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An Observation About the Abstract Cauchy Problem

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Abstract. The aim of this article: the establish the equivalence of condition (a) - (b) concerning the properly posed Cauchy problem for equations in Banach space:

$$u'(t) = Au(t), \quad t \geq 0 \tag{E}$$

with condition (a) - (b'); these conditions appear in [Fattorini [1, p.29–30]].

Let us consider a Banach space E over the real or complex field, and then a linear operator A, with dense domain $\mathcal{D}(A)$ in E, such that $x \in \mathcal{D}(A) \Rightarrow Ax \in E$.

Next, let $u(\cdot)$ be a function from $[0,\infty)$ to E, such that $u(t)$ is continuously differentiable for $t \geq 0$ (rigth - derivative only for $t = 0$!), and such that $u(t) \in \mathcal{D}(A)$ find a function $u(\cdot)$ as above, such that

$$\frac{du}{dt} = Au(t), \quad t \geq 0 \quad \text{and} \quad u(0) = u_0 \text{ - a given element of } E. \tag{1}$$

This problem is said be well posed (or properly posed) in $t \geq 0$, if

(a) There exists a dence (linear) subspace D of E, such that, for any $u_0 \in D$, there exists a solution $u(\cdot)$ of (1);

(b) There exists a nondecreasing, nonnegative function $C(t)$, $t \geq 0 \to \mathbb{R}$, such that

$$\|u(t)\| \leq C(t)\,\|u(0)\| \quad (t \geq 0) \tag{2}$$

for *any* solution of (E) (not only for solution in (a) where $u(0) \in D$.

It is obvious that $D \subset \mathcal{D}(A)$.

Our main goal in the following lines: to establish the following result.

Theorem 1. *The assumption (a) - (b) is equivalent with assumption (a) - (b') where*

(b') Let $\{u_n(\cdot)\}$ be a sequence of solutions of (E), such that $\lim_{n\to\infty} u_n(0) = \theta$. Then $u_n(t) \to \theta$ uniformly on compacts of $[0,\infty)$.

Proof. Let us assume that (2) holds; if the compact set in $[0,\infty)$ is contained in the interval $[0,T]$, we obtain, for all $t \in [0,T]$, the estimate

$$\|u_n(t)\| \leq C(t)\,\|u_n(0)\| \leq CT\,\|u_n(0)\| \tag{3}$$

which has (b') as a trivial consequence.

Next, we shall establish the converse implication.

Let $u(\cdot)$ be a solution of (1) with $u(0) = \theta$; consider the constant sequence $\{u_n(t)\}$, where $u_n(t) = u(t)\ \forall n = 1, 2, \ldots$. From (b') we derive $u(t) = \theta\ \forall t \geq 0$.

This implies uniqueness of the (linear) initial value problem (1).

Define now the mapping $S(t)$, $D \to \mathcal{D}(A)$ by the relation

$$S(t)\, u_0 = u(t) \quad \text{(the-only-solution with } u(0) = u_0), \quad \forall t \geq 0. \tag{4}$$

It is quite simple to prove that $S(t)$ is a linear operator on D ($\forall t \geq 0$). Afterwards, let us obtain the uniform estimate:

$$\|S(t)\, u_0\| < M(T)\, \|u_0\|, \quad \forall T > 0, \quad \forall t \in [0, T], \quad \forall u_0 \in D \tag{5}$$

(with some constant $M(T) > 0$).

In fact, if (5) would be untrue, there would exist some $T > 0$ in such a way that: $\forall n \in \mathbb{N}$, $\exists t_n \in [0, T]$ and $u_{0,n} \in D$, satisfying the lower estimates

$$\|S(t_n)\, u_{0,n}\| > n\, \|u_{0,n}\| \tag{6}$$

(thus $u_{0,n} \neq \theta\ \forall n = 1, 2, \ldots$).

Consider now elements $v_{0,n}$ defined by

$$v_{0,n} = \frac{1}{n\, \|u_{0,n}\|} \cdot u_{0,n}\ , \quad n \in \mathbb{N}. \tag{7}$$

We see that $v_{0,n} \in D$, $\forall n = 1, 2, 3 \ldots$.

