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A Class of Holomorphic Functions II

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Abstract. By using the operator $D^n f(z)$, $z \in U$, we shall introduce a class of holomorphic functions. We let $M_n(\alpha)$ denote this class and obtain some subordination results.
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1 Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}(U)$ be the space of holomorphic function in U.

We let:

$$A_n = \left\{f \in H(U), \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\right\}$$

with $A_1 = A$.

If f and g are analytic in U, then we say that f is subordinate to g, written $f \prec g$ or $f(z) \prec g(z)$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Let $K = \{f \in A : \ \text{Re}\, \dfrac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U\}$ denote the class of normalized convex functions in U.

We use the following subordination results.

Lemma A. (S. S. Miller and P. T. Mocanu [2]) *Let g be a convex function in U and let*

$$h(z) = g(z) + \alpha z g'(z)$$

where $\alpha > 0$.

If p is holomorphic in U and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec g(z),$$

where $g(z) = \dfrac{1}{\alpha z^{1/\alpha}} \displaystyle\int_0^z t^{1/\alpha - 1} h(t)dt$ and this result is sharp.

Definition 1. [3] *For $f \in A$ and $n \geq 0$ we define the operator $D^n f$ by*

$$D^n f(z) = f(z) * \frac{z}{(1-z)^{n+1}} = \frac{z}{n!}(z^{n-1}f(z))^{(n)}, \quad z \in U,$$

where $*$ stands for convolution.

Remark 1. We have

$$\begin{aligned} D^0 f(z) &= f(z), \ z \in U \\ D^1 f(z) &= zf'(z), \ z \in U \\ 2D^2 f(z) &= z[D'f(z)]' + D^1 f(z) \\ (n+1)D^{n+1}f(z) &= z[D^n f(z)]' + nD^n f(z) \end{aligned}$$

2 Main results

Definition 2. *For $\alpha < 1$ and $n \in N$, we let $M_n(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality:*

$$\mathrm{Re}(D^n f)'(z) > \alpha.$$

Theorem 1. *Let $h \in \mathcal{H}(U)$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2(n+1)}, \quad n \geq 0.$$

If $f \in A$ and verifies the differential subordination

$$[D^{n+1}f(z)]' \prec h(z), \quad z \in U \tag{1}$$

then

$$[D^n f(z)]' \prec g(z),$$

where

$$g(z) = \frac{n+1}{z^{n+1}} \int_0^z t^n h(t)dt.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordination technique [1], [2], shows that the function g is convex. From

$$(n+1)D^{n+1}f(z) = z\left[D^n f(z)\right]' + nD^n f(z)$$

we obtain

$$(n+1)\left[D^{n+1}f(z)\right]' = z\left[D^n f(z)\right]'' + (n+1)\left[D^n f(z)\right]'.$$

If we let $p(z) = [D^n f(z)]'$ then we obtain

$$\left[D^{n+1}f(z)\right]' = p(z) + \frac{1}{n+1}zp'(z)$$

and (1) becomes

$$p(z) + \frac{1}{n+1}zp'(z) \prec h(z).$$

By using Lemma A we have

$$p(z) \prec g(z) = \frac{n+1}{z^{n+1}} \int_0^z t^n h(t)dt$$

and g is the best dominant. ■

Theorem 2. *Let g be a convex function, $g(0) = 1$ and*

$$h(z) = g(z) + zg'(z).$$

If $f \in A$ and verifies the differential subordination

$$[D^n f(z)]' \prec h(z), \quad z \in U \tag{2}$$

then

$$\frac{D^n f(z)}{z} \prec g(z).$$

Proof. We let $p(z) = \dfrac{D^n f(z)}{z}$, $z \in U$ and we obtain

$$D^n f(z) = zp(z).$$

By differentiating we obtain

$$[D^n f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Then (2) becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma A we have

$$p(z) \prec g(z).$$

■

Theorem 3. *Let $h \in \mathcal{H}(U)$, $h(0) = 1$, $h'(0) \neq 0$ which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}.$$

If $f \in A$ verifies the differential subordination

$$[D^n f(z)]' \prec h(z), \quad z \in U \tag{3}$$

then

$$\frac{D^n f(z)}{z} \prec g(z)$$

where

$$g(z) = \frac{1}{z} \int_0^z h(t)dt.$$

The function g is convex and is the best dominant.

Proof. We let $p(z) = \dfrac{D^n f(z)}{z}$, $z \in U$, $z \neq 0$ and we obtain

$$D^n f(z) = zp(z).$$

By differentiating we obtain

$$[D^n f(z)]' = p(z) + zp'(z).$$

Then (3) becomes

$$p(z) + zp'(z) \prec h(z).$$

By using Lemma A we have

$$p(z) \prec g(z)$$

where

$$g(z) = \frac{1}{z}\int_0^1 h(t)dt,$$

where g is convex and is the best dominant. ■

Corollary 1. *If $f \in M_n(\alpha)$, then*

$$\operatorname{Re}\frac{D^n f(z)}{z} > 2\alpha - 1 + 2(1-\alpha)\ln 2.$$

Proof. From Theorem 3 we deduce

$$\frac{D^n f(z)}{z} \prec g(z) = \frac{1}{z}\int_0^z h(t)dt,$$

where $h(z) = \dfrac{1+(2\alpha-1)z}{1+z}$, $z \in U$.

Hence

$$\operatorname{Re}\frac{D^n f(z)}{z} > q(1) = 2\alpha - 1 + 2(1-\alpha)\ln 2.$$

■

References

[1] S. S. Miller, P. T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker Inc., New York Basel, 2000.

[2] S. S. Miller, P. T. Mocanu, *On some classes of first-order differential subordinations*, Michigan Math. J. 32(1985), 185-195.

[3] St. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. 49(1975), 109-115.