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Differential Properties of the Value Functions of Infinite-Horizon Optimal Control Problems

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Abstract. We use a certain monotonicity property of the value function of a general infinite-horizon optimal control problem to prove that it satisfies certain differential inequalities implying the fact that it is a particular type of viscosity solution of the associated Hamilton-Jacobi-Bellman equation.

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1 Introduction

The aim of this paper is to extend to infinite-horizon optimal control problems the differential properties of the value functions which, in the case of finite-horizon problem had been obtained in Şt. Mirică (1992; 2004b); in particular, we prove that the differential inequalities in this paper imply the fact that the value function is a particular type of viscosity solution as defined in Bardi and Capuzzo-Dolcetta (1997), Crandall and Lions (1983), Lions (1982), etc.

As a general idea, for this type of results we are replacing the so called "Dynamic Programming Principle" (actually "The Dynamic Programming Functional Equation") by the equivalent property of "Monotonicity of the Value Function" along admissible trajectories proved in I. Mirică (1997), thus, we are able to use the "Contingent Directional Derivatives" instead of the weaker concept of "Contingent semi-differential". Moreover, this approach allows the study of more general problems, defined by unbounded differential inclusions which are also subject to "active" phase (time-state) constraints.

The paper is organized as follows: in Section 2 we present some notations, definitions and preliminary results and in Section 3 we recall the monotonicity and asymptotic properties of the value function proved in I. Mirică (1997). Next, in Section 4 we recall some concepts and results in Şt. Mirică (1992a, 1992b, 2004a, 2004b) concerning the sets of generalized tangent and contingent directions to the trajectories of differential inclusions. In Sections 5 and 6 we prove the main results concerning first some "abstract" differential inequalities, expressed in terms of corresponding sets of generalized tangent directions, then more "explicit" differential inequalities under suitable additional hypotheses.

2 Notations and definitions

In this paper we are studying the *value function* of an infinite-horizon optimal control problem which consists in the *minimization* of each of the functionals

$$\mathcal{C}(s,y;x(\cdot)) := \int_s^\infty f_0(t,x(t),x'(t))dt \quad (s,y)\in E\subseteq \mathbf{R}\times\mathbf{R}^n \tag{2.1}$$

subject to:

$$x'(t)\in F(t,x(t)) \ a.e. \ (s,\infty) \quad x(s)=y \tag{2.2}$$

$$(t,x(t))\in E\subseteq \mathbf{R}\times\mathbf{R}^n \quad (\forall)\ t\in[s,\infty) \tag{2.3}$$

$$\begin{gathered}\hat{x}(\cdot) := (x(\cdot),x_0(\cdot))\in\Omega_\alpha\subseteq AC^{loc}([s,\infty);\mathbf{R}^n\times\mathbf{R}),\\ x_0(t) := \int_s^t f_0(\sigma,x(\sigma),x'(\sigma))d\sigma\end{gathered} \tag{2.4}$$

$$f_0(\cdot,x(\cdot),x'(\cdot))\in L^\infty([s,\infty);\mathbf{R}). \tag{2.5}$$

where $\Omega_\alpha\subseteq AC^{loc}$ is a specified class of AC^{loc} (i.e. locally absolutely continuous) mappings.

We note that *the data* of the problem are the following:
- the set $E\subseteq\mathbf{R}\times\mathbf{R}^n$ of *admissible phases* (time-state);
- the multifunction $F(\cdot,\cdot):E\to\mathcal{P}(\mathbf{R}^n)$ defining the "dynamics" in (2.2) and which, in particular, may be of a "continuously parameterized" type:

$$F(t,x)=f(t,x,U),\ \ f(\cdot,\cdot,u),u\in U,\ \text{continuous}; \tag{2.6}$$

- the real function, $f_0(\cdot,\cdot,\cdot):G(F(\cdot,\cdot))\to\mathbf{R}$ defining the cost functional in (2.1) and which, in the case of parameterized differential inclusion in (2.6) is replaced by $f_0(\cdot,\cdot,\cdot):E\times U\to\mathbf{R}$ so the cost functional in (2.1) takes the form:

$$\mathcal{C}(s,y;u(\cdot)) := \int_s^\infty f_0(t,x(t),u(t))dt,\quad x'(t)=f(t,x(t),u(t)),u(t)\in U a.e.(s,\infty); \tag{2.7}$$

- the class $\Omega_\alpha\in\{\Omega_{pc},\Omega_r,\Omega_p,p\in[1,\infty]\}$ of locally absolutely continuous admissible trajectories where $x(\cdot)\in\Omega_{pc}$ if $x'(\cdot)$ is piecewise continuous, $x(\cdot)\in\Omega_r$ if $x'(\cdot)$ is regulated and $x(\cdot)\in\Omega_p, p\in[1,\infty]$ if $x'(\cdot)\in L_p^{loc}([s,\infty),\mathbf{R}^n)$.

We point out the fact that if $h(\cdot):[s,\infty)\to\mathbf{R}$ is measurable then we adopt the measure-theoretic definition (e.g. Dunford and Schwartz (1958)) of the integral $\int_s^\infty h(t)dt$ so that it has the properties:

$$\int_s^\infty h(t)dt = \lim_{T\to\infty}\int_s^T h(t)dt,\quad \lim_{s\to\infty}\int_s^\infty h(t)dt=0. \tag{2.8}$$

We note that in the theory of normal integrands in Rockafellar (1976) as well as in Carlson et. al. (1991) one uses a different definition of this integral which may not have the properties in (2.8).

