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Functional Type Caristi-Kirk Theorems

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Abstract. A simplified proof is given for the functional type Caristi-Kirk fixed point result in Bae, Cho and Yeom [2]. In addition, technical extensions of the underlying statement are also proposed. Finally, some applications of the obtained facts to fixed point theorems for weakly contractive multi-maps are considered.

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1 Introduction

Let (M, d) be a complete metric space. Denote, for each $a \in M$, $\varepsilon > 0$,

$$M(a, \varepsilon) = \{x \in M; d(a, x) < \varepsilon\}, \quad M[a, \varepsilon] = \{x \in M; d(a, x) \leq \varepsilon\}.$$

(the open/closed sphere centered at a, with radius ε). Further, given the function $\varphi : M \to R$, put

$$\liminf_{x \to a} \varphi(x) = \sup_{\varepsilon > 0} \inf \varphi(M(a, \varepsilon)), \ \limsup_{x \to a} \varphi(x) = \inf_{\varepsilon > 0} \sup \varphi(M(a, \varepsilon)). \tag{1.1}$$

(the inferior/superior limit of φ at a). Clearly,

$$\liminf_{x \to a} \varphi(x) \leq \varphi(a) \leq \limsup_{x \to a} \varphi(x), \text{ for all such } a. \tag{1.2}$$

When the left (right) inequality becomes equality, we say that φ is *lower (upper) semicontinuous* at a (in short: *lsc (usc)* at a). And if $a \in M$ is generic, then φ is called *lsc (usc)* over M. The sequential characterization of these local properties is immediate, by definition; precisely,

$$\begin{aligned} &\varphi \text{ is lsc at } a \Longleftrightarrow \liminf_n \varphi(x_n) \geq \varphi(a), \text{ if } x_n \to a \\ &\varphi \text{ is usc at } a \Longleftrightarrow \limsup_n \varphi(x_n) \leq \varphi(a), \text{ if } x_n \to a. \end{aligned} \tag{1.3}$$

This in turn, gives a useful characterization of their global counterpars

$$\begin{aligned} &\varphi \text{ is lsc over } M \Longleftrightarrow \{x \in M; \varphi(x) \leq t\} \text{ is closed, for all } t \in R \\ &\varphi \text{ is usc over } M \Longleftrightarrow \{x \in M; \varphi(x) \geq t\} \text{ is closed, for all } t \in R. \end{aligned} \tag{1.4}$$

The verification of all these is obtainable in a direct manner; see, e.g., Barbu and Precupanu [3, Ch 2, Sect 1] for details.

We are now passing to the problem under consideration. Let $x \vdash Tx$ stand for a selfmap of M. The following 1975 fixed point theorem by Caristi and Kirk [7] is our starting point.

Theorem 1. *Suppose that some lsc (over M) function $x \vdash \varphi(x)$ from M to $R_+ = [0, \infty[$ may be found such that*

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad \textit{for all } x \in M. \tag{1.5}$$

Then, T has at least one fixed point in M.

The original proof of this result is by transfinite induction; see also Wong [19]. Note that, in terms of the associated (to φ) order in M

$$(x, y \in M) \ \ x \leq y \ \text{ iff } d(x, y) \leq \varphi(x) - \varphi(y) \tag{1.6}$$

the contractivity condition (1.5) reads

$$x \leq Tx, \text{ for all } x \in M \ \ \text{(i.e.: } T \text{ is } \textit{progressive}). \tag{1.7}$$

So, by the well known Bourbaki meta-theorem [4], this result is logically equivalent with the Zorn maximality principle subsumed to the order (1.6); i.e., with Ekeland's variational principle [9]. Hence, the sequential type argument used by the quoted author to get his principle is also working in our precised framework; see also Pasicki [13]. A proof of Theorem 1 involving the chains of the structure $(M, \leq)$ may be found in Turinici [17]; and its sequential translation has been developed in Dancs, Hegedus and Medvegyev [8]. Further aspects (involving the general case) may be found in Hyers, Isac and Rassias [10, Ch 5]; see also Brunner [6], Taskovic [16] and Valyi [18].

