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A class of second order boundary value problems concerning the capillarity theory

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1. INTRODUCTION

In this paper, we study the existence of solutions for the boundary value problem (possibly degenerate):

$$-(r(t)\partial\varphi(u'(t)))' + q(t)Au(t) \ni f(t), \quad a.e. \ on \ (0,T) \qquad (E)$$

$$[r\partial\varphi(u')_{|t=0}\,, \quad -r(T)\partial\varphi(u'(T))] \in \partial l(u(0), u(T)) \qquad (BC)$$

in a Hilbert space H, where $A : D(A) \subseteq H \to H$ is a maximal monotone operator, $\varphi : H \to (-\infty, +\infty]$, $l : H \times H \to (-\infty, +\infty]$ are lower semicontinuous, convex and proper functions and $\partial\varphi$, ∂l are their subdifferentials. Here we assume that $r \in C^1(0,T]$, $q \in L^\infty(0,T]$, $r(t) > 0$, $q(t) > 0$, $(\forall)t \in (0,T]$ and $f \in L^2(0,T;H)$.

In particular, this problem represents a model for capillarity in circular tubes (see [8]). Let $u = u(x,y)$ be the height of a liquid in a vertical tube situated above the reference plane $u = 0$. The classical equation of capillarity which describes the equilibrium configuration of the liquid surface with constant surface tension in a uniform gravity field ([6],[7],[8]) is

$$div\frac{\nabla u}{(1+\|\nabla u\|^2)^{1/2}} = Ku, \quad (x,y) \in \Omega. \qquad (1.1)$$

Here K is a positive constant and Ω is the domain of the plane $u = 0$, occuppied by the tube. Let γ be the angle between the liquid surface and the lateral surface of the tube (named the contact angle). Then, we associate to (1.1), the boundary condition

$$(1+\|\nabla u\|^2)^{-1/2}\cdot\frac{\partial u}{\partial n}=\cos\gamma, \tag{1.2}$$

where n is the outward normal to the boundary $\partial\Omega$. From a physical point of view, this boundary condition is a natural one.

If we consider the functional (see [3] for $\Omega = I\!R^N$)

$$\mathcal{J}(v)=\int_\Omega\{(1/2)\phi(\|\nabla v\|^2)+h(v)\}dx-\int_{\partial\Omega}\varphi v dx, \tag{1.3}$$

(Ω is a domain of $I\!R^N$, $N\geq 2$), then, formally, every critical point u of $\mathcal{J}$ (i.e. $\mathcal{J}'(u)=0$) is a solution of the boundary value problem

$$div(\phi'(\|\nabla u\|^2)\nabla u)=h'(u) \text{ in } \Omega \tag{1.4}$$

$$\phi'(\|\nabla u\|^2)\frac{\partial u}{\partial n}=\varphi, \text{ on } \partial\Omega. \tag{1.5}$$

The capillarity problem (1.1)-(1.2) is a particular case of (1.4), (1.5), if we take $N=2$ and

$$\phi(\alpha)=2((\alpha+1)^{1/2}-1),\ \ \varphi=\cos\gamma,\ \ h(\alpha)=K\alpha^2/2.$$

If we consider that the tube is circular and the height of the liquid u depends only on the distance r from the origin (in this case we say that u is radially symmetric), then Ω is the unit sphere $B(0,1)=\{x=(x_1,\ldots,x_N\}\in I\!R^N;\ \|x\|^2=x_1^2+\ldots+x_N^2<1\}$ and u depends only on $r=\|x\|$. So we may assume that $\varphi=C$ =constant. Physical, this is not a restriction because, in the case of a circular tube, the liquid surface is a rotation surface.

Supposing that $\Omega=B(0,1)\subseteq I\!R^N$, $N\geq 2$ and $u=u(r)$, we can use in (1.4), (1.5) spherical coordinates: $x_1=r\cos\theta_1$, $x_2=r\sin\theta_1\cos\theta_2$, $x_3=r\sin\theta_1\sin\theta_2\cos\theta_3,\ldots$, $x_{N-1}=r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2}\cos\theta_{N-1}$, $x_N=r\sin\theta_1\ldots\sin\theta_{N-2}\sin\theta_{N-1}$, where $r\in[0,1]$, $\theta_i\in[0,\pi]$, $i=\overline{1,N-2}$, $\theta_{N-1}\in[0,2\pi]$. Thus, (1.4), (1.5) become

$$(r^{N-1}u'(r)\phi'(u'(r)^2))'=r^{N-1}h'(u(r)),\ \ 0<r<1, \tag{1.6}$$

$$u'(1)\phi'(u'(1)^2)=c. \tag{1.7}$$

If we put

$$j(\alpha) = (1/2)\phi(\alpha^2), \quad G(\alpha) = j'(\alpha) = \alpha\phi'(\alpha^2), \quad H(\alpha) = h'(\alpha),$$

the problem (1.6)-(1.7) can be written in the form

$$(r^{N-1}G(u'(r)))' = r^{N-1}H(u(r)), \quad 0 < r < 1, \tag{1.8}$$

$$G(u'(1)) = c. \tag{1.9}$$

The existence and uniqueness of the solution of (1.8)-(1.9) was established, in the case $N = 2$ and $H(\alpha) = \alpha$, by A. Corduneanu and G. Moroşanu in [4], [12] and by Moroşanu in [9]. The same authors studied a more general problem (in [5], [10]):

$$(r^a G(u'(r)))' = r^b u(r), \quad 0 < r < 1, \tag{1.10}$$

$$\lim_{r \to 0_+} r^a G(u'(r)) = 0, \quad G(u'(1)) = c, \tag{1.11}$$

where $a, b \in \mathbb{R}$ such that $b + 1 > \max(0, a)$.

In [11], Moroşanu stated the existence of a unique solution $u \in C^1[0,1]$ for the degenerate boundary problem

$$(p(r)G(u'(r)))' = q(r)H(u(r)), \quad 0 < r < 1, \tag{1.12}$$

$$\lim_{r \to 0_+} p(r)G(u'(r)) = 0, \quad p(1)G(u'(1)) = c, \tag{1.13}$$

under very general assumptions, where $p(0) = q(0) = 0$. As the author suggested in [11], an open problem is the generalization of (1.12), (1.13) to a Hilbert space, in the spirit of [1]. This is the subject of our paper.

The problem (E)-(BC) which we considered, extends (1.12), (1.13) to a Hilbert space (where G and H become maximal monotone operators) and also generalizes the non-degenerate equation of Barbu ([1]) in a Hilbert space:

$$-(\partial\varphi(u'(t))' + Au(t) \ni f(t), \quad a.e. \ t \in (0,T), \tag{1.14}$$

with the boundary condition

$$[\partial\varphi(u'(0)), -\partial\varphi(u'(T))] \in \partial l(u(0), u(T)). \tag{1.15}$$

The structure of this work is the following one: in the second section, we recall some basic notions and properties about maximal monotone operators and subdifferentials mappings in Hilbert spaces. In Section 3, we state our main result, which will be proved in Section 4. Here we give an auxiliary result which is important in itself. In Section 5, we give some particular cases and examples.