Denoting by $\Omega_\alpha(s,y)$ the set of trajectories $x(\cdot):[s,\infty)\to\mathbf{R}^n$ satisfying (2.2)-(2.5), the *value function* of the problem is defined by:

$$W(s,y) := \inf_{x(\cdot)\in\Omega_\alpha(s,y)} \mathcal{C}(s,y;x(\cdot)),\ (s,y)\in E, \tag{2.9}$$

hence, using the convention $\inf\emptyset=+\infty$, one obtains: $W(s,y)=+\infty$ iff $\Omega_\alpha(s,y)=\emptyset$.

For each $(s,y)\in E$, the (possibly empty) set of *optimal trajectories* corresponding to the initial point (s,y) is defined by:

$$\widetilde{\Omega}_\alpha(s,y) := \{\widetilde{x}(\cdot)\in\Omega_\alpha(s,y);\ \mathcal{C}(s,y;\widetilde{x}(\cdot))=W(s,y)\} \tag{2.10}$$

Thus the "phase space" $E\subseteq\mathbf{R}\times\mathbf{R}^n$ admits the partition:

$$\begin{gathered} E=E^{+\infty}\cup E^{-\infty}\cup E^R,\quad E^{\pm\infty}:=\{(s,y)\in E;\ \ W(s,y)=\pm\infty\},\\ E^R=\operatorname{dom}W(s,y):=\{(s,y)\in E;\ \ W(s,y)\in\mathbf{R}\}. \end{gathered} \tag{2.11}$$

Moreover, the effective domain, E^R, of the value function may be partitioned by:

$$E^R=\widetilde{E}\cup E^i,\ \widetilde{E}:=\operatorname{dom}\widetilde{\Omega}_\alpha(\cdot,\cdot):=\{(s,y)\in E^R;\widetilde{\Omega}_\alpha(s,y)\neq\emptyset\},\ E^i:=E^R\setminus\widetilde{E} \tag{2.12}$$

Throughout the paper we shall assume the following:

Hypothesis 2.1.

(i) The subset $E\subseteq\mathbf{R}\times\mathbf{R}^n$ is nonempty and $F(\cdot,\cdot):E\to\mathcal{P}(\mathbf{R}^n)$ is a multifunction with nonempty closed values;

(ii) The real function $f_0(\cdot,\cdot,\cdot):G(F(\cdot,\cdot)):=\{(t,x,x');\ (t,x)\in E,\ x'\in F(t,x)\}\to\mathbf{R}$ is a "normal integrand" in the sense of Rockafellar (1986).

3 Monotonicity and asymptotic properties

In this section we recall the basic, monotonicity and asymptotic properties in I. Mircă (1997) of the value function in (2.9), which are expressed in terms of the larger class of *locally admissible trajectories* defined by:

$$\begin{gathered} \Omega_\alpha^{loc}(s,y):=\{x(\cdot):I\subseteq\mathbf{R}\to\mathbf{R}^n;\ \widehat{x}(\cdot)=(x(\cdot),x_0(\cdot))\in\Omega_\alpha^{loc},\\ x'(t)\in F(t,x(t))\,a.e.(I), s\in I, x(s)=y,\ f_0(\cdot,x(\cdot),x'(\cdot))\in L_1^{loc}(I;\mathbf{R})\}, (s,y)\in E \end{gathered} \tag{3.1}$$

and of the associated "extended" real functions

$$\omega_x(t):=W(t,x(t))+\int_s^t f_0(r,x(r),x'(r))dr,\ x(\cdot)\in\Omega_\alpha^{loc}(s,y),\ t\in I=\operatorname{dom}x(\cdot) \tag{3.2}$$

Theorem 3.1. ([8]) *If Hypothesis 2.1 is satisfied then the value function* $W(\cdot,\cdot)$ *in* (2.9) *has the following properties:*

(i) (**Monotonicity**). *For any* $(s,y)\in E,\ x(\cdot)\in\Omega_\alpha^{loc}(s,y)$, *the extended real function* $\omega_x(\cdot)$ *in* (3.2) *is increasing* (i.e. $\omega_x(t_1)\le\omega_x(t_2),\ (\forall)s\le t_1<t_2<+\infty,\ t_1,t_2\in I$).

(ii) (**Asymptotic properties**). *For any* $(s,y)\in E\setminus E^{+\infty}$, $x(\cdot)\in\Omega_\alpha(s,y)$, *there exists* $\lim\limits_{t\to+\infty}W(t,x(t))$ *and it satisfies the following relations:*

$$0\geq\lim_{t\to+\infty}W(t,x(t))\geq W(s,y)-\mathcal{C}(s,y;x(\cdot))\tag{3.3}$$

(iii) (**Optimality**). *If* $(s,y)\in\widetilde{E}$ *and* $\widetilde{x}(\cdot)\in\Omega_\alpha(s,y)$ *then* $\widetilde{x}(\cdot)\in\widetilde{\Omega}_\alpha(s,y)$ (i.e. *it is optimal) iff the real function* $\omega_{\tilde{x}}(\cdot)$ *in* (3.2) *is constant and satisfies:*

$$\omega_{\tilde{x}}(t)=\omega_{\tilde{x}}(s)=W(s,y),\ (\forall)t\in[s,\infty),\ \lim_{t\to+\infty}W(t,\tilde{x}(t))=0\tag{3.4}$$

