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Fully Developed Flow of a Second-grade Fluid

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Abstract. In this paper we will study the fully developed flow of a second-grade fluid near an oscillating plate. Exact solution is determined for arbitrary frequency of the oscillation. The results obtained are then compared with those of Newtonian fluid. The velocity and pressure profiles are obtained for various dimensionless numbers.

Keywords: Second-grade fluid, Oscillating plate.

1 Introduction

Two distinct features of many non-Newtonian fluids are either the fact that many of them exhibit normal stress differences or the fact that their viscosity depends on the shear rate. Perhaps the simplest model which can predict the normal stress differences is the second-grade fluid, or the Rivlin-Ericksen fluid of grade two([1],[2]). This model has been used and studied extensively [3], and is a special case of fluids of differential type [4].

The constitutive relation for the second grade fluid is given by [5]

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{1}$$

where $\mathbf{T}$ is the Cauchy stress tensor; p is the indeterminate part of the stress due to the constraint of incompressibility, μ is the coefficient of viscosity, and α_1 and α_2 are material moduli usually referred to as the normal stress coefficients. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first and the second Rivlin-Ericksen tensors [1], respectively, and are given by

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T, \tag{2}$$

$$\mathbf{A}_2 = \frac{d}{dt}\mathbf{A}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1, \tag{3}$$

$$\mathbf{L} = \boldsymbol{\nabla}\mathbf{v}, \tag{4}$$

where $\mathbf{v}$ denotes the velocity field, ∇ is the gradient operator, and d/dt is the material time derivative which is defined as

$$\frac{d(.)}{dt} = \frac{\partial(.)}{\partial t} + [\nabla(.)]\mathbf{v}, \tag{5}$$

where $\partial/\partial t$ is the partial time derivative.

2 Mathematical analysis

We consider here a $h-$ high mass of an incompressible second-grade fluid bounded by an infinite flat plate occupying plane $y=0$ and oscillating in its own plane along $x-$ axis. The horizontal homogeneity of the problem shows that the flow quantities depend on y and t only, t being the time variable. For a fully developed flow, we seek velocity of the form

$$\mathbf{v}=v(y,t)\mathbf{i} \tag{6}$$

where $\mathbf{i}$ is the unit vector in the $x-$ direction (the direction of the flow). Substituting (1) and (6) in the balance of the linear momentum,

$$\varrho\frac{d\mathbf{v}}{dt}=div\mathbf{T}+\varrho\mathbf{b}, \tag{7}$$

and using the fact that the fluid can undergo only isochoric motions (incompressibility constraint), i.e. $div\mathbf{v}=0$, we obtain

$$\begin{aligned}
&\varrho\frac{\partial v}{\partial t}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 v}{\partial y^2}+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right),\\
&0=\frac{\partial}{\partial y}\left[-p+(\alpha_2+2\alpha_1)\left(\frac{\partial v}{\partial y}\right)^2\right],\\
&0=\frac{\partial p}{\partial z}.
\end{aligned} \tag{8}$$

If we define a modified pressure p^* through [6]

$$p^*=p-(2\alpha_1+\alpha_2)\left(\frac{\partial v}{\partial y}\right)^2, \tag{9}$$

equations (8) become

$$\begin{aligned}
&\varrho\frac{\partial v}{\partial t}=-\frac{\partial p^*}{\partial x}+\mu\frac{\partial^2 v}{\partial y^2}+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right),\\
&\frac{\partial p^*}{\partial y}=\frac{\partial p^*}{\partial z}.
\end{aligned} \tag{10}$$

Let us look at the boundary conditions for this problem. At the surface of the plane, i.e. at $y=0$, we impose the no-slip condition. That is

$$v=a\cos(\omega t+\varepsilon), \qquad at \quad y=0. \tag{11}$$

At the free surface, we impose the traction-free condition. That is,

$$t_x=t_y=0, \qquad at \quad y=h. \tag{12}$$

Now, $\mathbf{t}=\mathbf{T}^T.\mathbf{n}$, which implies

$$\mu\frac{\partial v}{\partial y}(h,t)+\alpha_1\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial y}(h,t)\right)=0, \tag{13}$$

$$-p+(2\alpha_1+\alpha_2)\left(\frac{\partial v}{\partial y}(h,t)\right)^2=0, \qquad t>0$$