2. PRELIMINARY RESULTS

Through this paper, we are given a real Hilbert space H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let A be a maximal monotone operator in H, of domain $D(A)$, which means that

$$(y_2 - y_1,\ x_2 - x_1) \geq 0,\ (\forall) y_i \in Ax_i,\ \ x_i \in D(A),\ \ i = 1,2,$$

and A admits no proper monotone extensions. It is well known that a monotone mapping A is maximal monotone iff $R(I + A) = H$.

One can define the operators $\mathcal{J}_\lambda = (I + \lambda A)^{-1}$ and $A_\lambda = (I - \mathcal{J}_\lambda)/\lambda$ on H $(\lambda > 0)$, named the resolvent of A, respectively the Yosida approximation of A. It is known that $A_\lambda u \in A(\mathcal{J}_\lambda u)$ (see [2]). Let $|\ \ |$ and $\langle\cdot,\cdot\rangle$ the norm and the inner product of $L^2(0,T;H)$.

Given a proper, convex and lower semicontinuous (lsc) function $\varphi : H \to (-\infty, +\infty]$ and $x \in D(\varphi) = \{u \in H,\ \ \varphi(u) < \infty\}$, we denote by

$$\partial\varphi(x) = \{v \in H,\ \ \varphi(x) - \varphi(y) \leq (v, x - y),\ \ (\forall) y \in H\}. \tag{2.1}$$

Such elements $v \in H$ are called subgradients of φ at x and the multivalued mapping $u \to \partial\varphi(u)$ is called the subdifferential of φ.
If φ is Gateaux differentiable at x, then $\partial\varphi(x)$ has a single point and coincides with the Gateaux differential at x. Note that $\partial\varphi$ is maximal monotone in $H \times H$.

We set the regularized function φ_λ associated with φ, given by

$$\varphi_\lambda(u) = \inf\{\|u - v\|^2/2\lambda + \varphi(v),\ \ v \in H\}. \tag{2.2}$$

Now, we recall some basic results, which will be used later (for proof, see [2]).

Lemma 2.1. The function φ_λ given above is convex, Fréchet differentiable on H and $\partial\varphi_\lambda = (\partial\varphi)_\lambda$. Moreover,

$$\varphi_\lambda(u) = \|u - \mathcal{J}_\lambda u\|^2/2\lambda + \varphi(\mathcal{J}_\lambda u),\ \ (\forall)\lambda > 0,\ \ (\forall) u \in H, \tag{2.3}$$

$$\lim_{\lambda \downarrow 0} \varphi_\lambda(u) = \varphi(u), \quad (\forall) u \in H, \tag{2.4}$$

$$\varphi(\mathcal{J}_\lambda u) \leq \varphi_\lambda(u) \leq \varphi(u), \quad (\forall) u \in H, (\forall)\lambda > 0. \tag{2.5}$$

Lemma 2.2. The function $\varphi : H \to (-\infty, +\infty]$ is lsc with respect to some topology, if and only if, all the level sets of the form

$$\{x \in H, \ \varphi(x) \leq \lambda\}, \quad \lambda \in \mathbb{R} \tag{2.6}$$

are closed with respect to that topology.

If φ is convex, these level sets are convex subsets of H.

Lemma 2.3. If φ is a proper, convex, lsc function on H, then φ is bounded from below by an affine function, i.e., there is $y \in H$ and $\mu \in \mathbb{R}$, such that

$$\varphi(x) \geq (x, y) + \mu, \quad \text{for every } \ x \in H.$$

Lemma 2.4. If $\varphi : H \to (-\infty, +\infty]$ is proper, lsc and convex on H and

$$\lim_{\|x\| \to \infty} \varphi(x) = +\infty,$$

then there is a $x_0 \in H$ such that $\varphi(x_0) = \inf\{\varphi(x), \ x \in H\}$.

Lemma 2.5. Let $\varphi : H \to (-\infty, +\infty]$ be proper, convex, lsc on H. Then, the following conditions are equivalent:

$$(a) \quad \lim_{\|x\| \to \infty} \varphi(x)/\|x\| = +\infty; \tag{2.7}$$

$$(b) \quad R(\partial\varphi) = H \ \text{ nd } \ (\partial\varphi)^{-1} \ \text{ is bounded.} \tag{2.8}$$

Lemma 2.6. Let A be maximal monotone on $H \times H$ and let $\varphi : H \to (-\infty, +\infty]$ be a lsc, convex, proper function, such that, there exists $h \in H$ and C real such that

$$\varphi((I + \lambda A)^{-1}(x + \lambda h)) \leq \varphi(x) + C\lambda(1 + \varphi(x)) \tag{2.9}$$

for all $\lambda > 0$ and $x \in H$. Then $A + \partial\varphi$ is maximal monotone.

The operator A is coercive if $(Ax, x) \geq C\|x\|^2$, for all $x \in H$ (C-constant). If A is maximal monotone and coercive, then A is bijective, i.e. $R(A) = H$.

For other results on convex functions and maximal monotone operators, we refer to the book of Barbu [2].

3. THE MAIN RESULT

Let H be a real Hilbert space, with the norm $\|\cdot\|$ and the inner product $(\cdot,\cdot)$. Set $W^{1,p}(0,T) = \{u \in L^p(0,T;H),\ \ u' \in L^p(0,T;H)\}$, where $1 \le p \le \infty$, $T > 0$ is given and the derivative is in the sense of H-valued vectorial distributions on $(0,T)$. We recall that a function $u \in W^{1,p}$ coincides a.e. on $(0,T)$ with an absolutely continuous function (denoted again u) on $[0,T]$, which is a.e. differentiable on $(0,T)$ and whose derivative u' belongs to $L^p(0,T;H)$. So, $W^{1,p}(0,T)$ is a Banach space with the norm$\|\ \|_{1,p}$, defined by

$$\|u\|_{1,p}^p = \|u(0)\|^p + \int_0^T \|u'(t)\|^p dt,\ \ 1 \le p < \infty \tag{3.1}$$

and the corresponding norm from $W^{1,\infty}(0,T)$.

Let A be a maximal monotone operator from H into itself, $\varphi : H \to (-\infty,+\infty]$, $l : H \times H \to (-\infty,\infty]$ proper, convex, lsc functions, $f \in L^2(0,T;H)$, r,q two real functions, $r \in C^1(0,T]$, $q \in L^\infty(0,T)$, $r(t) > 0$, $q(t) > 0$, for every $t \in (0,T]$.

We shall study the existence of the solution for the problem (E)-(BC), under the following hypotheses:

(H1) Suppose that there exists $\alpha > 1$, such that

$$\varphi(u)/\|u\|^\alpha \to \infty \text{ as } \|u\| \to \infty, \tag{3.2}$$

$$l(u,v)/(\|u\| + \|v\|)^\alpha \to \infty \text{ as } \|u\| \to \infty \text{ and } \|v\| \to \infty. \tag{3.3}$$

(H2) There exists $a > 1$ such that

$$\varphi(\lambda u) = \lambda^a \varphi(u), \text{ for every } \lambda > 0 \text{ and } u \in D(\varphi). \tag{3.4}$$

(H3) Denote by D_φ the set $D_\varphi = \{[u(0),u(T)];\ u \in W^{1,1}(0,T;H),\ r\cdot\varphi(u') \in L^1(0,T)\}$.