Remark 3.2. As it is easy to prove, the monotonicity property (i) in Theorem 3.1 is equivalent with the so called "functional equation of Dynamic Programming",

$$W(s,y)=\inf_{x(\cdot)\in\Omega_\alpha^T(s,y)}[W(T,x(T))+\int\limits_s^T f_0(t,x(t),x'(t))dt],\ (\forall)\ T>s,\ (T,y)\in E\tag{3.5}$$

where the set of "truncated" (locally admissible) trajectories is defined by:

$$\Omega_\alpha^T(s,y):=\{x(\cdot)\in\Omega_\alpha^{loc}(s,y);\ [s,T]\subset\operatorname{dom}x(\cdot)\}.\tag{3.6}$$

We recall that the "functional equation" in (3.5) ("Dynamic Programming Principle") is frequently used in the theory of viscosity solutions of Hamilton-Jacobi equations associated to the problem (2.1)-(2.5) (e.g. Bardi and Cappuzo-Dolcetta (1997), Crandall and Lions (1983), Lions (1982), etc.). However, as we shall see in what follows, the (equivalent) monotonicity property (i) in Theorem 3.1 may lead to stronger results than those in the theory of viscosity solutions.

Remark 3.3. The monotonicity property (i) in Theorem 3.1 implies the fact that for any $(s,y)\in E^{\mathbf{R}}$, $x(\cdot)\in\Omega_\alpha^{loc}(s,y)$ one has:

$$\frac{d}{dt}W(t,x(t))+f_0(t,x(t),x'(t))\geq 0\ \ a.e.(s,\infty),\tag{3.7}$$

which, in the case $W(\cdot,\cdot)$ *is differentiable* at $(t,x(t))$ *a.e.* (s,∞), takes the form:

$$\frac{\partial W}{\partial t}(t,x(t))+\frac{\partial W}{\partial x}(t,x(t))\cdot x'(t)+f_0(t,x(t),x'(t))\geq 0\ \ a.e.(s,\infty).\tag{3.8}$$

On the other hand, if $(s,y)\in\widetilde{E}$ and $\widetilde{x}(\cdot)\in\widetilde{\Omega}_\alpha(s,y)$ is optimal and the same condition is satisfied, then from (3.4) it follows:

$$\frac{\partial W}{\partial t}(t,\widetilde{x}(t))+\frac{\partial W}{\partial x}(t,\widetilde{x}(t))\cdot\widetilde{x}'(t)+f_0(t,\widetilde{x}(t),\widetilde{x}'(t))=0\ \ a.e.(s,\infty).\tag{3.9}$$

The properties in (3.8), (3.9) suggest the fact under certain hypotheses, at differentiability points, the value function satisfies the well known "Partial Differential Equation of Dynamic Programming":

$$\frac{\partial W}{\partial t}(t,x)+H(t,x,\frac{\partial W}{\partial x}(t,x))=0,\ H(t,x,p)=\inf_{v\in F(t,x)}[<p,v>+f_0(t,x,v)]\tag{3.10}$$

known also as the Hamilton-Jacobi-Bellman (HJB) equation associated to the problem (2.1)-(2.5).

However, as simple examples show, the value function in (2.9) has scarce regularity properties and, in particular, may not be differentiable at each point of its effective domain.

4 Generalized tangent and contingent directions

The differential properties in (3.8)-(3.9) may be take a more "explicit" form if we use the sets of generalized tangent, quasitangent, contingent and peritangent directions to the trajectories of a differential inclusion of the form in (2.2).

To simplify the exposition we shall use in what follows only the following sets of generalized tangent directions introduced in Şt. Mirică (1992a, 2004b).

Definition 4.1. ([10, 12]) *If* $F(\cdot,\cdot)$ *defines the differential inclusion in* (2.2) *and* $\Omega_{\alpha}^{loc}(s,y)$, $(s,y)\in E$, *denotes the set of locally admissible trajectories then:*

(i) *The sets of unilateral tangent directions at* $(s,y)\in E$ *are the sets defined by:*

$$T_F^{\pm}(s,y):=\{v\in\mathbf{R}^n;\ (\exists)\ x(\cdot)\in\Omega_1^{loc}(s,y):(\exists)x'_{\pm}(s):=\lim_{\theta\to 0\pm}\frac{x(s+\theta)-y}{\theta}=v\}\tag{4.1}$$

(ii) *The sets of unilateral contingent directions at* $(s,y)\in E$ *are the sets defined by:*

$$\begin{aligned}K_F^{1,\pm}(s,y)&:=\{v\in\mathbf{R}^n;\ (\exists)\ x(\cdot)\in\Omega_1^{loc}(s,y),\ \theta_m\to 0\pm:\ \tfrac{x(s+\theta_m)-y}{\theta_m}\to v\}\\ K_F^{0,\pm}(s,y)&:=\{v\in\mathbf{R}^n;\ (\exists)\ x(\cdot)\in\Omega_1^{loc}(s,y),\ (\theta_m,r_m)\to(0\pm,0+):\\ &\qquad \tfrac{x(s+\theta_m\cdot r_m)-y}{\theta_m}\to v\}\end{aligned}\tag{4.2}$$

(iii) *The sets of unilateral generalized contingent directions at* $(s,y)\in E$ *are the sets defined by:*

$$\begin{aligned}GK_F^{1,\pm}(s,y)&:=\{v\in\mathbf{R}^n;\ (\exists)\ x_m(\cdot)\in\Omega_1^{loc}(s,y),\ \theta_m\to 0\pm:\ \tfrac{x_m(s+\theta_m)-y}{\theta_m}\to v\}\\ GK_F^{0,\pm}(s,y)&:=\{v\in\mathbf{R}^n;\ (\exists)\ x_m(\cdot)\in\Omega_1^{loc}(s,y),\ (\theta_m,r_m)\to(0\pm,0+):\\ &\qquad \tfrac{x_m(s+\theta_m\cdot r_m)-y}{\theta_m}\to v\}\end{aligned}\tag{4.3}$$

Other sets of generalized tangent directions have been recently introduced in studied in Şt. Mirică (2004b).