Now, the Caristi-Kirk fixed point theorem found (especially via Ekeland's approach) some basic applications to control and optimization, generalized differential calculus, critical point theory and normal solvability; see the above references for details. So, it must be not surprising that, soon after its formulation, many extensions of Theorem 1 were proposed. For example, in the 1982 paper by Ray and Walker [14], the following result of this type was obtained. Take some function $b : R_+ \to R_+$ with

$$b \text{ is decreasing and } \int_0^\infty b(\tau) d\tau = \infty. \tag{1.8}$$

Theorem 2. *Suppose that some lsc function $\varphi : M \to R_+$ and some $a \in M$ may be found so that*

$$b(d(a, x)) d(x, Tx) \leq \varphi(x) - \varphi(Tx), \quad \textit{for all } x \in M. \tag{1.9}$$

Then, conclusions of Theorem 1 are holding.

Clearly, Theorem 2 includes Theorem 1, to which it reduces when $b = 1$. The reciprocal inclusion also holds (cf. Park and Bae [12]). Summing up,

$$\text{Theorem 1} \Longleftrightarrow \text{Theorem 2} \quad \text{(from a logical viewpoint).} \tag{1.10}$$

Hence, this extension of the Caristi-Kirk fixed point theorem has a technical significance only. However, it may be sometimes useful for concrete applicatons; we refer to the quoted papers for details.

Finally, in the 1994 paper of Bae, Cho and Yeom [2], a lot of "functional" Caristi-Kirk fixed point statements was established, each of these including Theorem 1 as a particular case. In the light of the remarks above, one may ask of whether the reciprocal inclusion is also true. It is our aim in this exposition to give an appropriate (positive) answer to this question; details will be presented in Section 3. The basic tool of our investigations is a general (non-transitive) maximality principle equivalent with Brezis-Browder's [5] (cf. Section 2). The usefulness of such techniques is made clear in Section 4 where some applications are given to weakly contractive closed multi-maps.

2 General Brezis-Browder principles

Let M be some nonempty set. Take a *quasi-order* (i.e.: reflexive and transitive relation) $(\leq)$ over M; as well as a function $x \vdash \varphi(x)$ from M to R_+. Call the point $z \in M$, $(\leq, \varphi)$-*maximal* when

$$w \in M \text{ and } z \leq w \text{ imply } \varphi(z) = \varphi(w).$$

[This may be compared with the property of z being *(strongly)* $(\leq)$-*maximal* (in the sense: $w \in M$ and $z \leq w$ imply $z = w$). But, in general, these are distinct concepts]. A basic result about the existence of such points is the 1976 Brezis-Browder ordering principle [5]:

Proposition 1. *Suppose that*

$$\text{each ascending sequence in } M \text{ has an upper bound} \tag{2.1}$$

$$\varphi \text{ is } (\leq)\text{-decreasing } (x \leq y \Longrightarrow \varphi(x) \geq \varphi(y)). \tag{2.2}$$

Then, for each $u \in M$ there exists a $(\leq, \varphi)$-maximal $v \in M$ with $u \leq v$. So, if (2.2) holds in the stronger sense

$$\varphi \text{ is strictly } (\leq)\text{-decreasing } (x \leq y, x \neq y \Longrightarrow \varphi(x) > \varphi(y)), \tag{2.3}$$

the $(\leq, \varphi)$-maximal v above is $(\leq)$-maximal; i.e., $(\leq)$ is a Zorn quasi-order.

This principle includes the one due to Ekeland [9]. It found some useful applications to convex and nonconvex analysis (cf. the above references). So, technical extensions of it may be not without interest. Roughly speaking, these are obtainable by relaxing the quasi-order conditions imposed upon $(\leq)$. For the developments below, it will suffice considering the non-transitive case. So, take a *reflexive* relation $(\triangleleft)$ over M ($x \triangleleft x$, for all $x \in M$); as well as a function $\varphi : M \to R_+$. Call the point $z \in M$, $(\triangleleft, \varphi)$-*maximal* when

$$w \in M \text{ and } z \triangleleft w \text{ imply } \varphi(z) = \varphi(w).$$

[As before, this is comparable with the property of z being *(strongly)* $(\triangleleft)$-*maximal* (in the sense: $w \in M$ and $z \triangleleft w$ imply $z = w$). But, in general, these are distinct ones]. To establish under which conditions such elements exist, we need a convention. Call the sequence (x_n) in M, $(\triangleleft)$-*ascending*, when

$$x_n \triangleleft x_{n+1}, \quad \text{for all ranks } n.$$

Also, we say that $u \in M$ is an *asymptotic* $(\triangleleft)$-*upper bound* of (x_n) if

$$\text{for each } n, \text{ there exists } m \geq n \text{ with } x_m \triangleleft u.$$