Suppose that there exists a pair $[u_1,u_2] \in D(l) \cap D_\varphi$ such that one of the following two conditions holds:

$$u_1 \in \operatorname{int}\{u \in H;\ [u,u_2] \in D(l)\}, \tag{3.5}$$

$$u_1 \in \operatorname{int}\{u \in H;\ [u,u_2] \in D_\varphi\}. \tag{3.6}$$

(H4) qA is $\partial\varphi$-monotone, i.e.

$$\varphi((I+\lambda qA)^{-1}x - (I+\lambda qA)^{-1}y) \le \varphi(x-y), \text{ for all } \lambda > 0,\ x,y \in H, \tag{3.7}$$

and, there exists $[h_1, h_2] \in D_\varphi$ such that, for every $\lambda > 0$ and $x, y \in H$,

$$l((I + \lambda qA)^{-1}(x + \lambda h_1),\ (I + \lambda qA)^{-1}(y + \lambda h_2)) \leq l(x, y). \tag{3.8}$$

First, we give the following auxiliary result:

Lemma 3.1. Let q be a real function, $q \in W^\infty(0,T)$, $q(t) > 0$, $(\forall)t \in (0,T]$, $A : D(A) \subseteq H \to H$ a maximal monotone operator and $\tilde{A}$ the realization of A in $L^2(0,T;H)$, which means

$$\begin{aligned}\tilde{A} = \{[u,v] \in L^2(0,T;H) \times L^2(0,T;H);\ u(t) \in D(A)\ \ a.e.\ \ t \in (0,T)\ \text{ and} \\ v(t) \in Au(t),\ \text{ a.e. } t \in (0,T)\}.\end{aligned} \tag{3.9}$$

Then, $q\tilde{A}$ is maximal monotone in $L^2(0,T;H)$.

Proof. The monotonicity of $q\tilde{A}$ is obvious because A is monotone and $q(t) > 0$, a.e. $t \in (0,T]$. Let us prove now the maximal monotonicity of $q\tilde{A}$ in $L^2(0,T;H)$. To this end, assume that $q\tilde{A}$ is not maximal monotone, so, there is a pair $[u_0, v_0] \in L^2(0,T;H) \times L^2(0,T;H)$, $[u_0, v_0] \notin q\tilde{A}$, such that

$$\langle u - u_0, v - v_0\rangle \geq 0, \ \text{ for all } [u,v] \in q\tilde{A}. \tag{3.10}$$

It is known that $\tilde{A}$ is maximal monotone in $L^2(0,T;H)$ and so $R(I + \tilde{A}) = L^2(0,T;H)$. Then, for $v_0/q + u_0 \in L^2(0,T;H)$, there is a pair $[f,g] \in \tilde{A}$, such that

$$f + g = v_0/q + u_0. \tag{3.11}$$

But, from $g \in \tilde{A}f$, it follows $qg \in q\tilde{A}f$, thus, in (3.10), we may put $u = f$, $v = qg$. In view of (3.11), we infer

$$\langle f - u_0,\ qu_0 - qf\rangle \geq 0. \tag{3.12}$$

This is equivalent with $\int_0^T q(t)\|f(t) - u_0(t)\|^2 dt \leq 0$, so $f = u_0$. From (3.11), we have $qg = v_0$. We obtain $[u_0, v_0] = [f, qg] \in q\tilde{A}$, which is a contradiction.

Now we are able to state the main result (an existence result for the problem (E)-(BC)).

Theorem 3.2. If assumptions (H1)-(H4) are satisfied, then there exist two functions $u \in W^{1,\infty}(0,T;H)$ and $p \in W^{1,2}(0,T;H)$ such that

$$p(t) \in \partial\varphi(u'(t)),\ \text{ a.e. on } (0,T), \tag{3.13}$$

$$-(r(t)p(t))' + q(t)Au(t) \ni f(t), \text{ a.e. on } (0,T), \tag{3.14}$$

$$[rp|_{t=0}, -r(T)p(T)] \in \partial l(u(0), u(T)). \tag{3.15}$$

4. The proof of Theorem 3.2. We shall use the method of Barbu [1]. Define the convex proper function $F : L^2(0,T;H) \to (-\infty, +\infty]$,

$$F(u) = \begin{cases} \int_0^T r(t)\varphi(u'(t))dt + l(u(0), u(T)), & \text{if } u \in W^{1,1}(0,T;H) \text{ and } r\varphi(u') \in L^1(0,T); \\ +\infty, & \textit{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.1. The function F is lsc on $L^2(0,T;H)$.

Proof. First, we show that $D(F) \subseteq W^{1,\alpha}(0,T;H)$. Let $u \in D(F) \subseteq \{u \in L^2(0,T;H);\ \varphi(u'(t)) < \infty$, a.e. on $(0,T)$, $l(u(0),u(T)) < \infty\}$. We shall prove that $u \in W^{1,\alpha}(0,T;H)$ endowed with the norm $|\ |_{1,\alpha}$ given by (3.1). If not, we have $| u'(t) | \to \infty$ on a subsequence (and (H1) leads to $\varphi(u'(t)) \to \infty$ on that subsequence) or $| u(0) | \to \infty$ and $| u(T) | \to \infty$ (and (H1) leads to $l(u(0),u(T)) \to \infty$). In both cases, we obtain a contradiction. Thus, we have shown that $D(F) \subseteq W^{1,\alpha}(0,T;H)$.

Next, we prove that F is lsc on $W^{1,\alpha}(0,T;H)$, namely the level sets $C = \{u \in L^2(0,T;H),\ F(u) \leq \lambda\}$, $\lambda \in I\!R$, are closed in $W^{1,\alpha}(0,T;H)$ (with the norm $|\ |_{1,\alpha}$). Let $(u_n) \subseteq C$, $u_n \to u$ in $W^{1,\alpha}(0,T;H)$, as $n \to \infty$. Then, $u_n(0) \to u(0)$ in H as $n \to \infty$ and there exists $(u_{nk}) \subseteq (u_n)$,

$$\lim_{n_k \to \infty} u'_{n_k}(t) = u'(t), \text{ a.e. } t \in [0,T].$$

From Lemma 2.3, there are $a \in H$, $\beta \in I\!R$ such that $\varphi(u) \geq (a,u) + b$, for every $u \in H$. Then, the fact that φ is lsc and Fatou's Lemma imply

$$\int_0^T r(t)\varphi(u'(t))dt \leq \liminf_{n_k \to \infty} \int_0^T r(t)\varphi(u'_{n_k}(t))dt. \tag{4.2}$$

We also have

$$l(u(0), u(T)) \leq \liminf_{n_k \to \infty} l(u_{n_k}(0), u_{n_k}(T)), \tag{4.3}$$

since l is lsc. From (4.2) and (4.3), we deduce that

$$F(u) \leq \liminf_{n_k \to \infty} F(u_{n_k}) \leq \lambda,$$

so $u \in C$. This leads to the fact that F is lsc on $W^{1,\alpha}(0,T;H)$.

Also, hypothesis (H1) implies that every level set is bounded and therefore weakly compact in $W^{1,\alpha}(0,T;H)$. Since F is convex, F is also lsc on $L^2(0,T;H)$, as claimed.

Lemma 4.2. Define the multivalued operator $\mathcal{A} : L^2(0,T;H) \to L^2(0,T;H)$,

$$\mathcal{A}u = -(rp)', \quad \text{for } u \in W^{1,\infty}(0,T;H), \tag{4.4}$$

where $p \in W^{1,2}(0,T;H)$ is such that

$$p(t) \in \partial\varphi(u'(t)) \text{ a.e. on } (0,T), \tag{4.5}$$

$$[rp|_{t=0}, -r(T)p(T)] \in \partial l(u(0),u(T)). \tag{4.6}$$

If (H1) and (H3) hold, then $\mathcal{A} = \partial F$.

