LIBERTAS MATHEMATICA vol XXV (2005)

An Extention of the Univalent Condition for an Integral Operator

Daniel BREAZ

Abstract. In this paper is presented an extention of the univalent condition for the integral operator $H_{\alpha,\beta}$ defined by Pescar in [3].

2000 Mathematics Subject Clasification: 30C45.

Introduction

Let $U = \{z : |z| < 1\}$ the unit disk. We consider the class of analytical function in the unit disk, denoted by with $A = \left\{f : f = z + a_2 z^2 + a_3 z^3 + \ldots\right\}, z \in U$. Also we consider S the class of univalent and regular functions with the conditions $f(0) = f'(0) - 1 = 0$.

The Schwartz Lemma. *Let the analytic function g be regular in the unit disc U and $g(0) = 0$. If $|g(z)| \leq 1$, $\forall z \in U$, then*

$$|g(z)| \leq |z|, \quad \forall z \in U \tag{1}$$

and equality holds only if $g(z) = \varepsilon z$, where $|\varepsilon| = 1$.

We have the following result proved by Pascu in [2] and that we shall use in proving the results contained in this paper.

Teorema A. [2] *Let $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ and $f \in A$. If*

$$\frac{1 - |z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha} \left|\frac{zf''(z)}{f'(z)}\right| \leq 1, \quad \forall z \in U \tag{2}$$

then $\forall \beta \in \mathbb{C}$, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, the function

$$F_\beta(z) = \left[\beta \int_0^z t^{\beta-1} f'(t)\, dt\right]^{1/\beta} \tag{3}$$

is univalent.

In [3], Pescar proved the next result:

Theorem B. [3] *Let $g \in A$, that satisfies the condition*

$$\left|\frac{z^2 g'(z)}{g^2(z)} - 1\right| < 1, \quad z \in U \tag{4}$$

and $\alpha \in \mathbb{C}$ *so that*

$$\operatorname{Re}\alpha \geq \frac{3}{|\alpha|}. \tag{5}$$

If $|g(z)| \leq 1$, $\forall z \in U$, *then* $\forall \beta \in \mathbb{C}$, $\operatorname{Re}\beta \geq \operatorname{Re}\alpha$, *the function*

$$H_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{g(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{6}$$

This result of Pescar is extended in [1] by the author in the following theorem:

Theorem C. *Let* $f_i \in A$, $f_i(z) = z + a_2^i z^2 + \ldots$, $\forall i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, *that satisfies the next conditions*:

$$\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| < 1, \quad z \in U, \quad \forall i = \overline{1,n} \tag{7}$$

$$\alpha, \beta \in \mathbb{C}, \quad \operatorname{Re}\alpha > \frac{3n}{|\alpha|}, \quad \operatorname{Re}\beta \geq \operatorname{Re}\alpha. \tag{8}$$

If $|f_i(z)| \leq 1$, $\forall z \in U$, $\forall i = \overline{1,n}$, *then* $\forall \beta \in \mathbb{C}$, *satisfying the above conditions we have*:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \ldots \cdot \left(\frac{f_n(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{9}$$

We notice that for $n = 1$ we obtain the Theorem B.

Next we shall give a generalization of Theorem C, considering a real number $M \geq 1$ and using the condition $|f_i(z)| \leq M$, $\forall z \in U$, $\forall i = \overline{1,n}$.

Main results

Theorem 1. *Let* $M \geq 1$, $f_i \in A$, $f_i(z) = z + a_2^i z^2 + \ldots$, $\forall i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, *that satisfies the next conditions*:

$$\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| < 1, \quad z \in U, \quad \forall i = \overline{1,n} \tag{10}$$

$$\alpha, \beta \in \mathbb{C}, \quad \operatorname{Re}\alpha > \frac{n(2M+1)}{|\alpha|}, \quad \operatorname{Re}\beta \geq \operatorname{Re}\alpha. \tag{11}$$

If $|f_i(z)| \leq M$, $\forall z \in U$, $\forall i = \overline{1,n}$, *then* $\forall \beta \in \mathbb{C}$, *satisfying the above conditions we have*:

$$F_{\alpha,\beta}(z) = \left\{ \beta \int_0^z t^{\beta-1} \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \ldots \cdot \left(\frac{f_n(t)}{t} \right)^{\frac{1}{\alpha}} dt \right\}^{\frac{1}{\beta}} \in S. \tag{12}$$

Proof. We consider the function $h(z) = \int_0^z \left(\frac{f_1(t)}{t} \right)^{\frac{1}{\alpha}} \cdot \ldots \cdot \left(\frac{f_n(t)}{t} \right)^{\frac{1}{\alpha}} dt$. We observe that $h(0) = h'(0) - 1 = 0$.

