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About Subtractible Elements in a Standard H-Cone

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Abstract. In [4] was proved that in a P -harmonic space, a hyperharmonic function is the supremum of an increasing sequence of potentials which are harmonic outside a compact set. In a balayage space [5], the same is true for a bounded harmonic function. In the previous cases the notions of potential and harmonic are defined using the local structure.

We extend this result to the case of an H-cone. Here it is not quite clear what a harmonic function and consequential, a potential are. The subtractible elements and the pure potentials defined in [3], whose properties are analogous to the harmonic functions, respectively potentials, represent a possible substitute of them.

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Introduction

Throughout the paper S will denote a standard H-cone of functions on a set X [2]. S is endowed with the pointwise defined order, such that $s \wedge t = \inf(s,t)$, for any $s, t \in S$ and for any family $F \subset S$, there exists $\bigwedge F \in S$, which is the semi-continuous regularisation of the pointwise infimum. For any increasing family $F \subset S$, there exists $\bigvee F = \sup F \in S$.

S_0 denotes the set of universally continuous elements of S. For any $s \in S$ there exists an increasing sequence $s_n \in S_0$ such that $s = \vee s_n$.

The natural topology on X is the coarsest topology which makes continuous the functions of S_0 and the fine topology on X is the coarsest topology which makes continuous all the functions of S.

For any $A \subset X$ and $s \in S$, B_s^A is the element of S defined by

$$B_s^A = \bigwedge\{t \in S \mid t \geq s, \text{ on } A\}.$$

If A is fine open, the map $s \to B_s^A$ is a balayage on S.

$[S] = S - S$ is a vector lattice relative to the pointwise defined order; for any $f, g \in [S]$, the element $f \wedge g \in [S]$ may be considered pointwise defined, hence $f \wedge g(x) = \inf(f(x), g(x)),\ \forall x \in X$

We recall the definition of the *specific order.*

$$s, t \in S, \quad s \prec t \quad \text{if } \exists u \in S, \text{such that } s + u = t.$$

S endowed with the specific order is a lattice. We use the notations and for the supremum and infimum with respect to the specific order.

For $s \in S$ *the harmonic carrier* relative to the natural closure of X is

$$carr\ s = \{x \in \overline{X} \mid B^{X \setminus V} s \neq s,\ \forall V \text{ natural neighbourhood of } x\}.$$

In [3] was defined the B_f balayage related to another order on S. For any $f \in [S]_+$ and $s_1, s_2 \in S$ we denote

$$s_1 \leq_f s_2 \Leftrightarrow (s_1 - s_2)_+ \wedge f = 0.$$

The map B_f from s to S, defined by

$$B_f s = \bigwedge \{t \in S \mid s \leq_f t\}$$

is a balayage on S.

We recall that the reduite of an element $f \in [S]$ is the element of S denoted by Rf which means

$$Rf = \bigwedge \{s \in S \mid f \leq s\}.$$

In [2, Theorem 2.2.9] ir was proved that the balayage B_f satisfies

$$B_f s = \bigvee_{n \in N} R(s \wedge n\ f),$$

for any $f \in [S]_+$. Using the previous considerations we get the next result.

Lemma 1. *If $f \in [S]_+$ and $s \in S$ then*

$$B_f s = B^{[f>0]} s.$$

Proof. Because f is finely continuous, the set $[f > 0]$ is finely open and then $B^{[f>0]}$ is a balayage. We prove first that

$$B_f s \leq B^{[f>0]} s.$$

For any $x \in X$ such that $f(x) > 0$, $\inf(s, nf)(x) \leq s(x) = B^{[f>0]} s(x)$ and if $f(x) \leq 0$, $\inf(s, nf)(x) \leq 0$. Hence

$$R(\inf(s, nf) \leq B^{[f>0]} s,$$

since $B^{[f>0]} s \in S$ and $B^{[f>0]} \geq \inf(s, n\ f)$ on X. It follows that

$$B_f \leq B^{[f>0]} s.$$

Conversely, let $x \in X$ such that $f(x) > 0$; if we suppose that s is bounded, there exists $n_0 \in N$ such that $s(x) \leq n_0 f(x)$. Hence $\inf(s(x), nf(x)) = s(x)\ \forall n \geq n_0$; it follows that

$$s(x) = \sup_{n \geq n_0} R(\inf(s(x), nf(x)) = B_f s(x).$$

Hence

$$B_f s \geq B^{[f>0]} s.$$

It results that for any bounded s

$$B_f s = B^{[f>0]} s.$$

For arbitrary s, let be $s_n = \min(s, n)$, which increases to s. Then we have

$$B_f s = B_f \vee (s_n) = \bigvee B_f s_n = \bigvee B^{[f>0]} s_n = B^{[f>0]} s.$$

■

The subtactible elements and the pure potentials were defined and studied in [3].

An element $h \in S$ is called *subtractible* if for any $s \in S$ for which $h \leq s$ it results $h \prec s$. S_H denotes the set of all subtractible elements of S. Hence

$$S_H = \{h \in S \mid \forall s \in S, h \leq s \Rightarrow h \prec s\},$$

An element $p \in S$ is called *pure potential* if for any subtractible element h, we have $s \quad h = 0$. S_P denotes the set of all pure potentials

$$S_P = \{p \in S \mid p \quad h = 0, \ \forall h \in S_H\}.$$

Recall that any element $s \in S$ can be written uniquely of the form $s = h + p, \ h \in S_H, \ p \in S_P$.