Proof. It is easy to establish that $\mathcal{A} \subseteq \partial F$, using the definition of $\mathcal{A}$ and of the subdifferential mapping. To prove that $\mathcal{A} = \partial F$, it suffices to show that $\mathcal{A}$ is maximal monotone on $L^2(0,T;H)$, i.e. $R(I+\mathcal{A}) = L^2(0,T;H)$. This is equivalent with the proof of the fact that, for every $f \in L^2(0,T;H)$, there exist $u \in W^{1,\infty}(0,T;H)$ and $p \in W^{1,2}(0,T;H)$, such that

$$p(t) \in \partial\varphi(u'(t)), \text{ a.e. on } [0,T], \tag{4.7}$$

$$-(r(t)p(t))' + u(t) = f(t), \text{ a.e. on } (0,T), \tag{4.8}$$

$$[rp|_{t=0}, -r(T)p(T)] \in \partial l(u(0),u(T)). \tag{4.9}$$

In order to prove this, for every $\lambda > 0$, let $\varphi_\lambda : H \to (-\infty,+\infty]$, $l_\lambda : H\times H \to (-\infty,+\infty]$ be the regularized convex functions associated with φ and l (defined in Section 2).

Put

$$F_\lambda(u) = \int_0^T r(t)\varphi_\lambda(u'(t))dt + \int_0^T [\|u(t)\|^2/2 - (f(t),u(t))]dt + l_\lambda(u(0),u(T)), \tag{4.10}$$

$$D(F_\lambda) = \{u \in W^{1,1}(0,T;H),\ r\cdot\varphi_\lambda(u') \in L^1(0,T)\}. \tag{4.11}$$

A positive constant k may be found such that

$$\frac{\varphi_\lambda(u)}{\|u\|^{k+1}} \to \infty \text{ as } \|u\| \to \infty, \tag{4.12}$$

$$\frac{l_\lambda(u_1,u_2)}{(\|u_1\|+\|u_2\|)^{k+1}} \to \infty \text{ as } \|u_1\|,\ \|u_2\| \to \infty. \tag{4.13}$$

This is because for every $k > 0$, from Lemma 2.1 (relation (2.5)), it follows

$$\frac{\varphi_\lambda(u)}{\|u\|^{k+1}} \geq \frac{\varphi(\mathcal{J}_\lambda u)}{\|\mathcal{J}_\lambda u\|^\alpha} \cdot \frac{\|\mathcal{J}_\lambda u\|^\alpha}{\|u\|^{k+1}},$$

where $\|\mathcal{J}_\lambda u\| \to \infty$, as $\|u\| \to \infty$. We may choose $k > 0$ such that $\|\mathcal{J}_\lambda u\|^\alpha/\|u\|^{k+1}$ has not the limit 0 and, since hypothesis (H1) implies that $\varphi(\mathcal{J}_\lambda u)/\|\mathcal{J}_\lambda u\|^\alpha \to \infty$ as $\|u\| \to \infty$, we deduce (4.12). Similarly, one arrives at (4.13).

Using the same argument as that used for F, we can show that F_λ is lsc from $L^2(0,T;H)$ to $(-\infty, +\infty]$ and, according to (4.12), (4.13), we have

$$F_\lambda(u) \to \infty, \quad \text{as} \quad |u| \to \infty \ (\lambda > 0), \tag{4.14}$$

(in fact, $F_\lambda(u)/ \mid u \mid^{k+1} \to \infty$ as $\mid u \mid \to \infty$). In view of Lemma 2.4, we find $u_\lambda \in W^{1,1}(0,T;H)$ such that

$$F_\lambda(u_\lambda) \leq F_\lambda(v), \quad (\forall) v \in W^{1,1}(0,T;H). \tag{4.15}$$

But φ_λ, l_λ are Frechet differentiable (in the Hilbert space H), hence (4.15) gives us

$$\int_0^T r(\partial\varphi_\lambda(u'_\lambda), v')dt + \int_0^T (u_\lambda - f, v)dt + ((\partial l_\lambda(u_\lambda(0), u_\lambda(T)), [v(0), v(T)])) = 0, \tag{4.16}$$

for every $v \in W^{1,\infty}(0,T;H)$. Here $((\ ,\ ))$ is the inner product in $H \times H$. In particular, we obtain the following equation in the sense of distributions over $(0,T)$:

$$-(r\partial\varphi_\lambda(u'_\lambda))' + u_\lambda - f = 0. \tag{4.17}$$

Since $(r\partial\varphi_\lambda(u'_\lambda))' = u_\lambda - f \in L^2(0,T;H)$, we arrive at

$$-(r\partial\varphi_\lambda(u'_\lambda))' + u_\lambda = f, \quad \text{a.e. on} \quad (0,T). \tag{4.18}$$

Multiplying (4.18) by $v \in W^{1,2}(0,T;H)$, integrating over $(0,T)$ by parts, we infer via (4.16):

$$(r\partial\varphi_\lambda(u'_\lambda)_{|t=0}, v(0)) - r(T)(\partial\varphi_\lambda(u'_\lambda(T)), v(T)) = ((\partial l_\lambda(u_\lambda(0), u_\lambda(T)), [v(0), v(T)])).$$

But v is arbitrary in $W^{1,2}(0,T;H)$, hence

$$[r\partial\varphi_\lambda(u'_\lambda)_{|t=0}, -r(T)\partial\varphi_\lambda(u'_\lambda(T))] = \partial l_\lambda(u_\lambda(0), u_\lambda(T)). \tag{4.19}$$

In view of Lemma 2.5 and (4.12), $(\partial\varphi_\lambda)^{-1}$ is defined on all of H and it is bounded on bounded subsets of H. Since $r\partial\varphi_\lambda(u'_\lambda) \in W^{1,2}(0,T;H)$, we get $u'_\lambda \in L^\infty(0,T;H)$, so $u_\lambda \in W^{1,\infty}(0,T;H)$. From (4.15), (4.10) and the boundedness of $F_\lambda(v)$, we see that

$$\int_0^T \|u_\lambda(t)\|^2 dt \le C, \quad (\forall)\lambda > 0. \tag{4.20}$$

Indeed, we have

$$\int_0^T \|u_\lambda(t)\|^2 dt \le 2\int_0^T (f(t), u_\lambda(t))dt - 2\int_0^T r(t)\varphi_\lambda(u'_\lambda(t))dt - 2l_\lambda(u_\lambda(0), u_\lambda(T)) \le$$

$$\le 2 \mid f \mid \cdot \mid u_\lambda \mid -2C_1\int_0^T r(t)dt - 2C_2,$$

where C_1, C_2 are positive constants, which are inferior bounds for φ_λ and l_λ:

$$\varphi_\lambda(u) \ge \varphi(\mathcal{J}_\lambda u) \ge \inf\{\varphi(y),\ y \in H\} = C_1 > -\infty \tag{4.21}$$

and analogously for l_λ. These imply (4.20).