The following 1994 ordering principle involving these data established in Bae, Cho and Yeom [2] is available:

Proposition 2. *Suppose that* (2.2) (*with* $(\triangleleft)$ *in place of* $(\leq)$) *is true and*

$$\text{each } (\triangleleft)\text{-ascending sequence has an asymptotic } (\triangleleft)\text{-upper bound.} \tag{2.4}$$

Then, for each $u \in M$ *there exists a* $(\triangleleft, \varphi)$*-maximal* $v \in M$ *with*

$$\begin{gathered} u = x_1 \triangleleft \ldots \triangleleft x_k = v \ \ (\text{i.e.: } x_i \triangleleft x_{i+1}, \ \ i = 1, ..., k-1), \\ \text{for some } k \geq 2 \text{ and some } x_1, ..., x_k \in M. \end{gathered} \tag{2.5}$$

So, if in addition, (2.3) *is admitted* (*with* $(\triangleleft)$ *in place of* $(\leq)$), *the* $(\triangleleft, \varphi)$*-maximal* v *above is* $(\triangleleft)$*-maximal; i.e.,* $(\triangleleft)$ *is a Zorn reflexive relation.*

Clearly, Proposition 2 includes Proposition 1, to which it reduces when $(\triangleleft)$ is a quasi-order on M. The reciprocal inclusion also holds; wherefrom

$$\text{Proposition 1} \Longleftrightarrow \text{Proposition 2 (from a logical viewpoint).} \tag{2.6}$$

This follows at once from the

Proof of Proposition 2. Let $(\leq)$ stand for the quasi-order on M induced by $(\triangleleft)$:

$$\begin{gathered} x \leq y \text{ iff } x = u_1 \triangleleft ... \triangleleft u_k = y \quad \text{(see above)} \\ \text{for some } k \geq 2 \text{ and some } u_1, ..., u_k \in M. \end{gathered} \tag{2.7}$$

By the admitted hypothesis, it is clear that conditions (2.1)+(2.2) hold for the pair $(\leq, \varphi)$. So, for the starting point $u \in M$ there exists $v \in M$ with

$$u \leq v \quad \text{ and } v \text{ is } (\leq, \varphi)\text{-maximal.}$$

The former of these is just (2.5). And the latter one, in conjunction with

$$z \text{ is } (\leq, \varphi)\text{-maximal} \Longrightarrow z \text{ is } (\triangleleft, \varphi)\text{-maximal}$$

yields the remaining properties of the statement. □

As a consequence, the non-transitive type extension of the Brezis-Browder ordering principle assured by Proposition 2 has a technical significance only. This also tells us that, in all concrete fixed point theorems constructed by means of the Bae-Cho-Yeom ordering principle, a substitution of it by Proposition 1 is always possible. Note finally that this conclusion is also true for arbitrary type relations. Further aspects will be discussed elsewhere.

3 Functional Caristi-Kirk theorems

Let $c : R_+ \to R$ be some function. Denote, for $\alpha \in R_+$

$$\liminf_{t \to \alpha+} c(t) = \sup_{\varepsilon > 0} \inf c([\alpha, \alpha + \varepsilon]), \ \limsup_{t \to \alpha+} c(t) = \inf_{\varepsilon > 0} \sup c([\alpha, \alpha + \varepsilon]) \tag{3.1}$$

(the *right inferior/superior limit* of c at α). Clearly,

$$\liminf_{t \to \alpha+} c(t) \leq c(\alpha) \leq \limsup_{t \to \alpha+} c(t), \text{ for all such } \alpha. \tag{3.2}$$

When the left (right) inequality becomes equality, we say that c is *right lower/upper semicontinuous* at α (in short: *right lsc/usc* at α). And, if $\alpha \in R_+$ is generic, then c is called *right lsc/usc* on R_+. The sequential characterization of these local properties is immediate, by definition; precisely,

$$\begin{aligned} &c \text{ is right lsc at } \alpha \Longleftrightarrow \liminf_n c(t_n) \geq c(\alpha), \text{ when } t_n \downarrow \alpha \\ &c \text{ is right usc at } \alpha \Longleftrightarrow \limsup_n c(t_n) \leq c(\alpha), \text{ when } t_n \downarrow \alpha. \end{aligned} \tag{3.3}$$

(Here, $t_n \downarrow \alpha$ means: $t_n \to \alpha$ and $t_n \geq \alpha$, for all n).