Comparing $(13)_2$ and (9) indicates $p^* = 0$ at $y = h$ which implies $\partial p^*/\partial x = 0$. Therefore, equation $(10)_1$ becomes

$$\varrho \frac{\partial v}{\partial t} = \mu \frac{\partial^2 v}{\partial y^2} + \alpha_1 \frac{\partial}{\partial t}\left(\frac{\partial^2 v}{\partial y^2}\right). \tag{14}$$

Introducing the dimensionless quantities

$$V = \frac{v}{a}, \quad Y = \frac{y}{h}, \quad T = \frac{\mu t}{\varrho h^2}, \quad R = \frac{\mu}{\varrho h^2}, \quad R1 = \frac{\alpha_1}{\varrho h^2}, \quad \Omega = \frac{\varrho h^2}{\mu}\omega, \tag{15}$$

we obtain from (14)

$$\frac{\partial V}{\partial T} = \frac{\partial^2 V}{\partial Y^2} + R1 \frac{\partial}{\partial T}\left(\frac{\partial^2 V}{\partial Y^2}\right), \tag{16}$$

with the boundary

$$\begin{aligned} V &= \cos(\Omega T + \varepsilon), \qquad \text{at} \quad Y = 0, \\ 0 &= \frac{\partial V}{\partial Y} + R1 \frac{\partial}{\partial T}\left(\frac{\partial V}{\partial Y}\right), \qquad \text{at} \quad Y = 1. \end{aligned} \tag{17}$$

To solve (16) we postulate,

$$V(Y,T) = F(Y)\exp\{i(\Omega T + \varepsilon)\} + \bar{F}(Y)\exp\{-i(\Omega T + \varepsilon)\}, \tag{18}$$

and obtain

$$F''(Y) - \Omega \frac{\Omega R1 + i}{1 + \Omega^2 R1^2} F(Y) = 0. \tag{19}$$

The solution to (19) is given by

$$F(Y) = C_1 \exp\{\Lambda Y\} + C_2 \exp\{\Lambda Y\},$$

where C_1, C_2 are arbitrary complex constant, and

$$\Lambda^2 = \Omega \frac{\Omega R1 + i}{1 + \Omega^2 R1^2}.$$

Introducing in (18) and ask the conditions (17) we obtain the solution in the form

$$\begin{aligned} V(Y,T) \;=\; & [A\cos(\beta Y + \Omega T + \varepsilon) - B\sin(\beta Y + \Omega T + \varepsilon)]\exp\{\alpha Y\} \\ & -[A\cos(\beta Y - \Omega T - \varepsilon) + B\sin(\beta Y - \Omega T - \varepsilon)]\exp\{-\alpha Y\} \\ & + \cos(\beta Y - \Omega T - \varepsilon)\exp\{\alpha Y\} \end{aligned} \tag{20}$$

where

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2}}\sqrt{\frac{\Omega}{1 + \Omega^2 R1^2}}\sqrt{\sqrt{1 + \Omega^2 R1^2} + \Omega R1}, \\ \beta &= \frac{1}{\sqrt{2}}\sqrt{\frac{\Omega}{1 + \Omega^2 R1^2}}\sqrt{\sqrt{1 + \Omega^2 R1^2} - \Omega R1}, \end{aligned} \tag{21}$$