Now, from (4.18) and (4.20), it follows that

$$(r \cdot \partial\varphi_\lambda(u'_\lambda))' \text{ is bounded in } L^2(0,T;H). \tag{4.22}$$

Next, let us prove that

$$\|r\partial\varphi_\lambda(u'_\lambda)_{|_{t=0}}\| + \|r(T)\partial\varphi_\lambda(u'_\lambda(T))\| \text{ is bounded in } H, \text{ as } \lambda \downarrow 0. \tag{4.23}$$

To this end, considere the two cases from the hypothesis (H3).

Case I: Suppose that condition (3.5) holds. Then, there exists $w \in W^{1,1}(0,T;H)$ such that $\varphi(w') \in L^1(0,T)$ and

$$w(0) \in \text{int}\{y \in H,\ [y, w(T)] \in D(l)\}. \tag{4.24}$$

So, $w(0)$ belongs to the interior of the effective domain of the function $y \to l(y, w(T))$. But it is well known that l is continuous on $\text{int}D(l)$, hence $y \to l(y, w(T))$ is locally bounded at $y = w(0)$. Then, there are $\rho, c > 0$ constants,

$$l(w(0) + \rho\omega, w(T)) \le C, \quad (\forall)\omega \in H, \ \|\omega\| = 1. \tag{4.25}$$

Denoting $p_\lambda(t) = \partial\varphi_\lambda(u'_\lambda(t))$, (4.19) implies

$$l_\lambda(u_\lambda(0), u_\lambda(T)) - l_\lambda(v_1, v_2) \le (([rp_{\lambda_{|_{t=0}}}, -r(T)p_\lambda(T)], [u_\lambda(0), u_\lambda(T)] - [v_1, v_2])). \tag{4.26}$$

For $v_1 = w(0) + \rho\omega$ and $v_2 = w(T)$, one obtains:

$$l_\lambda(u_\lambda(0), u_\lambda(T)) \le l_\lambda(w(0)+\rho\omega, w(T)) + (rp_{\lambda|_{t=0}}, u_\lambda(0)-w(0)-\rho\omega) - (r(T)p_\lambda(T), u_\lambda(T)-w(T)). \tag{4.27}$$

On the other hand, taking the inner product of (4.18) by $u_\lambda - w$ and integrating by parts over $(0,T)$, we get

$$-\int_0^T (r\partial\varphi_\lambda(u'_\lambda), u'_\lambda - w')dt + \int_0^T (f - u_\lambda, u_\lambda - w)dt =$$
$$= (rp_{\lambda|_{t=0}}, u_\lambda(0) - w(0)) - (r(T)p_\lambda(T), u_\lambda(T) - w(T)). \tag{4.28}$$

From the definition of $\partial\varphi_\lambda(u'_\lambda(t))$ and (4.21), it follows

$$-\int_0^T (r\partial\varphi_\lambda(u'_\lambda), u'_\lambda - w')dt \le \int_0^T [r\varphi_\lambda(w') - r\varphi_\lambda(u'_\lambda)]dt \le \int_0^T [r\varphi(w') - rC_1]dt,$$

so (4.28) implies (with the aid of (4.20)):

$$(rp_{\lambda|_{t=0}}, u_\lambda(0) - w(0)) - (r(T)p_\lambda(T), u_\lambda(T) - w(T)) \le C_3, \ \ (\forall)\lambda > 0. \tag{4.29}$$

We set $\omega = (rp_{\lambda|_{t=0}})/\|rp_{\lambda|_{t=0}}\|$ in (4.27) and we use (4.29) to deduce that

$$\rho\left(rp_{\lambda|_{t=0}}, \frac{rp_{\lambda|_{t=0}}}{\|rp_{\lambda|_{t=0}}\|}\right) + l_\lambda(u_\lambda(0), u_\lambda(T)) \le l_\lambda(w(0) + \rho\frac{rp_{\lambda|_{t=0}}}{\|rp_{\lambda|_{t=0}}\|}, w(T)) + C_3. \tag{4.30}$$

But $l_\lambda(u_1, u_2) \ge C_2$ and $l_\lambda(u_1, u_2) \le l(u_1, u_2)$, for every $u_1, u_2] \in H \times H$. Thus, (4.30) gives us the boundedness of $rp_{\lambda|_{t=0}}$ in H and (4.22) implies that $r(T)p_\lambda(T)$ is bounded in H, for every λ. We have shown (4.23).

Case II: Suppose that condition (3.6) holds. There exists a function $w \in W^{1,1}(0,T;H)$ such that $l(w(0), w(T)) < \infty$ and

$$w(0) \in \text{int}\{y \in H, \ \ [y, w(T)] \in D\varphi\}. \tag{4.31}$$

Let ϕ be the function

$$\phi(y_1, y_2) = \inf\{\int_0^T [r\varphi(y') + \frac{1}{2}\|y\|^2 - (f, y)]dt, \ \ y \in W^{1,1}(0,T;H), \ \ y(0) = y_1, \ \ y(T) = y_2\}.$$

We note that ϕ is convex and lsc on $H \times H$, $D(\phi) = D_\varphi$ and the infimum from the definition of ϕ is attained. It follows from (4.31) that there exist $\rho > 0$ and $C > 0$, such that

$$\phi(w(0) + \rho\omega, w(T)) \le C, \ \ (\forall)\omega \in H, \ \ \text{with } \|\omega\| = 1. \tag{4.32}$$

We multiply again (4.18) by $u_\lambda - w$ and integrate from 0 to T, to find

$$-(rp_{\lambda_{|t=0}}, u_\lambda(0) - w(0) - \rho\omega) + (r(T)p_\lambda(T), u_\lambda(T) - w(T)) \geq$$

$$\geq \int_0^T [r\varphi_\lambda(u'_\lambda) + \frac{\|u_\lambda\|^2}{2} - (f, u_\lambda)]dt - \phi(w(0) + \rho\omega, w(T)). \tag{4.33}$$

Now, we use (4.26) for $v_1 = w(0)$, $v_2 = w(T)$. We have

$$-(rp_{\lambda_{|t=0}}, u_\lambda(0) - w(0)) + (r(T)p_\lambda(T), u_\lambda(T) - w(T)) \leq l_\lambda(w(0), w(T)) - l_\lambda(u_\lambda(0), u_\lambda(T)).$$

This together with (4.33) implies

$$-\rho(rp_{\lambda_{|t=0}}, \omega) \leq l(w(0), w(T))) + C_4. \tag{4.34}$$

Taking $\omega = -rp_{\lambda|_{t=0}}/\|rp_{\lambda|_{t=0}}\|$, it follows that $rp_{\lambda_{|t=0}}$ is bounded in H with respect to λ and, consequently (4.23) in both cases.