We recall from [3] two properties of these classes of elements.

Proposition 2. *Let $h \in S_H$ and $p \in S$ be such that $p \quad h = 0$. Then the balayage B_f, where $f = (h - p)_+$ satisfies the following properties*

$$B_f h = h, \quad B_f p \leq h.$$

Proposition 3. *Let B_n be a decreasing sequence of balayages on S such that there exists a weak unit u of S such that*

$$\bigwedge_{n \in N} B_n u = 0.$$

Then any element $h \in S$ for which

$$B_n h = h, \forall n \in N$$

is subtractible and we have $u \quad h = 0$.

We suppose that X is saturated. In this case it is possible to prove the next theorem [2].

Proposition 4. *Let B be a balayage on $X, B \neq 0$. Then for any $s \in S$ we have*

$$Bs = \sup\{q \in S_0 | Bq = q, q \leq s\}.$$

We shall suppose that S satisfies *the axiom A*; this axiom was introduced in [3] and it stipulates that any universally continuous element of S is a pure potential, hence $S_0 \subset S_P$. This is equivalent with the fact that there is no absorbent point in X with respect to S.

Main result

Theorem 5. *Let $h \in S_H$ be a natural continuous element, such that there exists $x \in X$ for which $h(x) > 0$. Then for any compact $K \subset X$ there exists a sequence $q_i \in S_0, \quad q_i \leq h$ for which $B^{X \setminus K} q_i = q_i$ and*

$$h = \sup q_i.$$

PROOF. First we shall construct a suitable balayage in order to apply Proposition 2. Let $\alpha = \sup_{x\in K} h(x)$; using ([1] Theorem 1.3), for $M > 1$ there exists $p \in S_0$, such that $p = M$ on K and $p \le M$. Let be $f = h - \alpha\frac{p}{M-1}$. We observe that $K \subset [f < 0]$. Indeed for any $x \in K$, we have

$$(M-1)h(x) \le \alpha(M-1) < \alpha p.$$

From Proposition 2 and Lemma 1 we get

$$B^{[f>0]}h = h.$$

Let us observe that $B^{[f>0]} \ne 0$; indeed, if we suppose that $f(x) \le 0$, $\forall x \in X$ then $h(x) \le \alpha\frac{p(x)}{M-1}$ and because $p \in S_P$ and $h \in S_H$ it results that $h = 0$, which is impossible. Now from Proposition 4

$$h = B^{[f>0]}h = \sup\{q \in S_0 | B^{[f>0]}q = q, q \le h\}.$$

From $K \subset [f < 0]$, we deduce

$$q = B^{[f>0]}q \le B^{X\setminus K}q \le q,$$

hence $B^{X\setminus K}q = q$.

Let us prove that the previous family is increasing. Let $q_i \in S_0$, such that $q_i \le h$ and $B^{[f>0]}q_i = q_i$. We recall from ([2] Proposition 4.1.2), that $q_i \vee q_2 \in S_0$. We have $q_1 \vee q_2 \le h$; obviously

$$B^{[f>0]}(q_1 \vee q_2) \le q_1 \vee q_2.$$

Conversely

$$q_i = B_f q_i = \bigvee_{n\in N} R\ (q_i \wedge nf) \le \bigvee_{n\in N} R\ ((q_1 \vee q_1) \wedge nf) = B_f(q_1 \vee q_2).$$

Hence

$$B^{[f>0]}(q_1 \vee q_2) = q_1 \vee q_2.$$

Now, using ([2] Proposition 4.2.1), there exists a countable family $q_i \in S_0$ such that

$$h = \sup_i\{q_i \in S_0 | B^{X\setminus K}\ q_i = q_i, q_i \le h\}$$

We observe that the natural interior of K, denoted *int* (K) has the property

$$int\ (K) \cap carr_n q = \emptyset,$$

for any q from the previous family. Indeed if the natural interior is nonempty, for any $x \in int\ (K)$

$$B^{X\setminus int\ (K)}q \ge B^{X\setminus K}q = q$$

hence $x \notin carr_n\ q$. ■

We suppose the following condition is fulfilled.

(I) There exists an increasing net of compacts K_n such that

$$\bigwedge_{n\in N} B^{X\setminus K_n}u = 0,$$

where u is a weak unit of S.

Theorem 6. *Let be $h \in S$, for which there exists an increasing sequence $q_i \in S_0$ with the properties:*

1. $h = \sup q_i$

2. for any compact K_n from (I) and for any i, there exists $m \geq i$ such that $B^{X\setminus K_n} q_m = q_m$.

Then $h \in S_H$.

PROOF. We shall use Proposition 3. Indeed, $B_n = B^{X\setminus K_n}$ is a decreasing sequence of balayages, for which by (I)

$$\bigwedge_{n \in N} B_n u = 0.$$

We have

$$B_n h = B_n(\sup q_i) = \sup B_n q_i \leq \sup B_n q_m = \sup q_m = h.$$

It follows that $h \in S_H$. ■

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