and

$$\begin{aligned}
A &= \frac{1}{\Omega\sqrt{1+\Omega^2 R1^2}\sin(2\beta)}[(\beta^2+\alpha^2\Omega^2R1^2)\cos(\beta+\Omega T+\varepsilon)\sin(\beta-\Omega T-\varepsilon) \\
&\quad +(\alpha^2+\beta^2\Omega^2R1^2)\sin(\beta+\Omega T+\varepsilon)\cos(\beta-\Omega T-\varepsilon) \\
&\quad +\frac{\Omega}{2}\cos 2(\Omega T+\varepsilon)-\frac{\Omega^2 R1}{1+\Omega^2R1^2}\sin 2(\Omega T+\varepsilon)], \qquad (22) \\
B &= \frac{1}{\Omega\sqrt{1+\Omega^2 R1^2}\sin(2\beta)}[(\alpha^2+\beta^2\Omega^2R1^2)\cos(\beta+\Omega T+\varepsilon)\cos(\beta-\Omega T-\varepsilon) \\
&\quad +(\beta^2-\alpha^2\Omega^2R1^2)\sin(\beta+\Omega T+\varepsilon)\sin(\beta-\Omega T-\varepsilon) \\
&\quad -\frac{\Omega}{2}\sin 2(\Omega T+\varepsilon)-\frac{\Omega^2 R1}{1+\Omega^2R1^2}\cos 2(\Omega T+\varepsilon)].
\end{aligned}$$

In order to gain an understanding of the flow pattern, we have carried out the numerical computation for the velocity given in (20)-(22).

Figures 1 and 2 show the variation of the dimensionless velocity V with time T for different values of the parameter $R1$ at distance $Y = 0.5$, and $Y = 1$, respectively, from the oscillating plate. The results are given for $R1 = 0$ (Newtonian fluid), 0.4, 0.6 when $\Omega = 0.8$ and $\varepsilon = \pi$. Figure 3 shows the results for the dimensionless velocity at the free surface ($Y = 1$) as a function of time T, respectively for $R1 = 0$ and $\Omega = 0.8$, $R1 = 0$ and $\Omega = 1.2$, $R1 = 0.6$ and $\Omega = 0.8$, and $R1 = 0.6$ and $\Omega = 1.2$. The dimensionless pressure profiles as a function of Y, for $\Omega = 0.8$ at time $T = 4$ is given in Fig. 4.

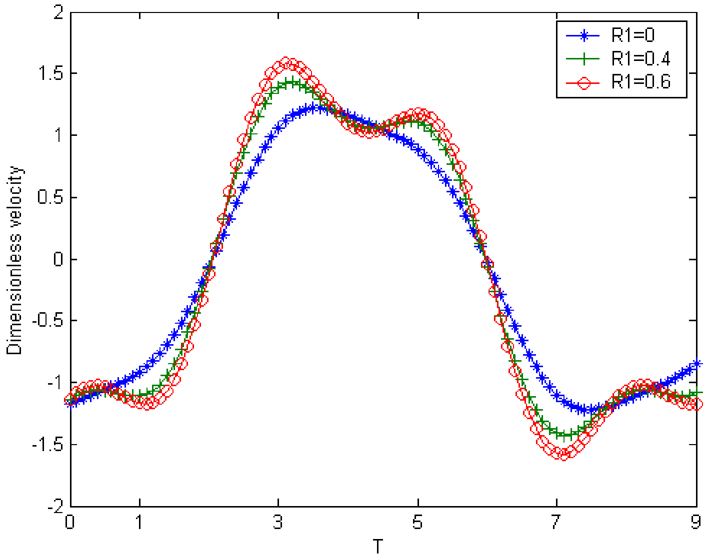

Fig.1 Typical dimensionless velocity profiles for several values of R1 with $\Omega = 0.8, \varepsilon = \pi$ for Y=0.5