Observe that (4.22), (4.23) and the equality

$$r(T)\partial\varphi_\lambda(u'_\lambda(t)) = r(t)\partial\varphi_\lambda(u'_\lambda(t)) + \int_t^T (r\partial\varphi_\lambda(u'_\lambda))'(s)ds \tag{4.35}$$

lead to the boundedness of $\{r\partial\varphi_\lambda(u'_\lambda)\}$ in $L^\infty(0,T;H)$, and therefore

$$\{u'_\lambda\} \text{ is bounded in } L^\infty(0,T;H), \tag{4.36}$$

because $\varphi(\mathcal{J}_\lambda u'_\lambda) \leq \varphi_\lambda(u'_\lambda) \leq \varphi_\lambda(u_0) + (\partial\varphi_\lambda(u'_\lambda), u'_\lambda - u_0)$ and because (3.2) is valid. Now, we may pass to the limit in (4.20), (4.22), (4.35), 4.36) on a subsequence (denoted again u_λ):

$$u_\lambda \to u \text{ weakly-star in } L^\infty(0,T;H) \tag{4.37}$$

$$u'_\lambda \to u' \text{ weakly-star in } L^\infty(0,T;H) \tag{4.38}$$

$$rp_\lambda = r\partial\varphi_\lambda(u'_\lambda) \to rp \text{ weakly-star in } L^\infty(0,T;H) \tag{4.39}$$

$$(rp_\lambda)' = (r\partial\varphi_\lambda(u'_\lambda))' \to (rp)' \text{ weakly in } L^2(0,T;H), \tag{4.40}$$

as $\lambda \downarrow 0$.

But, for every $\varphi \in C^1(0,T;H)$, (4.40) and (4.39) imply:

$$\int_0^T ((rp)', \varphi)(t)dt = \lim_{\lambda\downarrow 0}\int_0^T ((rp_\lambda)', \varphi)(t)dt = \lim_{\lambda\downarrow 0}[(rp_\lambda, \varphi)\mid_0^T - \int_0^T (rp_\lambda, \varphi')(t)dt] =$$

$$\lim_{\lambda\downarrow 0}[(r(T)p_\lambda(T),\varphi(T))-(rp_{\lambda_{|t=0}},\varphi(0))]-r(T)(p(T),\varphi(T))+(rp_{|t=0},\varphi(0))+\int_0^T((rp)',\varphi)(t)dt,$$

so,

$$\lim_{\lambda\downarrow 0}[r(T)(p_\lambda(T)-p(T),\varphi(T))-(rp_{\lambda_{|t=0}}-rp_{|t=0},\varphi(0))]=0,\ \text{ or} \tag{4.41}$$

$$[r\partial\varphi_\lambda(u'_\lambda)_{|t=0},r(T)\partial\varphi_\lambda(u'_\lambda(T))]\to[rp_{|t=0},r(T)p(T)]\ \text{ weakly in} H\times H. \tag{4.42}$$

Similarly, we have

$$[u_\lambda(0),u_\lambda(T)]\to[u(0),u(T)]. \tag{4.43}$$

Now, let us prove the strong convergence of u_λ in $L^2(0,T;H)$ and then, pass to the limit in (4.18), (4.19).

Let u_λ and u_μ be two solutions of (4.18)-(4.19). Subtracting the corresponding equations, multiplying by $u_\lambda-u_\mu$ and integrating by parts over $[0,T]$, one finds:

$$((\partial l_\lambda(u_\lambda(0),u_\lambda(T))-\partial l_\mu(u_\mu(0),u_\mu(T)),[u_{\lambda_{|t=0}}-u_{\mu_{|t=0}},u_\lambda(T)-u_\mu(T)]))+$$

$$+\int_0^T\|u_\lambda(t)-u_\mu(t)\|^2dt+\int_0^T(r\partial\varphi_\lambda(u'_\lambda)-r\partial\varphi_\mu(u'_\mu),u'_\lambda-u'_\mu)(t)dt=0. \tag{4.44}$$

If we denote $B=\partial\varphi$, then, from Lemma 2.1, $B_\lambda=\partial\varphi_\lambda$ and $B_\lambda u'_\lambda\in B(\mathcal{J}_\lambda u'_\lambda)$, where $\mathcal{J}_\lambda=(I+\lambda B)^{-1}$. Since $u'_\lambda=\mathcal{J}_\lambda u'_\lambda+\lambda B_\lambda u'_\lambda$, we deduce that

$$(r\partial\varphi_\lambda(u'_\lambda)-r\partial\varphi_\mu(u'_\mu),u'_\lambda-u'_\mu)=r(B_\lambda u'_\lambda-B_\mu u'_\mu,\mathcal{J}_\lambda u'_\lambda-\mathcal{J}_\mu u'_\mu)+$$

$$+r(B_\lambda u'_\lambda-B_\mu u'_\mu,\lambda B_\lambda u'_\lambda-\mu B_\mu u'_\mu)\geq -r(\lambda+\mu)(B_\lambda u'_\lambda,B_\mu u'_\mu)\geq-(\lambda+\mu)\cdot C,$$

because $rB_\lambda u'_\lambda$ is bounded in $L^\infty(0,T;H)$. Passing to the limit as $\lambda,\mu\downarrow 0$, we get

$$\limsup_{\lambda,\mu\downarrow 0}(r\partial\varphi_\lambda(u'_\lambda)-r\partial\varphi_\mu(u'_\mu),u'_\lambda-u'_\mu)(t)\geq 0$$

and, analogously

$$\limsup_{\lambda,\mu\downarrow 0}((\partial l_\lambda(u_\lambda(0),u_\lambda(T))-\partial l_\mu(u_\mu(0),u_\mu(T)),[u_{\lambda_{|t=0}}-u_{\mu_{|t=0}},u_\lambda(T)-u_\mu(T)]))\geq 0.$$

Then, (4.44) holds if and only if

$$u_\lambda\to u\ \text{ strongly in }\ L^2(0,T;H) \tag{4.45}$$

$$\limsup_{\lambda,\mu\downarrow 0}\int_0^T(r\partial\varphi_\lambda(u'_\lambda)-r\partial\varphi_\mu(u'_\mu),\ u'_\lambda-u'_\mu)dt=0 \tag{4.46}$$

$$\limsup_{\lambda,\mu\downarrow 0}((\partial l_\lambda(u_\lambda(0),u_\lambda(T)) - \partial l_\mu(u_\mu(0),u_\mu(T)),[u_\lambda(0)-u_\mu(0),u_\lambda(T)-u_\mu(T)])) = 0. \tag{4.47}$$

Taking into account (4.46), (4.38) and (4.39), we may pass to the limit in $(\partial\varphi_\lambda)(u'_\lambda(t)) \in \partial\varphi(\mathcal{J}_\lambda u'_\lambda(t))$, so

$$p(t) \in \partial\varphi(u'(t)) \text{ a.e. on } (0,T).$$

The relations (4.42), (4.43) and (4.47) are used for to pass to the limit in (4.19) and we conclude

$$[rp|_{t=0}, -r(T)p(T)] \in \partial l(u(0),u(T)).$$

Finally, passing to the limit in (4.18), it follows that

$$-(r(t)p(t))' + u(t) = f(t), \text{ a.e. } (0,T),$$

so u is a solution of $\mathcal{A}u + u \ni f$, for every fixed $f \in L^2(0,T;H)$. This completes the proof of the lemma.