Remark 4.2. On may note here that the sets $K_F^{0,\pm}(s,y)$, $GK_F^{0,\pm}(s,y)$ may by considered "unbounded" *generalized tangent directions* since in the case $F(\cdot,\cdot)$ is *locally bounded* at $(s,y)\in E$ then

$$K_F^{0,\pm}(s,y)\subseteq GK_F^{0,\pm}(s,y)\subseteq\{0\}\tag{4.4}$$

while if $F(\cdot,\cdot)$ it is not locally bounded at $(s,y)\in\mathrm{Int}(E)$ then $K_F^{0,\pm}(s,y)$, $GK_F^{0,\pm}(s,y)\neq\{0\}$ are cones with vertex at $0\in\mathbf{R}^n$.

Using the well known "mean value theorem" for the Lebesgue integral, in Şt. Mirică (1992a) it is proved that the above generalized tangent directions satisfy the following *upper estimates:*

$$\begin{cases} T_F^{\pm}(s,y) \subseteq K_F^{1,\pm}(s,y) \subseteq GK_F^{1,\pm}(s,y) = F^{co}(s,y) := \bigcap\limits_{\delta>0} \overline{co}F(B_\delta(s,y)) \\ K_F^{0,\pm}(s,y) \subseteq GK_F^{0,\pm}(s,y) \subseteq F_\infty^{co}(s,y) := \bigcap\limits_{\delta,r>0} [0,r)\cdot \overline{co}F(B_\delta(s,y)) \end{cases} \tag{4.5}$$

which are useful in the theory of sufficient optimality conditions (e.g. Şt. Mirică (2004a)).

In the theory of necessary conditions considered in this paper, certain lower estimates and even exact characterizations are obtained under one of the following hypothesis.

Hypothesis 4.3. The multifunction $F(\cdot,\cdot)$ is Hausdorff-continuous with closed convex values.

Hypothesis 4.4. $F(\cdot,\cdot)$ is Hausdorff-continuous with compact convex values.

Hypothesis 4.5. $F(\cdot,\cdot)$ is Hausdorff-continuous (with respect to both variables) and locally-Lipschitz with respect to the second variable.

Hypothesis 4.6. $F(\cdot,\cdot)$ is upper hemicontinuous and continuously parametrized in the sense of (2.6) where $f(\cdot,\cdot,u)$, $u \in U$, are continuous and U is a Hausdorff topological space.

Under each of these hypothesis, the following lower estimates and exact characterizations are obtained in Şt. Mirică (1992a, 2004b).

Theorem 4.7.

(i) *If* $F(\cdot,\cdot)$ *satisfies Hypothesis 4.3 then*

$$T_F^{\pm}(s,y) = K_F^{1,\pm}(s,y) = F(s,y),\ (\forall)\ (s,y) \in \mathrm{Int}(E) \tag{4.6}$$

(ii) *If* $F(\cdot,\cdot)$ *satisfies Hypothesis 4.4 then*

$$GK_F^{1,\pm}(s,y) = F(s,y),\ GK_F^{0,\pm}(s,y) = \{0\},\ (\forall)\ (s,y) \in \mathrm{Int}(E) \tag{4.7}$$

(iii) *If Hypothesis 4.5 is satisfied then*

$$T_F^{\pm}(s,y) \subseteq K_F^{1,\pm}(s,y) \subseteq GK_F^{1,\pm}(s,y) = \overline{co}F(s,y),\ (\forall)\ (s,y) \in \mathrm{Int}(E) \tag{4.8}$$

(iv) *If Hypothesis 4.6 is satisfied then*

$$K_F^{1,\pm}(s,y) = GK_F^{1,\pm}(s,y) = \overline{co}F(s,y) = \overline{co}f(s,y,U),\ (\forall)\ (s,y) \in \mathrm{Int}(E) \tag{4.9}$$

(v) *If either of the Hypotheses 4.5, 4.6 is satisfied then*

$$K_F^{0,\pm}(s,y) = D^\infty[\overline{co}F(s,y)] := \{v = \lim_{m\to\infty} r_m \cdot v_m;\ r_m \to 0+,\ v_m \in \overline{co}F(s,y)\}. \tag{4.10}$$

We note that at the boundary points $(s,y) \in \partial E := E \backslash \mathrm{Int}(E)$ the sets of generalized tangent directions in (4.1)-(4.2) are difficult to evaluate in the general case.

Remark 4.8. In the next sections we shall use the "extended" differential inclusion:

$$\widehat{x}' \in \widehat{F}(t,\widehat{x}) := \{(v, f_0(t,x,v)); v \in F(t,x)\}, if\ \widehat{x} = (x,x_0), (t,x) \in E, x_0 \in \mathbf{R}, \tag{4.11}$$

since the "extended trajectory", $\widehat{x}(\cdot) = (x(\cdot), x_0(\cdot))$ in (2.4), is a solution of (4.11); therefore the sets of generalized tangent directions in (4.1)-(4.3) are related to the multifunction $\widehat{F}(\cdot,\cdot)$ defined in (4.11), which are defined in the same way replacing $x(\cdot)$ by $\widehat{x}(\cdot) = (x(\cdot), x_0(\cdot))$.