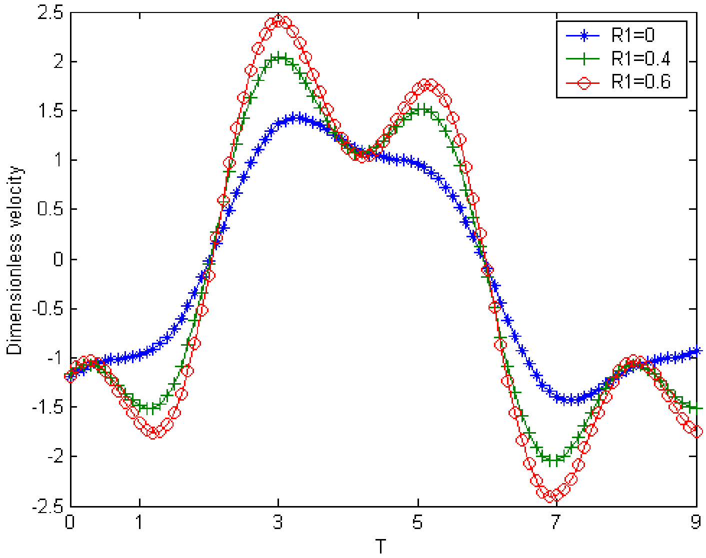

Fig. 2 Typical dimensionless velocity for several values of R1 with $\Omega = 0.8, \varepsilon = \pi$ for $Y = 1$.

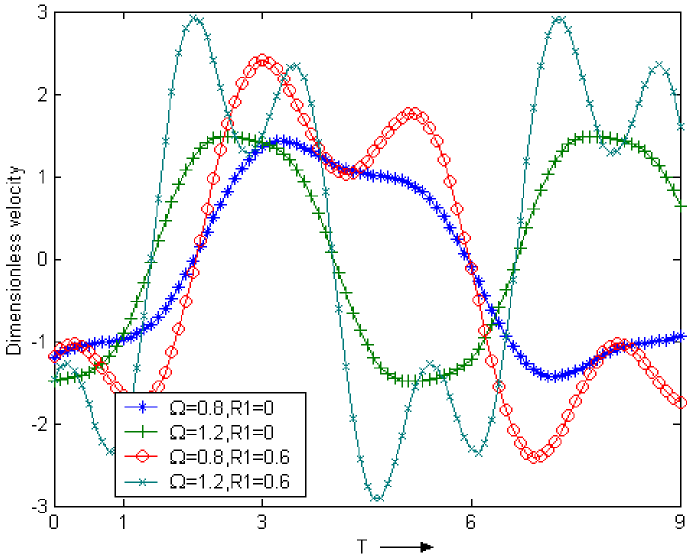

Fig. 3 Typical dimensionless velocity profiles with $\varepsilon = \pi$ and for $Y = 1$.

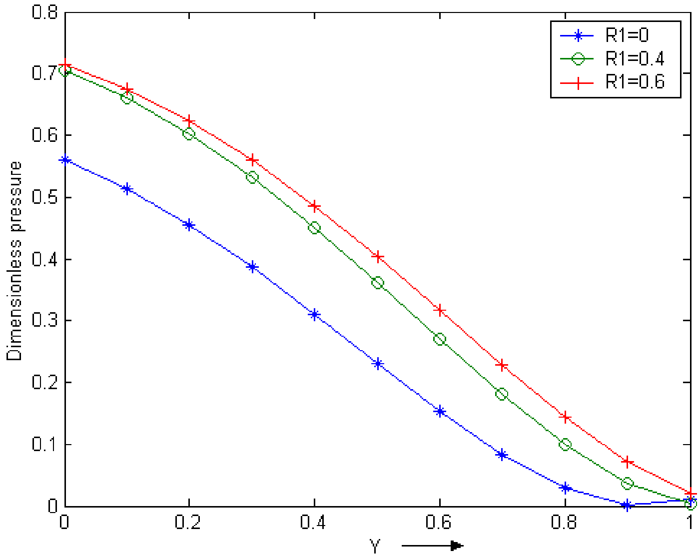

Fig. 4 Dimensionless pressure profiles for several values of R1 with $\Omega = 0.8, \varepsilon = \pi$ at $T = 4$.

3 Conclusions

The fully developed film flow of a second-grade fluid near an oscillating plate is studied. The results in term of dimensionless velocity and pressure profiles are presented for various dimensionless numbers. The significant parameters in this problem are the dimensionless number Ω which is related to the frequency of the oscillations and the dimensionless number $R1$ which is related to the normal stress coefficient α_1.

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