The following simple consequence of these facts is to be noted.

Proposition 3. *Let the notations above be in use.*

i) If c is right usc at α then it is right locally bounded above at α:

$$\textit{there exists } \lambda = \lambda(\alpha) > 0 \textit{ such that } \sup c([\alpha, \alpha + \lambda]) < \infty \tag{3.4}$$

ii) If c is right lsc at α then it is right locally bounded below at α:

$$\textit{there exists } \lambda = \lambda(\alpha) > 0 \textit{ such that } \inf c([\alpha, \alpha + \lambda]) > -\infty. \tag{3.5}$$

Some basic developments now come into discussion. Let (M, d) be a complete metric space; and $x \vdash Tx$, some selfmap of it. The following "functional" Caristi-Kirk fixed point theorem established in Bae, Cho and Yeom [2] is available.

Theorem 3. *Suppose that some lsc (over M) function $\varphi : M \to R_+$ and some right usc (on R_+) function $c : R_+ \to R_+$ may be found so as*

$$d(x, Tx) \leq \max\{c(\varphi(x)), c(\varphi(Tx))\}[\varphi(x) - \varphi(Tx)], \quad x \in M. \tag{3.6}$$

Then, T has at least one fixed point in M.

Proof. We shall follow the authors' approach, with some modifications. For the moment note that, without loss, one may assume

$$c(t) \geq 1, \quad \text{for all } t \in R_+. \tag{3.7}$$

For otherwise, the function $c^* = c + 1$ (satisfying this property) is also right usc on R_+; and, combining with

$$\varphi(x) \geq \varphi(Tx), \quad x \in M, \quad \text{whenever (3.6) holds} \tag{3.8}$$

it is clear that the underlying condition takes place with c substituted by c^*. Let $(\triangleleft)$ stand for the reflexive relation (over M)

$$(x, y \in M)\ x \triangleleft y \text{ iff } d(x, y) \leq \max\{c(\varphi(x)), c(\varphi(y))\}[\varphi(x) - \varphi(y)]. \tag{3.9}$$

We intend to show that conditions of Proposition 2 are fulfilled by our data. This is clearly true for (2.2) and (2.3); so, it remains to verify that (2.4) holds too. Let (x_n) be some $(\triangleleft)$-ascending sequence in M:

$$d(x_n, x_{n+1}) \leq \max\{c(\varphi(x_n)), c(\varphi(x_{n+1}))\}[\varphi(x_n) - \varphi(x_{n+1})], \text{ for all } n. \tag{3.10}$$

By (3.8), $(\varphi(x_n))$ is descending (hence Cauchy) in R_+; wherefrom

$$\alpha = \lim_n \varphi(x_n) \text{ exists (and } \alpha \leq \varphi(x_n), \text{ for all } n).$$

Combining with the right usc at α property, one gets

$$\limsup_n c(\varphi(x_n)) \le c(\alpha);\ \text{so, } c(\varphi(x_n)) \le \mu,\ \text{for all } n \text{ and some } \mu \ge c(\alpha).$$

This, along with (3.10), gives

$$d(x_n, x_{n+1}) \le \mu[\varphi(x_n) - \varphi(x_{n+1})],\ \forall n; \text{ hence } (x_n) \text{ is Cauchy.}$$

As (M, d) is complete and φ is lsc (on M)

$$x_n \to y \text{ (hence } (\alpha =) \lim_n \varphi(x_n) \ge \varphi(y)), \text{ for some } y \in M. \tag{3.11}$$

We now claim that y is an asymptotic $(\triangleleft)$-upper bound for (x_n), in the sense

$$\forall m, \exists n \ge m : d(x_n, y) \le \max\{c(\varphi(x_n)), c(\varphi(y))\}[\varphi(x_n) - \varphi(y)]. \tag{3.12}$$

In fact, assume that the opposite relation holds:

$$\exists m, \forall n \ge m : d(x_n, y) > \max\{c(\varphi(x_n)), c(\varphi(y))\}[\varphi(x_n) - \varphi(y)]. \tag{3.13}$$