Moreover, the "extended orientor field" $\widehat{F}(\cdot,\cdot)$ in (4.11) is said to be *continuously parameterized* if there exist a topological space, U, and the continuous mappings
$f(\cdot,\cdot,\cdot) : E \times U \to \mathbf{R}^n$, $f_0(\cdot,\cdot,\cdot) : E \times U \to \mathbf{R}$ such that

$$\widehat{F}(t, \widehat{x}) := \{(f(t,x,u), f_0(t,x,u));\ u \in U\},\ (\forall)\ (t,x) \in E. \tag{4.12}$$

5 Abstract differential inequalities

The monotonicity and optimality properties in Theorem 3.1 lead to certain differential inequalities expresed in terms of the *extreme contingent derivatives:* if $g(\cdot) : X \subset \mathbf{R}^n \to \mathbf{R}$ is a real function, $x \in X$ and the (unilateral) contingent cones are defined by:

$$K_x^{\pm} X := \{v \in \mathbf{R}^n;\ (\exists)\ (\theta_m, v_m) \to (0\pm, v) :\ x + \theta_m \cdot v_m \in X,\ (\forall)\ m \in \mathbf{N}\}, \tag{5.1}$$

then the extreme contingent derivatives of $g(\cdot)$ at x in direction $v \in K_x^{\pm} X$ are defined by:

$$\begin{aligned} \overline{D}_K^{\pm} g(x;v) &:= \limsup_{\substack{(\theta,u)\to(0\pm,v)\\ x+\theta\cdot u\in X}} \tfrac{g(x+\theta\cdot u)-g(x)}{\theta} \\ \underline{D}_K^{\pm} g(x;v) &:= \liminf_{\substack{(\theta,u)\to(0\pm,v)\\ x+\theta\cdot u\in X}} \tfrac{g(x+\theta\cdot u)-g(x)}{\theta} \end{aligned} \tag{5.2}$$

The first main result of this paper is the following:

Theorem 5.1. *If* $\widehat{F}(\cdot,\cdot)$ *is the extended vector field in* (4.11), (4.12) *and* $K_{\widehat{F}}^{r,\pm}(\cdot,\cdot)$, $GK_{\widehat{F}}^{r,\pm}(\cdot,\cdot)$, $r \in \{0,1\}$, *are the generalized tangent directions in* (4.2)-(4.3) *then the value function,* $W(\cdot,\cdot)$, *in* (2.9) *satisfies the following differential inequalities at each point* $(s,y) \in E^{\mathbf{R}} = \operatorname{dom} W(\cdot,\cdot)$:

$$\overline{D}_K^{\pm} W((s,y);(1,v)) + v_0 \geq 0,\ (\forall)\ \widehat{v} = (v, v_0) \in GK_{\widehat{F}}^{1,\pm}(s,y) \tag{5.3}$$

$$\overline{D}_K^{\pm} W((s,y);(0,v)) + v_0 \geq 0,\ (\forall)\ \widehat{v} = (v, v_0) \in GK_{\widehat{F}}^{0,\pm}(s,y) \tag{5.4}$$

Further, if $(s,y) \in \widetilde{E}$ (i.e. *it has an optimal trajectory*) *then:*

$$\min\{\inf_{(v,v_0)\in K_{\widehat{F}}^{1,+}(s,y)} [\underline{D}_K^{+} W((s,y);(1,v)) + v_0],\ \inf_{(v,v_0)\in K_{\widehat{F}}^{0,+}(s,y)} [\underline{D}_K^{+} W((s,y);(0,v)) + v_0]\} \leq 0 \tag{5.5}$$

and if $(s,y) \in \widetilde{E}$ *is not the left and point of are optimal trajectory in the sense that:*

$$(s,y) \in \widetilde{E}^{-} := \{(s,y) \in \widetilde{E};\ (\exists)(t_0, x_0) \in \widetilde{E},\ \widetilde{x}(\cdot) \in \widetilde{\Omega}(t_0, x_0),\ t_0 < s,\ \widetilde{x}(s) = y\} \tag{5.6}$$

then one has also:

$$\min\{\inf_{(v,v_0)\in K^{1,-}_{\widehat{F}}(s,y)}[\underline{D}^{-}_{K}W((s,y);(1,v))+v_0],\inf_{(v,v_0)\in K^{0,-}_{\widehat{F}}(s,y)}[\underline{D}^{-}_{K}W((s,y);(0,v))+v_0]\}\leq 0 \tag{5.7}$$

Finally, if the following condition is satisfied:

$$K^{0,\pm}_{\widehat{F}}(s,y)\subseteq\{(0,0)\} \tag{5.8}$$

(*in particular, if* $\widehat{F}(\cdot,\cdot)$ *is locally-bounded at* (s,y)) *then the inequalities in* (5.5) *and* (5.7) *may be replaced by the inequalities:*

$$\inf_{\widehat{v}=(v,v_0)\in K^{1,+}_{\widehat{F}}(s,y)}[\underline{D}^{+}_{K}W((s,y);(1,v))+v_0]\leq 0,\ (\forall)\ (s,y)\in\widetilde{E} \tag{5.9}$$

$$\inf_{\widehat{v}=(v,v_0)\in K^{1,-}_{\widehat{F}}(s,y)}[\underline{D}^{-}_{K}W((s,y);(1,v))+v_0]\leq 0,\ (\forall)\ (s,y)\in\widetilde{E}^{-} \tag{5.10}$$