With this notation (6) implies

$$\|S(t_n)\, v_{0,n}\| > 1, \quad n = 1, 2, \ldots \quad \text{and} \quad \|v_{0,n}\| = \frac{1}{n} \to 0 \quad \text{as} \quad n \to \infty. \tag{8}$$

From assumption (a) we can derive; there exists a solution $v_n(\cdot)$ of (E) such that $v_{0,n} = v_n(0)$. As $v_{0,n} \to \theta$ for $n \to \infty$, using (b') we infer that

$$v_n(t) \to \theta \quad \text{as} \quad n \to \infty, \quad \text{uniformly on } [0, T]. \tag{9}$$

Actually, from the very definition of the mapping $S(t)$, we deduce that

$$v_n(t) = S(t)\, v_{0,n}\ , \quad t \geq 0, \quad n \in \mathbb{N} \tag{10}$$

and it follows that

$$\sup_{[0,T]} \|v_n(t)\| = \sup_{[0,T]} \|S(t)\, v_{0,t}\| \geq \|S(t_n)\, v_{0,n}\| > 1, \quad n \in \mathbb{N}. \tag{11}$$

This estimate is in contradiction with (9); we proved in this manner the uniform estimate (5).

Our next step consists in the introduction of the non-nenative function $C(t)$ where

$$C(t) = \sup_{\substack{0 \leq \zeta \leq t \\ u_0 \neq \theta}} \frac{\|S(\zeta)\, u_0\|}{\|u_0\|}\ . \tag{12}$$

We see that $C(t) < +\infty$ because, from (5), we have

$$\frac{\|S(\zeta)\, u_0\|}{\|u_0\|} \leq M(t)\,, \quad \forall \zeta \in [0, t] \quad \text{and} \quad \forall u_0 \in D, \quad u_0 \neq \theta.$$

Therefore $C(t) \leq M(t) < \infty$, $\forall t \geq 0$ and $\|S(t)\, u_0\| \leq C(t)\, \|u_0\|$, $\forall t \geq 0$, $\forall u_0 \in D$. Now, the newly defined function $C(t)$ is non-deceasing: In fact, if $0 \leq t_1 < t_2$ we get obviously

$$C(t_1) = \sup_{\substack{0 \leq \zeta \leq t_1 \\ u_0 \neq \theta}} \frac{\| S(\zeta) u_0 \|}{\| u_0 \|} \leq \sup_{\substack{0 \leq \zeta \leq t_2 \\ u_0 \neq \theta}} \frac{\| S(\zeta) u_0 \|}{\| u_0 \|} = C(t_2). \tag{13}$$

This way we have a proof of (b) for the solutions with initial datum in D. In the final part of the proof we show that (b) actually holds for any solution of (E).

Note first that, for every $t \geq 0$ the operator $S(t)$ is a linear continuous mapping (in E-norm), from D into $\mathcal{D}(A)$ (this is a consequence of (5)). Since D is a dense subset of E we can extend, $\forall t \geq 0$, the operator $S(t)$ to a linear continuous operator, $E \to E$, which we denote by $\tilde{S}(t)$; thus we obtain

$$\tilde{S}(t) u_0 = S(t) u_0 \quad \forall u_0 \in D \quad \text{and} \quad \left\| \tilde{S}(t) \right\|_{\mathcal{L}(E)} \leq C(t), \quad \forall t \geq 0. \tag{14}$$

Next, we shall se that if $u(\cdot)$ is any solution of (E), then

$$u(t) = \tilde{S}(t) u(0), \quad \forall t \geq 0. \tag{15}$$

In fact; if $u(0) \in D$, $u(t) = S(t) u(0)$ (use (4)). If $u(0) \notin D$, consider a sequence $\{u_{0,n}\}$ in D where $u_{0,n} \to u(0)$. It follows that $\tilde{S}(t) u_{0,n} = S(t) u_{0,n} \to \tilde{S}(t) u(0)$ (by the very construction of the extension $\tilde{S}(t)$). We consider the solutions: $u_n(\cdot)$ given by: $u_n(t) = S(t) u_{0,n}$, where $u_n(0) = u_{0,n}$. The sequence of solutions: $\{u_n(t) - u(t)\}$ is such that $u_n(0) - u(0) \to \theta$. Apply (b') and obtain that $u_n(t) \to u(t)$, uniformly on compacts of $[0, \infty)$. Therefore $\tilde{S}(t) u_{0,n} \to u(t)$, $\forall t \geq 0$ and also, as seen above $\tilde{S}(t) u_{0,n} \to \tilde{S}(t) u(0)$. It results: $u(t) = \tilde{S}(t) u(0)$, $\forall t \geq 0$. Then (14) gives

$$\| u(t) \| \leq C(t) \| u(0) \|, \quad \forall t \geq 0$$

which is in fact assumption (b). ■

References

[1] H. O. Fattorini, *The Cauchy Problem* (Encyclopedia of Mathematics and its applications), vol. 18, Addison - Wesley, Massachusetts, 1983.

[2] S. Zaidman, *Abstract differential equations*, Pitman, London, 1979.