This cannot happen under the alternative $\alpha > \varphi(y)$. For, passing to limit as $n \to \infty$ in (3.13) yields (via (3.7))

$$0 = d(y, y) \ge \lim_n [\varphi(x_n) - \varphi(y)] = \alpha - \varphi(y) > 0, \quad \text{contradiction.}$$

So, necessarily

$$(\alpha =) \lim_n \varphi(x_n) = \varphi(y) \quad \text{(if we take (3.11) into account).} \tag{3.14}$$

Now, two cases are open before us.

i) The property below is retainable

$$\text{there exists } k \ge m \text{ such that } c(\varphi(x_n)) \le c(\varphi(x_k)), \text{ for all } n \ge k. \tag{3.15}$$

From (3.10) we therefore get

$$d(x_n, x_{n+1}) \le c(\varphi(x_k))[\varphi(x_n) - \varphi(x_{n+1})], \text{ for all } n \ge k;$$

wherefrom (by the triangle inequality)

$$d(x_k, x_p) \le c(\varphi(x_k))[\varphi(x_k) - \varphi(x_p)], \text{ for all } p \ge k.$$

Performing the limit as $p \to \infty$ gives (via (3.14))

$$d(x_k, y) \le c(\varphi(c_k))[\varphi(x_k) - \varphi(y)] \le \max\{c(\varphi(x_k)), c(\varphi(y))\}[\varphi(x_k) - \varphi(y)];$$

in contradiction with the working condition (3.13).

ii) The written property above is not retainable:

$$\text{for each } k \ge m \text{ there exists } n > k \text{ such that } c(\varphi(x_n)) > c(\varphi(x_k)). \tag{3.16}$$

By repeatedly applying it, one gets

$$\sup\{c(\varphi(x_n)); n \ge m\} = \limsup_n c(\varphi(x_n)) \le c(\varphi(y)).$$

Combining with (3.10) yields

$$d(x_n, x_{n+1}) \le c(\varphi(y))[\varphi(x_n) - \varphi(x_{n+1})], \quad \text{for all } n \ge m; \text{ wherefrom}$$
$$d(x_m, x_p) \le c(\varphi(y))[\varphi(x_m) - \varphi(x_p)], \quad \text{for all } p \ge m.$$

Performing the limit as $p \to \infty$ yields (via (3.14))

$$d(x_m, y) \leq c(\varphi(y))[\varphi(x_m) - \varphi(y)] \leq \max\{c(\varphi(x_m)), c(\varphi(y))\}[\varphi(x_m) - \varphi(y)];$$

in contradiction with the working condition (3.13).

From this discussion it follows that the claim (3.12) involving the point y is retainable. By Proposition 2 there exists, for the starting point $x \in M$, some $(\triangleleft)$-maximal element $z \in M$ with the property (2.5). But then, (3.6) gives at once $z = Tz$. The proof is complete. □

It is to be noted here the very specific character of these developments. Precisely, given the functions $c : R_+ \to R_+$ and $H : R_+^2 \to R_+$, let us consider the functional (Caristi-Kirk type) contractivity condition

$$d(x, Tx) \leq H(c(\varphi(x)), c(\varphi(Tx)))[\varphi(x) - \varphi(Tx)], \quad x \in M. \tag{3.17}$$

The argument we just exposed (patterned, as above said, after Bae, Cho and Yeom [op. cit.]) is applicable only to the choices

$$c \text{ is right usc on } R_+; \quad H(t, s) = \max\{t, s\}. \tag{3.18}$$

So, it is natural to ask of what hapens when these restrictions are no longer applicable. The appropriate answer to this is closely related to the regularity properties introduced via Proposition 3 above. Precisely, call the function $H : R_+^2 \to R_+$, *locally bounded* if

$$\text{the image of each bounded part in } R_+^2 \text{ is bounded (in } R). \tag{3.19}$$

The following "functional" Caristi-Kirk fixed point theorem involving such data may be formulated:

Theorem 4. *Suppose that some lsc function $\varphi : M \to R_+$, a right locally bounded from above $c : R_+ \to R_+$ and some locally bounded function $H : R_+^2 \to R_+$ may be found so as* (3.17) *holds. Then, T has at least one fixed point in M.*

