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A Set Theoretic Property of Maximal Sublattices

Amir M. RAHIMI

Abstract. Every lattice L contains the least element 0. All sublattices (unless otherwise indicated) contain 0. For any lattice L, $J(L)$ the Jacobson radical of L is the intersection of all maximal sublattices of L. It is shown that a lattice L satisfies the ascending chain condition on sublattices if and only if every sublattice of L is finitely generated. If L is finitely generated, then every proper sublattice of L is contained in a maximal sublattice of L. A verssion of Nakayama's lemma for finitely generated lattices is presented. Finally, by a constructive method, it is shown that in any lattice that satisfies the ascending chain condition on sublattices if $x \ni J(L)$, then x belongs to some minimal generating set of L.

1 Preliminaries

In this paper, every lattice L contains the least element 0; i.e., $0 \leq a$ for all $a \in L$. All sublattices (unless otherwise indicated) contain 0. A sublattice B of a lattice L is said to be proper whenever B is properly contained in L. For any nonempty subset X of L, the sublattice generated by X is defined to be the intersection of all sublattices of L which contain X and is denoted by (X).

Definition. *A lattice L is said to satisfy the maximum (resp., minimum) condition on sublattices if every nonempty set of sublattices of L contains a maximal (resp., minimal) element with respect to the set theoretic inclusion.*

The proof of Theorem 1.4 in Chapter 8 in [1], which is stated for submodules, is merely set theoretic and we can easily pattern after for sublattices as in the following result.

Theorem 1.1. *A lattice L satisfies the ascending (resp., descending) chain condition on sublattices if and only if L satisfies the maximum (resp., minimum) condition on sublattices.*

Proof. Follow the pattern of the argument in Theorem 1.4 in Chapter 8 of [1]. ■

Theorem 1.2. *A lattice L satisfies the ascending chain condition on sublattices if and only if every sublattices of L is finitely generated.*

Proof. Suppose B is a sublattice of a lattice L. Then let S be the set of all finitely generated sublattices of L which is contained in B. Since $\mathcal{S}$ is nonempty, then $\mathcal{S}$ contains a maximal element C by Theorem 1.1. Thus, C is finitely generated. Say, by $x_1, x_2, \ldots, x_n$. For each $x \in B$, let D_x be the sublattice of L which is contained in B and generated by $x_1, x_2, \ldots, x_n$. Then D_x is an element of $\mathcal{S}$ and $C \subseteq D_x$. Since C is maximal, then $D_x = C$ for every $x \in B$. Whence, $x \in D_x = C$ for every $x \in B$ and $B \subseteq C$. Since $C \subseteq B$ by construction, $B = C$. Thus, B is finitely generated. Conversely, given a chain of sublattices $B_1 \subseteq B_2 \subseteq \ldots$ of L. Then it is easy to verify that $\bigcup\limits_{i \geq 1} B_i$ is a sublattice of L. Therefor, finitely, generated. Say, by $x_1, x_2, \ldots, x_n$. Since each $x_i \in B_j$, then there is an index n such that $x_i \in B_n$ for $i = 1, 2, \ldots, k$. Consequently, $\bigcup B_i$ is a subset of B_n. Whence, $B_i = B_n$ for any $i \geq n$. ■

2 Maximal Sublattices and the Jacobson Radical

Definition. *Let L be a lattice. A proper sublattice M of L is said to be maximal provided that for any sublattice N of L with $M \subseteq N \subseteq L$, then either $M = N$ or $N = L$.*

Theorem 2.1. *Let M be a sublattice of a lattice L. Then M is maximal if and only if for each $x \in L \setminus [M$, $(x, M) = L$.*

PROOF. The necessary part is clear. For the sufficient part, let N be a sublattice of L such that M is properly contained in N. Thus, for any $x \in N \setminus M$, $L = (x, M) \subseteq N$ implies $N = L$. ■

The next result is easily obtained simply by using a proof quite similar to the standard proof given for maximal ideals in a ring with identity.

Theorem 2.2. *Let L be a finitely generated lattice. Then every proper sublattice of L is contained in a maximal sublattice of L.*

PROOF. Let $L = (a_1, a_2, \ldots, a_n)$ and let B be a proper sublattice of L. Let $\mathcal{S}$ denote the set of all proper sublattices S of L such that $B \subseteq S$. Clearly, $\mathcal{S}$ is nonempty since B is an element of $\mathcal{S}$. Parttially order $\mathcal{S}$ by set theoretic inclosion. Let $\{C_i\}_{i \in I}$ be a chain of sublattices in $\mathcal{S}$. Consider $C = \bigcup\limits_{i \in I} C_i$. Since $\{C_i\}_{i \in I}$ is a chain, then C is a sublattice of L. Clearly, $B \subseteq C$. To show that C is properly contained in L, it suffices to show that for some k, $1 \leq k \leq n$ we have $a_k \ni C$. Suppose that $a_j \in C$ for $j = 1, 2, \ldots, n$. Then for each j, $a_j \in C_{i_j}$ for some C_{i_j} an element of $\{C_i\}_{i \in I}$. However since the C_{i_j}'s are ordered, and since there are only a finite number of a_j's, then for some C_{i_k} we have $a_j \in C_{i_k}$ for each $j = 1, 2, \ldots, n$. Thus, $L = (a_1, a_2, \ldots, a_n) \subseteq C_{i_k}$. This contradicts C_{i_k} being an element of $\mathcal{S}$. Hence, $a_k \ni C$ for some $1 \leq k \leq n$ and so C is an element of $\mathcal{S}$. Clearly, C is an upper bound for the chain $\{C_i\}_{i \in I}$. By Zorne's lemma, $\mathcal{S}$ contains a maximal element which clearly is a maximal sublattice that contains B. ■

Definition. *Let L be a lattice. The Jacobson radical of L denoted $J(L)$ is the intersection of all maximal sublattices of L. If no maximal sublattices of L exist, then we set $J(L) = L$.*

The following corollary gives a necessary and sufficient condition for a finitely generated lattice to be trivial.