Proof of Theorem 3.2. Denote by A_1 the realization of A in $L^2(0,T;H)$, i.e.

$$A_1 = \{[u,v] \in L^2(0,T;H)\times L^2(0,T;H);\ u(t)\in D(A) \text{ a.e. on } (0,T) \text{ and}$$

$$v(t) \in Au(t), \text{ a.e. on } (0,T)\}.$$

In view of Lemma 3.1, qA_1 is a maximal monotone set in $L^2(0,T;H)\times L^2(0,T;H)$, and according to Lemma 4.2, the problem (3.13)-(3.15) can be written as

$$qA_1u + \partial F(u) \ni f. \tag{4.48}$$

Let $[h_1,h_2] \in D_\varphi$ be as in the hypothesis (H4). Then, there exists $h \in W^{1,1}(0,T;H)$ such that $r\varphi(h') \in L^1(0,T)$, $h(0) = h_1$, $h(T) = h_2$. The assumptions (H2) and (H4) imply

$$\varphi([J_\lambda(u(t)+\lambda h(t))]') \le \varphi(u'(t)+\lambda h'(t)), \text{ a.e. on } (0,T), \tag{4.49}$$

for every $u \in W^{1,1}(0,T;H)$, where $J_\lambda = (I+\lambda qA)^{-1}$. This is because

$$\varphi([J_\lambda(u(t)+\lambda h(t))]') = \varphi(\lim_{s\to 0}[J_\lambda(u(t+s)+\lambda h(t+s)) - J_\lambda(u(t)+\lambda h(t))]/s) =$$

$$\liminf_{s\to 0}\frac{1}{s^a}\varphi(J_\lambda(u(t+s)+\lambda h(t+s)) - J_\lambda(u(t)+\lambda h(t))) \le$$

$$\liminf_{s\to 0}\varphi\left(\frac{u(t+s)-u(t)}{s}+\lambda\frac{h(t+s)-h(t)}{s}\right)=\varphi(u'(t)+\lambda h'(t)).$$

Now, since φ is a conex function and hypothesis (H2) holds, we have

$$\varphi(u+\lambda v)\le\frac{1}{1+\lambda}\varphi((1+\lambda)u)+\frac{\lambda}{1+\lambda}\varphi((1+\lambda)v)=(1+\lambda)^{a-1}\cdot\varphi(u)+\lambda(1+\lambda)^{a-1}\cdot\varphi(v),$$

for every $u,v\in H$ and $\lambda>0$. This and (4.49) lead to

$$\varphi([J_\lambda(u(t)+\lambda h(t))])\le(1+b(\lambda))\varphi(u'(t))+\lambda(1+\lambda)^{a-1}\cdot\varphi(h'(t)),\tag{4.50}$$

a.e. on $(0,T)$, where $\lim_{\lambda\to 0}b(\lambda)/\lambda=a-1$. This relation together with assumption (H4) imply that, for every $u\in W^{1,1}(0,T;H)$ and $\lambda>0$,

$$F((I+\lambda qA_1)^{-1}(u+\lambda h))\le(1+b(\lambda))F(u)+\lambda(1+\lambda)^{a-1}\cdot\int_0^T r(t)\varphi(h'(t))dt.\tag{4.51}$$

Hence, with the aid of Lemma 2.6, $qA_1+\partial F$ is maximal monotone in $L^2(0,T;H)\times L^2(0,T;H)$. As in the proof of Lemma 4.2, we have

$$F(u)/\mid u\mid\to\infty,\ \text{ as }\ \mid u\mid\to\infty,$$

so, $\partial F+qA_1$ is coercive. Thus, $\partial F+qA_1$ is surjective, i.e. $R(\partial F+qA_1)=L^2(0,T;H)$.

We have proved that, for every $f\in L^2(0,T;H)$, there is $u\in D(\partial F)\cap D(A_1)$, such that $\partial F(u)+qA_1u\ni f$, or, equivalently, equation (4.48) has at least a solution, as claimed.

5. Particular cases. Applications.

Taking $\varphi(u)=\frac{1}{2}\|u\|^2$ in the preceding theorem, we have $\partial\varphi(u)=u$ and the hypotheses (H1), (H2), (H4) for φ are satisfied. We have also $D_\varphi=H\times H$, so (H3) is valid. Now we can enunciate the following result:

Theorem 5.1. Let A be a maximal monotone operator of H and $l:H\times H\to(-\infty,\infty]$ a lsc, convex, proper function which satisfies the assumptions (3.3) and (3.8). Then, for every $f\in L^2(0,T;H)$, the problem

$$r(t)u''(t)+r'(t)u'(t)\in q(t)Au(t)+f(t),\ \text{ a.e. on }\ (0,T)\tag{5.1}$$

$$[ru'_{|t=0},-r(T)u'(T)]\in\partial l(u(0),u(T))\tag{5.2}$$

has a solution $u \in W^{2,2}(0,T;H)$ and this solution is unique up to an additive constant.

Proof. The existence is immediate from Theorem 3.2.

To prove the uniqueness, let u, v be two solutions of (5.1), (5.2) and $w = u - v$. Subtracting the corresponding equations and multiplying by w, we obtain

$$(rw'' + r'w', w) \geq 0.$$

We integrate this inequality by parts over $[0,T]$ and we arrive at $\int_0^T r(t)\|w'(t)\|^2 dt \leq r(w', w)\,|_0^T$, hence

$$\int_0^T r(t)\|u'(t) - v'(t)\|^2 dt \leq$$
$$\leq -(([(ru' - rv')_{t=0}, -r(T)u'(T) + r(T)v'(T)], [u(0) - v(0), u(T) - v(T)])) \leq 0,$$

because ∂l is monotone. From this, we deduce that $u - v$ =const.

Remark 5.1. Boundary condition (BC) includes as special cases many important two-point boundary condition. For example, if $l(u_1, u_2) = l_1(u_1) + l_2(u_2)$, with l_1, l_2 lsc, proper, convex functions from H to $(-\infty, \infty]$, then (BC) becomes

$$r\partial\varphi(u')_{|_{t=0}} \in \partial l_1(u(0)), -r(T)\partial\varphi(u'(T)) \in \partial l_2(u(T)). \tag{5.3}$$

If

$$l(u_1, u_2) = \begin{cases} 0, & \text{if } u_1 = a \text{ and } u_2 = b, \\ +\infty, & \text{otherwise,} \end{cases}$$

where $a, b \in D(A)$ are given and $\varphi(u) = \frac{1}{2}\|u\|^2$, then Theorem 5.1. leads to:

Corollary 5.2. If A is a maximal monotone operator in H, $a, b \in D(A)$ and $f \in L^2(0,T;H)$, then there exists a unique function $u \in W^{2,2}(0,T;H)$ such that

$$r(t)u''(t) + r'(t)u'(t) \in q(t)Au(t) + f(t), \text{ a.e. on } (0,T), \tag{5.4}$$

$$u(0) = a, \quad u(T) = b. \tag{5.5}$$

Now, we state an existence result for a partial differential equation:

Let Ω be a bounded open set, $\Omega \subseteq I\!R^n$ and Γ its smooth boundary. Let β be a maximal monotone graph in $I\!R \times I\!R$ and let $j : I\!R \to (-\infty, +\infty]$ be such that $\partial j = \beta$. Suppose that

$$j(r)/r^2 \geq M, \text{ for } |r| \text{ large enough,} \tag{5.6}$$

$$0 \in D(\beta), \; 0 \in \beta(0), \; j(\lambda r) = \lambda^a j(r), \text{ for all } \lambda > 0, \; r \in \mathbb{R}, \tag{5.7}$$

where $a > 1$ is given. Let $p \in [2, \infty)$ and p' its conjugate.