Proof. Let $(s,y)\in E^{\mathbf{R}}=\operatorname{dom}W(\cdot,\cdot)$ and $\widehat{v}=(v,v_0)\in GK^{1,-}_{\widehat{F}}(s,y)$; from the definitions in (4.3)-(4.11) it follows that there exist $\theta_m\to 0-$ and $\widehat{x}_m(\cdot)=(x_m(\cdot),x^0_m(\cdot))\in S_{\widehat{F}}(s,(y,0))$ (i.e. solutions of the extended differential inclusion in (4.11)) such that:

$$\begin{gathered}\lim_{m\to+\infty}\tfrac{x_m(s+\theta_m)-y}{\theta_m}=v,\\ \lim_{m\to+\infty}\tfrac{x^0_m(s+\theta_m)}{\theta_m}=\lim_{m\to+\infty}\tfrac{1}{\theta_m}\int\limits_{s}^{s+\theta_m}f_0(t,x(t),x'(t))dt=v_0\end{gathered} \tag{5.11}$$

On the other hand one obviously has $x_m(\cdot)\in\Omega^{loc}_1(s+\theta_m,x(s+\theta_m))$ hence, according to Theorem 3.1 (i) the real function $\omega_{x_m}(\cdot)$ in (3.2) is increasing; in particular, since $\theta_m<0$, one has

$$\omega_{x_m}(s+\theta_m)=W(s+\theta_m,x_m(s+\theta_m))+\int\limits_{s}^{s+\theta_m}f_0(t,x(t),x'(t))dt\leq\omega_{x_m}(s)=W(s,x_m(s))$$

Therefore, since $\theta_m\to 0-$ and $x_m(s)\to y$, we obtain

$$\frac{W(s+\theta_m,x_m(s+\theta_m))-W(s,y)}{\theta_m}+\frac{1}{\theta_m}\int\limits_{s}^{s+\theta_m}f_0(t,x(t),x'(t))dt\geq 0,(\forall)m\in\mathbf{N} \tag{5.12}$$

Next, from the definition in (5.2) of the upper left contingent derivative and from (5.11) it follows:

$$\overline{D}^{-}_{K}W((s,y);(1,v))\geq\limsup_{m\to\infty}\frac{W(s+\theta_m,\ x_m(s+\theta_m))-W(s,y)}{\theta_m}$$

hence from (5.12) and (5.11) it follows $\overline{D}^{-}_{K}W((s,y);(1,v))+v_0\geq 0$ and one of the two symmetric inequalities in (5.3) is proved; the other one as well as the inequalities in (5.4) follow in the same way.

In order to prove the inequality in (5.5) we consider $(s,y)\in\widetilde{E}$, $\widetilde{x}(\cdot)\in\widetilde{\Omega}_{\alpha}(s,y)$ (an optimal trajectory) and note that according to Theorem 3.1 the function $\omega_{\widetilde{x}}(\cdot)$ in (3.2) is constant hence

$$\omega_{\widetilde{x}}(s+\theta) = W(s+\theta, \widetilde{x}(s+\theta)) + \int_s^{s+\theta} f_0(t, \widetilde{x}(t), \widetilde{x}'(t))dt = W(s,y), \ (\forall)\ \theta > 0$$

and therefore:

$$\frac{W(s+\theta, \widetilde{x}(s+\theta)) - W(s,y)}{\theta} + \frac{1}{\theta}\int_s^{s+\theta} f_0(t, \widetilde{x}(t), \widetilde{x}'(t))dt = 0, \ (\forall)\ \theta > 0. \tag{5.13}$$

Denoting, as usual, $\widehat{y} = (y, 0)$, $\widehat{x}(\cdot) = (\widetilde{x}(\cdot), \widetilde{x}_0(\cdot))$, where $\widetilde{x}_0(\cdot)$ is the function in (2.4), we consider the following two complementary cases:

Case A): $(\exists)\theta_m \to 0+$ such that $\frac{\widehat{x}(s+\theta_m)-\widehat{y}}{\theta_m} \to \widehat{v} = (v, v_0) \in K^{1,+}_{\widehat{F}}(s,y)$

Case B): $\rho(\theta) := \frac{\|\widehat{x}(s+\theta)-\widehat{y}\|}{\theta} \to +\infty$ as $\theta \to +\infty$

In Case A), using the fact that

$$\underline{D}^{+}_{K}W((s,y);(1,v)) \le \liminf_{m\to+\infty} \frac{W(s+\theta_m,\ \widetilde{x}(s+\theta_m)) - W(s,y)}{\theta_m},$$

from (5.13) one obtains: $\underline{D}^{+}_{K}W((s,y);(1,v)) + v_0 \le 0$ and (5.5) is proved since $\widehat{v} = (v, v_0) \in K^{1,+}_{\widehat{F}}(s,y)$.

In Case B), since $\widehat{x}(\cdot) = (\widetilde{x}(\cdot), \widetilde{x}_0(\cdot))$ is at least absolutely continuous, one has: $\sigma(\theta) := \|\widehat{x}(s+\theta) - \widehat{y}\| \to 0+$ as $\theta \to 0+$ while for any sequence $\theta_m \to 0+$ the vectors $\widehat{v}_m := \frac{\widehat{x}(s+\theta_m)-\widehat{y}}{\sigma(\theta_m)}$ are bounded hence, taking possibly a subsequence, we may assume that $\widehat{v}_m \to \widehat{v} = (v, v_0) \in \mathbf{R}^n \times \mathbf{R}$, $\| \widehat{v} \| = 1$.