Proof. Denote for simplicity $\alpha = \inf[\varphi(M)]$. By the imposed assumption upon $t \vdash c(t)$, there must be a $\lambda = \lambda(\alpha) > 0$ in such a way that $\mu = \sup c([\alpha, \alpha + \lambda]) < \infty$. Given this μ there exists, by the hypothesis involving $(t, s) \vdash H(t, s)$, some $\nu = \nu(\mu) > 0$ with $H(\tau, \sigma) \leq \nu$, whenever $0 \leq \tau,\ \sigma \leq \mu$. Finally, take some $x_0 \in M$ with

$$\alpha \leq \varphi(x_0) \leq \alpha + \lambda \text{ (possible, by the choice of } \alpha);$$

and put $M_0 = \{x \in M; \varphi(x) \leq \varphi(x_0)\}$. Clearly, M_0 is nonempty (since it contains x_0); moreover, it is closed (hence complete), as φ is lsc (over M). Finally, by (3.8), (also valid in our setting) one gets

$$x \in M_0 \Longrightarrow T(x) \in M_0; \text{ i.e., } M_0 \text{ is } T\text{-invariant.}$$

In particular (cf. the introduced notation) we have for all such x

$$\alpha \leq \varphi(x), \varphi(Tx) \leq \alpha + \lambda; \quad \text{wherefrom}$$
$$c(\varphi(x)), c(\varphi(Tx)) \leq \mu \text{ (hence } H(c(\varphi(x)), c(\varphi(Tx))) \leq \nu).$$

Summing up, the contractivity condition (1.5) (with $\nu\varphi$ in place of φ) is retainable over M_0. This, along with the standard Caristi-Kirk fixed point statement (subsumed to Theorem 1) gives the conclusion we need. □

An interesting particular case of these developments coresponds to the choice $H(t,s) = t$. Precisely, we have (cf. Bae [1])

Theorem 5. *Suppose that, some lsc function $\varphi : M \to R_+$ and some right locally bounded above function $c : R_+ \to R_+$ may be found so that*

$$d(x, Tx) \le c(\varphi(x))[\varphi(x) - \varphi(Tx)], \quad \text{for all } x \in M. \tag{3.20}$$

Then, conclusion of Theorem 4 is retainable.

Finally, as a direct consequence of this, we get an "implicit" functional type Caristi-Kirk fixed point theorem (which also extends a similar one in Bae [op. cit.]). Call the function $g : R_+ \to R_+$, *locally bounded above* if

$$g \text{ is bounded above on } [0, a], \text{ for each } a > 0.$$

Theorem 6. *Suppose that some lsc function $\varphi : M \to R_+$ and a locally bounded above function $g : R_+ \to R_+$ may be found so as*

$$d(x, Tx) \le \min\{\varphi(x), g(d(x, Tx))[\varphi(x) - \varphi(Tx)]\}, \quad x \in M. \tag{3.21}$$

Then, T has at least one fixed point in M.

Proof. By the assumptions about $g : R_+ \to R_+$, the function

$$(c : R_+ \to R_+) \quad c(t) = \sup g([0, t]), \quad t \in R_+$$

is well defined and increasing (hence right locally bounded above) on R_+. Moreover, by (3.21) (the first half)

$$g(d(x, Tx)) \le c(d(x, Tx)) \le c(\varphi(x)), \quad x \in M.$$

This, further combined with (3.21) (the second half) and (3.8), gives the relation (3.20). The desired conclusion is now clear, by Theorem 5 above. □

4 Weakly contractive closed multi-maps

Let (X, d) be a complete metric space. Denote

$$\mathcal{C}(X) = \text{the class of all nonempty closed parts of } X;$$

and let Δ stand for the Hausdorff (generalized) metric over it

$$\Delta(P, Q) = \max\{\sup_{x \in P} \operatorname{dist}(x, Q), \sup_{y \in Q} \operatorname{dist}(y, P)\}, \ P, Q \in \mathcal{C}(X);$$

here, dist(., .) is the usual point to set distance attached to d. Any function $x \vdash Sx$ from X to $\mathcal{C}(X)$ will be refered to as a *closed multivalued map* (in short: *closed multi-map*). The properties of such objects to be considered here are as follows. Let $k : R_+ \to R_+$ be a function; we call it *contractive*, if

$$k(0) = 0 \text{ and } 0 < k(t) < t \quad \text{for all } t > 0. \tag{4.1}$$

Further, take a certain $D \in \mathcal{C}(X)$. The closed multi-map $S : X \to \mathcal{C}(X)$ will be termed *weakly contractive* (modulo k) over D in case

$$\Delta(Sx, Sy) \le d(x, y) - k(d(x, y)), \quad \text{for all } x, y \in D; \tag{4.2}$$

here, $k : R_+ \to R_+$ is the one of (4.1). It is our aim in the following to establish further conditions under the pair (k, S) so that our closed multi-map should have a fixed point $z \in D$ (in the sense: $z \in Sz$). This will necessitate new conventions. Let k_0 stand for the associated function

$$(k_0 : R_+ \to R_+):\quad k_0(0) = 0 \text{ and } k_0(t) = t/k(t) \quad \text{for } t > 0. \tag{4.3}$$