Define $A : W_0^{1,p}(\Omega) \to (W_0^{1,p}(\Omega))' = W^{-1,p'}(\Omega)$, $A = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(\|\frac{\partial u}{\partial x_i}\|^{p-2}\frac{\partial u}{\partial x_i})$ the differential operator defined by

$$(Au, v) = \sum_{i=1}^{n}\int_\Omega \|\frac{\partial u}{\partial x_i}\|^{p-2}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}dx, \text{ for all } u, v \in W_0^{1,p}(\Omega).$$

We denote again A the restriction of this operator to $D(A) = \{u \in L^2(\Omega);\; Au \in L^2(\Omega)\}$. It is known that $A = \partial\psi$, where $\psi : L^2(\Omega) \to (-\infty, +\infty]$ is defined as

$$\psi(u) = \begin{cases} \frac{1}{p}\cdot\sum_{i=1}^{n}\int_\Omega \|\frac{\partial u}{\partial x_i}\|^p dx, & \text{if } u \in W_0^{1,p}(\Omega) \\ +\infty, & \text{otherwise,} \end{cases}$$

and $\beta(u(x)) = (\partial\varphi(u))(x)$, a.e. on Ω, where $\varphi : L^2(\Omega) \to (-\infty, +\infty]$ is the lsc function given by

$$\varphi(u) = \begin{cases} \int_\Omega j(u(x))dx, & \text{if } j(u) \in L^1(\Omega) \\ +\infty, & \text{otherwise.} \end{cases}$$

Let β_1, β_2 two maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, such that $0 \in D(\beta_1)$, $j_1, j_2 : \mathbb{R} \to (-\infty, \infty]$ the lsc, convex functions with the property $\partial j_i = \beta_i$, $i = 1, 2$. Suppose that

$$j_1(r)/r^2 \geq M, \text{ for } |r| \text{ large enough,} \tag{5.8}$$

$$j_2(u) \in L^1(\Omega), \text{ for every } u \in L^2(\Omega). \tag{5.9}$$

Let $l(u, v) = \int_\Omega [j_1(u(x)) + j_2(u(x))]dx$, for all $u, v \in L^2(\Omega)$.

We shall apply Theorem 3.2, with $H = L^2(\Omega)$, φ, l, A defined above, $r \in C^1(0, T]$, $q \in L^\infty(0, T)$, $r(t) > 0$, $q(t) > 0$ on $(0, T]$, $f : [0, T]\times\Omega \to L^2(\Omega)$, $f \in L^2(0, T; L^2(\Omega))$. To this end, let us verify the hypotheses of this theorem. We observe that (H1), (H2) hold. For (H3), we need to impose the existence of $u_0, v_0 \in L^2(\Omega)$ and $u \in W^{1,1}(0, T; L^2(\Omega))$ such that

$$u(0, x) = u_0(x), \; u(T, x) = v_0(x), \text{ a.e. on } \Omega \tag{5.10}$$

$$r \cdot j(\partial u/\partial t) \in L^1((0, T) \times \Omega), \; j_1(u_0) \in L^1(\Omega). \tag{5.11}$$

Let us verify now the hypothesis (H4). Since $\beta_\lambda(0) = 0$ and β_λ is monotone, we find

$$\int_\Omega \beta_\lambda(u(x) - v(x))(Au(x) - Av(x))dx \geq 0, \text{ for every } u, v \in W_0^{1,p}(\Omega),$$

so,

$$(\partial\varphi_\lambda(u-v),\ qAu-qAv)\geq 0,\ \text{ for all } u,v\in D(A),\ \ \lambda>0,$$

or, equivalently,

$$\varphi((I+\lambda qA)^{-1}u-(I+\lambda qA)^{-1}v)\leq\varphi(u-v).$$

Similarly, we obtain condition (3.8). Now we can give

Corollary 5.3. Let A,φ,l be given as above, $f\in L^2(0,T;L^2(\Omega))$, $r\in C^1(0,T]$, $q\in L^\infty(0,T)$, $r(t)>0$, $q(t)>0$, for every $t\in(0,T]$. Then, there exists at least a solution $u(t,x)$ of the boundary value problem

$$\frac{\partial}{\partial t}(r(t)\beta(\frac{\partial u}{\partial t}(t,x))+q(t)\sum_{i=1}^{n}\frac{\partial}{\partial x_i}(\|\frac{\partial u}{\partial x_i}(t,x)\|^{p-2}\frac{\partial u}{\partial x_i}(t,x))\ni f(t,x)\ \text{ a.e. on } (0,T)\times\Omega \tag{5.12}$$

$$r(t)\beta(\frac{\partial u}{\partial t}(t,x))_{|_{t=0}}-\beta_1(u(0,x))\ni 0,\ \text{ a.e. on } \Omega \tag{5.13}$$

$$r(T)\beta(\frac{\partial u}{\partial t}(T,x))+\beta_2(u(T,x))\ni 0,\ \text{ a.e. on } \Omega \tag{5.14}$$

$$u(t,x)=0 \qquad\qquad \text{on } \Sigma, \tag{5.15}$$

which satisfies $\frac{\partial u}{\partial t}\in L^\infty(0,T;L^2(\Omega))$, $Au\in L^2((0,T)\times\Omega)$, $u\in L^p(0,T;W_0^{1,p}(\Omega))$.

Another application of the existence Theorem 3.2, is to the problem (1.12)-(1.13) which was studied by Moroşanu [11]. There, we have $H=I\!R$, $[0,T]=[0,1]$, $r=p\in C(0,1]$, $q\in L^1(0,1)$, $p>0$, $q>0$ a.e. on $(0,1)$, $f\equiv 0$, $\partial\varphi=G$, $A=H$, where $G,H:I\!R\to I\!R$ are continuous, strictly incresing, $G(0)=0$, $H(0)=0$, satisfying some other properties. This problem has some applications in the capillarity theory, as we mentioned in the first section.

References

[1] V. Barbu, Existence theorems for a class of two point boundary problems, J. Diff. Eq., 17(1975), 236-257.

[2] V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff, Leyden, 1976.

[3] L. Caffareli, N. Garofalo and F. Segala, A gradient bound for entire solutions of quasi-linear equations and its consequences, Comm. Pure Appl. Math. 47(11), 1994, 1453-1473.

[4] A. Corduneanu et G. Moroşanu, Une équation intégro-différentielle de la théorie de la capillarité, C.R. Acad. Sci. Paris, t. 319, Serie I (1994), 1171-1174.

[5] A. Corduneanu and G. Moroşanu, A nonlinear integro-differential equation related to a problem from capillarity theory, Comm. Appl. Nonlinear Anal. 3(1996), 51-60.

[6] R. Finn, Equilibrium Capillary Surfaces, Springer-Verlag, New-York Inc., 1986.

[7] D. Gilbarg and N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Second Edition, Springer, Berlin-Heidelberg-New-York-Tokyo, 1983.

[8] L. Landau et E. Lifchitz, Mécanique des fluides, Mir, Moscou, 1971.

[9] G. Morosanu, A variational approach to a problem arising in capillarity theory, J. Math. Anal. Appl., 206(1997), 442-447.

[10] G. Moroşanu, On a problem arising in capillarity theory, Analele Univ. "Ovidius" Constantza, Ser. Mat. 3(1995), 127-137.

[11] G. Moroşanu, On a second order boundary value problem related to capillary surfaces, in "Proceedings to the Conference on Analysis, Numerics and Applications of Differential and Integral Equations", Stuttgart, 1996, in print.

[12] G. Moroşanu and A. Corduneanu, An integro-differential equation from the capillarity theory, Libertas Mathematica 14(1994), 115-123.