Corollary 2.3. *Let L be a finitely generated lattice. Then $J(L) = L$ if and only if $L = (0)$.*

Proof. If $J(L) = L$, then L has no maximal sublattices. By Theorem 2.2, (0) is contained in some maximal sublattice unless $(0) = L$. Conversely if $L = (0)$, then clearly $J(L) = L$. ■

Definition 2.1. *An element u of a lattice L is said to be a unit provided that u is not contained in any maximal sublattices of L.*

Theorem 2.4. *Let L be a finitely generated lattice. Then u an element of L is a unit if and only if $(u) = L$.*

Proof. The sufficient part is immediate. For the necessary part, apply Theorem 2.2 above. ■

As a consequence of the above theorem, it follows that a finitely generated lattice L can be generated by a single element of L if the set of all maximal sublattice of L does not cover L.

Theorem 2.5. *Let L be a lattice (not necessarily finitely generated) such that L has a unit element u. If for any $x \in L$ there exists an element y in $J(L)$ such that $u \geq x \vee y$ (resp., $u \leq x \wedge y$), then $u \vee x$ (resp., $u \wedge x$) is a unit in L.*

Proof. Let u be a unit in a lattice L and x an element of L such that $u \geq (x \vee y)$ for some y in $J(L)$. Suppose that $u \vee x$ is in some maximal sublattice M of L. Then $u = u \vee (x \vee y) = (u \vee x) \vee y \in M$. This contradicts u being a unit. Hence, $u \vee x$ is not contained in any maximal sublattice of L and is therefore a unit. The proof of the other case is left to the reader. ■

Theorem 2.6. *Let L be a finitely generated lattice with $L = \left(\{a_i\}_{i \in I}\right)$ where I is an arbitrary index set. If for any $x \in L$ there exists an element y in $J(L)$ such that for some $k \in I$, $a_k \geq (x \vee y)$ (resp., $a_k \leq (x \wedge y)$), then $L = (a_k \vee x, \{a_i\}_{i \neq k})$ (resp., $L = (a_k \wedge x, \{a_i\}_{i \neq I})$).*

Proof. Let $L = \left(\{a_i\}_{i \in I}\right)$. Suppose for some $x \in L$ there exists an element y in $J(L)$ such that $a_k \geq (x \vee y)$ for some $k \in I$. Consider the sublattice B of L such that $B = \left(a_k \vee x, \{a_i\}_{i \neq k}\right)$. If $B \neq L$, then by Theorem 2.2, there exists a maximal sublattice M of L such that $B \subseteq M$. Thus, $a_k = a_k \vee (x \vee y) = (a_k \vee x) \vee y \in M$ implies $L \subseteq M$ which is a contradiction to the choice of M. The proof of the other case is left to the reader. ■

Definition. *A minimal generating set of a lattice L is a subset X of L such that $(X) = L$ and no proper subset of X generates L.*

Theorem 2.7. *Let L be a finitely generated lattice and B a subset of L. Then the following statements are equivalent.*

i) $B \subseteq J(L)$.

ii) If C is a sublattice of L such that $L = (B \cup C)$, then $C = L$.

Proof. (i) implies (ii). Suppose for some sublattice C of L, we have $L = (B \cup C)$. If $C \neg L$, then by Theorem 2.2, there exists a maximal sublattice M of L such that $C \subseteq M$. But $B \subseteq J(L) \subseteq M$ implies that $(B \cup C) \subseteq M \neq L$. Thus, $C = L$. (ii) implies (i). Suppose that for every sublattice C of L such that $L = (B \cup C)$, we have $C = L$. Suppose then that $B \nsubseteq J(L)$. Consequently, there exists a maximal sublattice N of L such that $B \nsubseteq N$. Thus, $(B \cup N) = L$. But, $N \neq L$. This contradicts the hypothesis. Hence, $B \subseteq J(L)$. ■

Remark. *The preceding theorem can be regarded as a version of Nakayama's lemma for finitely generated lattices.*

Theorem 2.8. *Let L be a lattice that satifies the ascending chain condition on sublattices. Then $a \in L$ is a member of some minimal generating set of L whenever a lies outside the Jacobson radical of L.*

Proof. Let $a \ni J(L)$. Then there is a maximal sublattice M of L such that $a \ni M$. Since L satisfies the ascending chain condition on sublattices, then by Theorem 1.2, M is finitely generated. Say, $M = (m_1, m_2, \ldots, m_n)$. Thus, $L = (a, M) = (a, m_1, m_2, \ldots, m_n)$. If necessary, reduce the set $\{a, m_1, m_2, \ldots, m_n\}$ to a minimal generating set for L. Note that a must belong to any such set. ■

REFERENCES

[1] T. W. Hungerford, *Algebra*, Holt, Rinehart and Winst, New York, 1974.