On the other hand, since $\rho(\theta_m) \to +\infty$ one has $r_m := \frac{1}{\rho(\theta_m)} \to 0+$ and $\theta_m := r_m \cdot \sigma_m$, $\sigma_m := \sigma(\theta_m) \to 0+$ hence $\widehat{v}_m := \frac{\widehat{x}(s+r_m\cdot\sigma_m)-\widehat{y}}{\sigma_m} \to \widehat{v} = (v, v_0) \in K^{0,+}_{\widehat{F}}(s,y)$.

Further, as in the previous case, from (5.13) it follows:

$$\frac{W(s+r_m\cdot\sigma_m,\ \widetilde{x}(s+r_m\cdot\sigma_m)) - W(s,y)}{r_m\cdot\sigma_m} + \frac{1}{r_m\cdot\sigma_m}\int_s^{s+r_m\cdot\sigma_m} f_0(t, \widetilde{x}(t), \widetilde{x}'(t))dt = 0$$

hence

$$\frac{W(s+r_m\cdot\sigma_m,\ \widetilde{x}(s+r_m\cdot\sigma_m)) - W(s,y)}{\sigma_m} + \frac{1}{\sigma_m}\int_s^{s+r_m\cdot\sigma_m} f_0(t, \widetilde{x}(t), \widetilde{x}'(t))dt = 0$$

and therefore, from (5.2), as in the previous case it follows: $\underline{D}^{+}_{K}W((s,y);(1,v)) + v_0 \le 0$, and (5.5) is completely proved.

In the case $(s,y) \in \widetilde{E}^{-}$ the inequality in (5.7) follows in the same way using the optimal trajectory $\widetilde{x}(\cdot) \in \widetilde{\Omega}(t_0, x_0)$ in (5.6) and a sequence $\theta_m \to 0-$. $\square$

We note that the inequalities in (5.3), (5.4) may be refined using Clarke's directional derivatives and the sets of "peritangent" directions in Şt. Mirică (2004b)

6 Explicit differential inequalities under additional hypotheses

The rather abstract differential inequalities in Theorem 5.1 may be made more "explicit" (i.e. expressed in terms of the data of the problem) under the additional hypotheses in Theorem 4.7 that allow upper estimates of the sets of generalized tangent directions.

The second main result of this paper is the following direct consequence of Theorems 4.7 and 5.1.

Theorem 6.1. *The value function* $W(\cdot,\cdot)$ *in* (2.9) *has the following properties:*

(i) *If* $\widehat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.3-4.6 then*

$$\overline{D}^{\pm}_K W((s,y);(1,v)) + v_0 \geq 0,\ (\forall)\ \widehat{v} = (v,v_0) \in \overline{co}\widehat{F}(s,y),\ (s,y) \in E^{\mathbf{R}} \bigcap \mathrm{Int}(E) \tag{6.1}$$

and if $\widehat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.5, 4.6 then one has:*

$$\begin{gathered}\overline{D}^{\pm}_K W((s,y);(1,v)) + v_0 \geq 0,\ (\forall)\ \widehat{v} = (v,v_0) \in D^{\infty}[\overline{co}\widehat{F}(s,y)],\\ (s,y) \in E^{\mathbf{R}} \bigcap \mathrm{Int}(E).\end{gathered} \tag{6.2}$$

(ii) *If* $\widehat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.3-4.6 then:*

$$\begin{gathered}\min\{\inf_{\widehat{v}\in\overline{co}\widehat{F}(s,y)} [\underline{D}^{+}_K W((s,y);(1,v)) + v_0],\\ \inf_{\widehat{v}\in D^{\infty}[\overline{co}\widehat{F}(s,y)]} [\underline{D}^{+}_K W((s,y);(0,v)) + v_0]\} \leq 0,\ (\forall)\ (s,y) \in \widetilde{E} \bigcap \mathrm{Int}(E)\end{gathered} \tag{6.3}$$

$$\begin{gathered}\min\{\inf_{\widehat{v}\in\overline{co}\widehat{F}(s,y)} [\underline{D}^{+}_K W((s,y);(1,v)) + v_0],\\ \inf_{\widehat{v}\in D^{\infty}[\overline{co}\widehat{F}(s,y)]} [\underline{D}^{+}_K W((s,y);(0,v)) + v_0]\} \leq 0,\ (\forall)\ (s,y) \in \widetilde{E}^{-} \bigcap \mathrm{Int}(E).\end{gathered} \tag{6.4}$$

The inequalities in Theorem 6.1 may be expressed in terms on the *contingent* (Fréchet) *semidifferentials* defined by:

$$\begin{cases} \overline{\partial}_K g(x) := \{p \in \mathbf{R}^n;\ \ < p, v > \geq \overline{D}^{+}_K g(x;v),\ (\forall)\ v \in K^{+}_x X\} \\ \underline{\partial}_K g(x) := \{p \in \mathbf{R}^n;\ \ < p, v > \leq \underline{D}^{+}_K g(x;v),\ (\forall)\ v \in K^{+}_x X\} \end{cases} \tag{6.5}$$

which, at the points $x \in \mathrm{Int}(X)$ allow several equivalent definitions and are essentially used in the theory of viscosity solutions.