We say that k is *normally contractive* if

$$k \text{ is contractive and } k_0 \text{ is right locally bounded above.} \tag{4.4}$$

The following answer to the posed question is now available (in case $D = X$).

Theorem 7. *Let the closed multi-map $S : X \to \mathcal{C}(X)$ and the contractive function $k : R_+ \to R_+$ be such that*

$$S \text{ is weakly contractive (modulo } k\text{) and } k \text{ is normally contractive.} \tag{4.5}$$

Then, S has at least one fixed point in X.

Proof. Let e stand for the product metric in $X \times X$:

$$e((x, z), (u, v)) = \max\{d(x, u), d(z, v)\}, \quad (x, z), (u, v) \in X \times X;$$

clearly, $(X \times X, e)$ is a complete metric space. Denote also

$$M = \{(x, z) \in X \times X; z \in Sx\} \quad \text{(the } graph \text{ of } S).$$

By the very choice of $x \vdash Sx$ and (4.5) (the first half), M is closed (hence complete) in $X \times X$. Now, assume by contradiction that

$$S \text{ has no fixed points (in } X\text{): } \operatorname{dist}(x, Sx) > 0, \text{ for all } x \in X. \tag{4.6}$$

Let $(x, z) \in M$ be arbitrary fixed. Clearly,

$$d(x, z) > 0 \text{ (hence } k(d(x, z)) > 0); \quad \text{by (4.6) above.}$$

Moreover, again via (4.5) (the first half)

$$\operatorname{dist}(z, Sz) \le \Delta(Sx, Sz) \le d(x, z) - k(d(x, z)) < d(x, z) - (1/2)k(d(x, z)).$$

So, by definition, there must be $v \in Sz$ with

$$d(z, v) \le d(x, z) - (1/2)k(d(x, z)); \quad \text{or, equivalently,}$$
$$d(x, z) \le 2k_0(d(x, z))[d(x, z) - d(z, v)] \quad \text{(hence } d(x, z) \ge d(z, v)). \tag{4.7}$$

Define the (univalued) map $T : M \to M$ and the function $\varphi : M \to R_+$ as

$$T(x, z) = (z, v), \quad \varphi(x, z) = d(x, z), \quad (x, z) \in M.$$

Clearly, T has no fixed points in M (by the choice of these data). On the othert hand, (4.7) (the second half) yields

$$e((x, z), T(x, z)) = e((x, z), (z, v)) = \max\{d(x, z), d(z, v))\} = d(x, z);$$

wherefrom, by the notations above and (4.7) (the first half)

$$e((x, z), T(x, z)) \le 2k_0(\varphi(x, z))[\varphi(x, z) - \varphi(T(x, z))].$$

This, along with φ being continuous over M shows that Theorem 5 applies to these data (with $c = 2k_0$). But then, T must have a fixed point in M, contradiction. The proof is thereby complete. $\square$

Now, (4.4) is assured for the contractive $k : R_+ \to R_+$, whenever

$$\begin{gathered} k \text{ is right locally strictly bounded below at each } \alpha > 0: \\ \text{there exists } \lambda = \lambda(\alpha) > 0 \text{ such that } \inf k([\alpha, \alpha + \lambda]) > 0 \end{gathered} \tag{4.8}$$

$$\begin{gathered} k_0 \text{ is right locally strictly bounded above at } \alpha = 0: \\ \text{there exists } \lambda > 0 \text{ such that } \sup k_0([0, \lambda]) < \infty. \end{gathered} \tag{4.9}$$

In particular, (4.8) is fulfilled under

$$k \text{ is right lsc on } R_+^0 =]0, \infty[\text{ (cf. Section 3).} \tag{4.10}$$

So, Theorem 7 above includes the one due to Bae [1]; see also Reich [15]. This inclusion is a strict one, as simple examples show.

