_X f_a X, V_i) + g(\tilde{\nabla}_{E_{ai}} \tilde{\nabla}_X X, E_{ai}) = 0$$

and

$$g(\tilde{\nabla}_{[X, E_{ai}]}, V_i) + g(\tilde{\nabla}_{[X, E_{ai}]}, E_{ai}) = 0.$$

Therefore, the assertion follows by using the formula for the curvature tensor field and the above relations. By direct calculation and by using (3.1) we get

Theorem 3.3. *If M is a totally contact umbilical semi-invariant submanifold of a generalised 3-Sasakian structure, then its sectional curvature is not positive.*

Proof. Let $X \in \Gamma(D \oplus \{\xi\})$ be an unit vector field, and $E_{ai} \in \Gamma(D^{\perp})$. Then by using the Codazzi equation and (3.2) we get

$$\begin{aligned} K(X, E_{ai}) &= g(K(X, E_{ai})E_{ai}, X) = g((K(X, E_{ai}) f_a X)^{\perp}, E_{ai}) = \\ &= g((\nabla_X f)(E_{ai}, f_a X) - (\nabla_{E_{ai}} h)(X, f_a X), V_i), \end{aligned} \tag{3.3}$$

where $X^{\perp}$ denote the normal part of X and

$$(\nabla_X f)(Y, Z) = \nabla^{\perp}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Now we evaluate the right side of (3.3). First, by using (1.3), (3.1) and Lemma 3.2 we derive

$$\begin{aligned} (\nabla_X h)(E_{ai}, f_a Y) &= \nabla^{\perp}_X h(E_{ai}, f_a X) - h(\nabla_X E_{ai}, f_a X) - h(E_{ai}, \nabla_X f_a X) = \\ &= -g(X, X) V_i = -V_i, \end{aligned}$$

and

$$(\nabla_{E_{ai}} h)(X, f_a X) = \nabla^{\perp}_{E_{ai}} h(X, f_a X) - h(\nabla_{E_{ai}} X, f_a X) - h(X, \nabla_{E_{ai}} f_a X) = 0.$$

Finally we obtain

$$K(X, E_{ai}) = -g(V_i, V_i) < 0.$$

From Theorem 3.3 we deduce

Corollary 3.1. *If* M *is a totally contact umbilical semi-invariant submanifold of a generalised 3-Sasakian manifold* $\tilde{M}$*, with positive sectional curvature, then* $D = \{0\}$ *or* $\mu^{\perp} = \{0\}$.

References

[1] M.Barros, B.Y.Chen, F.Urbano, *Quaternionic CR-submanifolds of quaternionic manifolds*, Kodai Math.J. 4(1981), 399-417.

[2] A.Bejancu, *Geometry of CR-submanifolds*, Kluwer, Dordrecht, 1986.

[3] A.Bejancu, *QR-submanifolds of quaternionic Kaehler manifolds*, Chinese J.Math. 14(1986), 81-94.

[4] A.Bejancu, N.Papaghiuc, *Semi-invariant submanifolds of a Sasakian manifold.* An. Şt. Univ. A.I. Cuza, Iaşi 27(1981) 163-170.

[5] A.Bejancu, H.R.Farran, *On totally umbilical QR-submanifolds of quaternionic Kaehlerian manifolds* Bull.Austral.Math.Soc. Vol 62(2000), 95-103.

[6] C. Călin, *QR-submanifolds of a manifold with a Sasakian 3-structure*. Proc. of 24th Nat. Conf. of Geom. and Top., Timişoara, Romania, July 5-9, 1994, 73-84.

[7] B.Y.Chen, *Totally umbilical submanifolds of quaternionic space forms* J.Austral. Math. Soc. S.a 26(1978), 154-162

[8] Y.Y.Kuo, *On almost contact 3-structure*, Tohoku Math. J., 22(1970) 325-332.