We note that due to the relations

$$\overline{D}^{-}_K g(x;v) = -\underline{D}^{+}_K g(x;-v),\ \underline{D}^{-}_K g(x;v) = -\overline{D}^{+}_K g(x;-v),\ (\forall) v \in K^{-}_x X = -K^{+}_x X \tag{6.6}$$

the semidifferentials in (6.5) may be equivalently characterized by

$$\begin{cases} \overline{\partial}_K g(x) := \{p \in \mathbf{R}^n;\ \ < p, v > \leq \underline{D}^{-}_K g(x;v),\ (\forall)\ v \in K^{-}_x X\} \\ \underline{\partial}_K g(x) := \{p \in \mathbf{R}^n;\ \ < p, v > \geq \overline{D}^{-}_K g(x;v),\ (\forall)\ v \in K^{-}_x X\} \end{cases} \tag{6.7}$$

and are a natural generalizations of the usual (Fréchet) derivative since $g(\cdot)$ is differentiable at $x \in \mathrm{Int}(X)$ iff $\underline{\partial}_K g(x), \overline{\partial}_K g(x) \neq \emptyset$ and in this case:

$$\underline{\partial}_K g(x) = \overline{\partial}_K g(x) = \{Dg(x)\}. \tag{6.8}$$

Using these concepts, the Hamiltonian $H(\cdot,\cdot,\cdot)$ in (3.10) and the "horizon Hamiltonian"

$$H^{\infty}(s,y,p) := \inf_{\hat{v}=(v,v_0)\in D^{\infty}[\overline{co}\hat{F}(s,y)]} [< p, v > + v_0] \tag{6.9}$$

(in the case $\hat{F}(\cdot,\cdot)$ is not locally-bounded) the inequalities in Theorem 6.1 lead to the following result

Corollary 6.2. *The semidifferentials* $\overline{\partial}_K W(\cdot,\cdot)$, $\underline{\partial}_K W(\cdot,\cdot)$ *of the value function* $W(\cdot,\cdot)$ *in* (2.9) *have the following properties:*

(i) *If* $\widehat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.3-4.6 then:*

$$\begin{gathered} p_0 + H(s,y,p) \geq 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \partial_K W(s,y) = \overline{\partial}_K W(s,y) \bigcup \underline{\partial}_K W(s,y), \\ (s,y) \in E^{\mathbf{R}} \bigcap \mathrm{Int}(E) \end{gathered} \tag{6.10}$$

$$\begin{gathered} \min\{p_0 + H(s,y,p),\ p_0 + H^{\infty}(s,y,p)\} \leq 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \underline{\partial}_K W(s,y), \\ (s,y) \in \widetilde{E} \bigcap \mathrm{Int}(E) \end{gathered} \tag{6.11}$$

$$\begin{gathered} \min\{p_0 + H(s,y,p),\ p_0 + H^{\infty}(s,y,p)\} \leq 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \partial_K W(s,y), \\ (s,y) \in \widetilde{E}^{-} \bigcap \mathrm{Int}(E). \end{gathered} \tag{6.12}$$

(ii) *If* $\widehat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.5, 4.6 then:*

$$p_0 + H^{\infty}(s,y,p)\} \geq 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \partial_K W(s,y),\ (s,y) \in E^{\mathbf{R}} \bigcap \mathrm{Int}(E) \tag{6.13}$$

hence:

$$\begin{gathered} \min\{p_0 + H(s,y,p),\ p_0 + H^{\infty}(s,y,p)\} = 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \partial_K W(s,y), \\ (s,y) \in \widetilde{E}^{-} \bigcap \mathrm{Int}(E). \end{gathered} \tag{6.14}$$

(iii) *If* $\hat{F}(\cdot,\cdot)$ *satisfies one of the Hypotheses 4.3-4.6 and the condition in* (5.8) (i.e. *if* $\widehat{F}(\cdot,\cdot)$ *is locally-bounded) then:*

$$p_0 + H(s,y,p) \geq 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \underline{\partial}_K W(s,y),\ (s,y) \in \widetilde{E} \bigcap \mathrm{Int}(E) \tag{6.15}$$

$$\begin{gathered} p_0 + H(s,y,p) = 0,\ (\forall)\ \widehat{p} = (p_0,p) \in \partial_K W(s,y) = \overline{\partial}_K W(s,y) \bigcup \underline{\partial}_K W(s,y), \\ (s,y) \in \widetilde{E}^{-} \bigcap \mathrm{Int}(E) \end{gathered} \tag{6.16}$$

Remark 6.3. We recall that a continuous function $W(\cdot,\cdot)$ is said to be a *viscosity subsolution* of the (HJB)-equation in (3.10) if

$$p_0 + H(s,y,p) \geq 0,\ (\forall)\ (p_0,p) \in \underline{\partial}_K W(s,y),\ (s,y) \in E, \tag{6.17}$$

is said to be a *viscosity supersolution* if

$$p_0 + H(s,y,p) \leq 0,\ (\forall)\ (p_0,p) \in \overline{\partial}_K W(s,y),\ (s,y) \in E, \tag{6.18}$$

and is said to be a viscosity solution of (3.10) if it is both, a viscosity subsolution and a viscosity supersolution.

Therefore, the value function $W(\cdot,\cdot)$ satisfying (6.15) may be interpreted as a *strict* viscosity subsolution on the domain $\widetilde{E} \subseteq E^{\mathbf{R}}$ while if $W(\cdot,\cdot)$ is satisfying (6.16) then it may be interpreted as a "*strict* viscosity solution" of the (HJB) equation in (3.10).

We note also that at some points $(s,y) \in E^{\mathbf{R}}$ the semidifferentials $\underline{\partial}_K W(s,y)$, $\overline{\partial}_K W(s,y)$ may have empty values while the differential inequalities in Theorem 6.1 may still provide some useful information.

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