Let us now return to our initial setting. We are interested to establish an appropriate answer to the posed question in the general (modulo D) case. This will require some new conventions. Call the function $k : R_+ \to R_+$, *strongly normally contractive* provided

$$k \text{ is contractive and } k_0 \text{ is locally bounded above (cf. Section 3).} \tag{4.11}$$

Denote also, for each $x \in D$,

$$I_D(x) = \{z \in X \setminus \{x\};]x, z[\cap D \neq \emptyset\}$$

(the *metrical inward set* of D at x). Here, by convention, (with x, z as before)

$$[x, z] = \{y \in X; d(x, z) = d(x, y) + d(y, z)\}; \quad]x, z[= [x, z] \setminus \{x, z\}.$$

(the *closed/open metrical segment* between x and z) The closed multi-map $S : X \to \mathcal{C}(X)$ is said to be *metrically inward* over D provided

$$Sx \subseteq I_D(x), \quad \text{for all } x \in D. \tag{4.12}$$

Theorem 8. *Let the subset $D \in \mathcal{C}(X)$, the closed multi-map $S : X \to \mathcal{C}(X)$ and the contractive function $k : R_+ \to R_+$ be such that*

$$\begin{gathered} S \textit{ is weakly contractive (modulo } k\textit{) over } D \\ \textit{and } k \textit{ is strongly normally contractive} \end{gathered} \tag{4.13}$$

$$S \textit{ is metrically inward over } D. \tag{4.14}$$

Then, S has at least one fixed point in D.

Proof. Let e be the same product metric as in Theorem 7; and

$$M_D = \{(x, z) \in M; x \in D\} \quad \text{(where } M = \text{the graph of } S\text{).}$$

Clearly, M_D is closed (hence complete) in M. Assume by contradiction that

$$S \text{ has no fixed points in } D\text{: } \operatorname{dist}(x, Sx) > 0, \text{ for all } x \in D. \tag{4.15}$$

Let $(x, z) \in M_D$ be arbitrary fixed. Clearly, $z \in Sx \subseteq I_D(x) \setminus \{x\}$ (cf. (4.14)); so, there must be some $u \in D$ with

$$u \neq x, z; d(x, z) = d(x, u) + d(u, z) \ \text{ (i.e.: } u \in]x, z[). \tag{4.16}$$

On the other hand, by (4.13) (the first half)

$$\mathrm{dist}(z, Su) \leq \Delta(Sx, Su) \leq d(x, u) - k(d(x, u)) < d(x, u) - (1/2)k(d(x, u)).$$

So, by definition, there must be $v \in Su$ with

$$d(z, v) \leq d(x, u) - (1/2)k(d(x, u))(\leq d(x, u)). \tag{4.17}$$

Combining with (4.16) and the triangle inequality gives

$$(1/2)k(d(x, u)) \leq d(x, z) - (d(u, z) + d(z, v)) \leq d(x, z) - d(u, v);$$

or, equivalently (cf. the definition of this function)

$$d(x, u) \leq 2k_0(d(x, u))[d(x, z) - d(u, v)] \tag{4.18}$$

Define the univalent map $T : M_D \to M_D$ and the function $\varphi : M_D \to R_+$ as

$$T(x, z) = (u, v), \quad \varphi(x, z) = d(x, z), \quad (x, z) \in M_D.$$

Clearly, T has no fixed point in M_D (by the choice of these data). On the other hand, (4.17) and the definition of u yield

$$e((x, z), T(x, z)) = \max\{d(x, u), d(z, v)\} = d(x, u) \leq d(x, z) = \varphi(x, z);$$

wherefrom (if we take (4.18) into account)

$$e((x, z), T(x, z)) \leq 2k_0(e((x, z), T(x, z)))[\varphi(x, z) - \varphi(T(x, z))].$$

This, along with φ being continuous over M_D, shows that Theorem 6 applies to our data (with $g = 2k_0$). Hence, T must have a fixed point in M_D, contradiction. This ends the argument. □

Now, a sufficient condition for the strongly normal contractivity of k is

$$k_0 \text{ is usc on } R_+ \text{ (in the usual sense).} \tag{4.19}$$

In this case, Theorem 8 above reduces to the one in Bae [op. cit.]. For other aspects we refer to Lim [11] and the references therein.

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