By calculating the derivatives of the order I and II for this function we obtain:

$$h'(z) = \left(\frac{f_1(t)}{t}\right)^{\frac{1}{\alpha}} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\frac{1}{\alpha}} \tag{13}$$

respectively

$$h''(z) = \frac{1}{\alpha} h'(z) \cdot B_1 + \ldots + \frac{1}{\alpha} h'(z) \cdot B_n \tag{14}$$

where

$$B_i = \left(\frac{z}{f_i(z)}\right) \cdot \frac{z f_i'(z) - f_i(z)}{z^2}, \quad i = \overline{1,n}. \tag{15}$$

We calculate the fraction $\frac{zh''(z)}{h'(z)}$. Thus we obtain:

$$\frac{zh''(z)}{h'(z)} = \frac{z \cdot \frac{1}{\alpha} h'(z) \cdot \sum_{i=1}^{n} B_i}{h'(z)} = z \cdot \frac{1}{\alpha} \cdot \sum_{i=1}^{n} B_i, \quad \forall z \in U. \tag{16}$$

Replacing $B_i, i = \overline{1,n}$, in formula (16) we obtain:

$$\frac{zh''(z)}{h'(z)} = \frac{1}{\alpha}\left(\frac{z f_1'(z)}{f_1(z)} - 1\right) + \ldots + \frac{1}{\alpha}\left(\frac{z f_n'(z)}{f_n(z)} - 1\right). \tag{17}$$

We evaluate the modulus and we multiply in both terms with $\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}$. Hence:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right|\frac{|f_i(z)|}{|z|} + 1\right). \tag{18}$$

Because $|f_i(z)| \le M$, $\forall z \in U$, $\forall i = \overline{1,n}$ and applying Schwarz Lemma we obtain that

$$\frac{|f_i(z)|}{|z|} \le M, \quad \forall z \in U, \quad \forall i = \overline{1,n}. \tag{19}$$

We apply this relation in the above inequality and we obtain:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| M + 1\right). \tag{20}$$

But

$$\left|\frac{z^2 f_i'(z)}{f_i^2(z)}\right| = \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1 + 1\right| \le \left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| + 1, \quad \forall z \in U, \quad \forall i = \overline{1,n}. \tag{21}$$

We apply this relation in the above inequality and we obtain:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{1-|z|^{2\operatorname{Re}\alpha}}{|\alpha|\operatorname{Re}\alpha}\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| M + M + 1\right). \tag{22}$$

But $\left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| < 1$, $\forall z \in U$, $\forall i = \overline{1,n}$.

Thus the inequality becomes:

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \le \frac{(2M+1)\,n}{|\alpha|\operatorname{Re}\alpha}. \tag{23}$$

Because $\operatorname{Re}\alpha > \frac{n(2M+1)}{|\alpha|}$, obtain that

$$\frac{1-|z|^{2\operatorname{Re}\alpha}}{\operatorname{Re}\alpha}\left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \forall z \in U. \tag{24}$$

Applying Theorem A we obtain that $F_{\alpha,\beta} \in S$.

Remark. For $M = 1$ we obtain Theorem C.

References

[1] Breaz, D., *Operatori integrali pe spatii de functii univalente*, Editura Academiei Romane, Bucuresti, 2004.

[2] Pascu, N.N., *An improvement of Becker's univalence criterion*, Proceedings of the Commemorative Session Simoon Stoilow, Brasov, (1987), 43-48.

[3] Pescar, V., *New criteria for univalence of certain integral operators*, Demonstratio Mathematica, vol. XXXIII, 1 (2000), 51-54.